A subset of generalized Pell's equations $x^2 - Dy^2 = \pm 4$ and it's *abc*-properties

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Abstract: The rules describing emergence of *abc*-triples formed by the set of roots for the specific family of Pell's equations $x^2 - Dy^2 = \pm 4$ are revealed.
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1 Introduction

The proposed article is planned as continuation of [1], where necessary background knowledge concerning radicals of numbers, *abc*-conjecture, continuants and continued fractions was summarized. Right there are the main theorems about primary/secondary *abc*-triples in ordinary Pell's equations $x^2 - Dy^2 = \pm 1$ as well as about genuine ambiguity.

2 The principal equation $x^2 - Dy^2 = \pm 4$

2.1 From genuine ambiguity to Property A

In [1] we described genuine ambiguity as the link between fundamental roots $K(\rho, \omega)/K(\omega)$ of the generalized Pell's equations

$$x^2 - D \cdot y^2 = \pm 2 \tag{1}$$

and fundamental roots $K(a_0, \pi)/K(\pi)$ of the corresponding Pell's resolvent

$$x^2 - D \cdot y^2 = 1, \tag{2}$$

which is of the form

$$\begin{cases} K(\pi) &=& K(\rho, \omega) \cdot K(\omega), \\ K(a_0, \pi) &=& \dfrac{1}{2} \cdot [K^2(\rho, \omega) + D \cdot K^2(\omega)]. \end{cases} \tag{3}$$

Both sequences of roots – from Pell's resolvent (2) and from the ambiguous one (1) – are not in the equivalent position, because ordinary positive Pell's equation (2) is the basis for all generalized Pell's equations. It is possible to construct a situation of root systems from two generalized Pell's equations, linked by relations, similar to (3). We take

$$x^2 - D \cdot y^2 = N \neq \pm 1, \tag{4}$$

it's fundamental solution is $\pm K(\rho', \omega') + \sqrt{D} \cdot K(\omega')$, and

$$x^2 - D \cdot y^2 = -N, \tag{5}$$

it's fundamental solution is $\pm K(\rho, \omega) + \sqrt{D} \cdot K(\omega)$. Both root systems of (4) and (5) have the same discriminant D and are linked by (3) analogue

$$\begin{cases} K(\omega') &= K(\rho, \omega) \cdot K(\omega), \\ K(\rho', \omega') &= \frac{1}{2} \cdot [K^2(\rho, \omega) + D \cdot K^2(\omega)]. \end{cases} \tag{6}$$

From (4), (5) and (6):

$$\begin{aligned} N &= K^2(\rho', \omega') - D \cdot K^2(\omega') \\ &= \frac{1}{4}[K^4(\rho, \omega) + 2D \cdot K^2(\rho, \omega) \cdot K^2(\omega) + D^2 \cdot K^4(\omega)] - D \cdot K^2(\rho, \omega) \cdot K^2(\omega) \\ &= \frac{1}{4}[K^2(\rho, \omega) - D \cdot K^2(\omega)]^2 = \frac{1}{4} \cdot (-N)^2 = \frac{1}{4}N^2. \end{aligned}$$

So $N = 4$ and this works for single N value. Henceforth we will call such system of equations (4) and (5) with $N = 4$, linked with conditions (6), as corresponding to Property A.

We need *abc*-equations, therefore coprimality requirement gives fundamental roots $K(\rho', \omega')$, $K(\omega')$, $K(\rho, \omega)$ and $K(\omega)$ as odd numbers. As odd squares are congruent to 1 modulo 8, we get as necessary condition $D \equiv 5 \pmod{8}$.

Generalized Pell's equations with $N = 4$ were studied already from the times of L. Euler. Here we will use one significant result of A. Cayley [2], that from odd fundamental solutions of equations (4) and (5) it is possible to obtain fundamental solutions for ordinary Pell's equations $x^2 - D \cdot y^2 = \pm 1$. In continuant expressions it is the following:

$$\begin{cases} K(a_0, \pi) &= \frac{1}{2} \cdot [K^2(\rho, \omega) + 3] \cdot K(\rho, \omega), \\ K(\pi) &= \frac{1}{2} \cdot [K^2(\rho, \omega) + 1] \cdot K(\omega), \end{cases} \tag{7}$$

and

$$\begin{cases} K(a_0, \pi, 2a_0, \pi) &= \frac{1}{2} \cdot [K^2(\rho', \omega') - 3] \cdot K(\rho', \omega'), \\ K(\pi, 2a_0, \pi) &= \frac{1}{2} \cdot [K^2(\rho', \omega') - 1] \cdot K(\omega'). \end{cases} \tag{8}$$

Remark. Fundamental investigation of Diophantine equations $u^2 - D \cdot v^2 = \pm 4N$ was published by B. Stolt [3], but *abc*-conjecture emerged more than 30 years later.

It can be easily checked that from (6), (7) and (8) follows:

$$K^2(a_0,\pi) - D \cdot K^2(\pi) = -1, \tag{9}$$

and

$$K^2(a_0,\pi,2a_0,\pi) - D \cdot K^2(\pi,2a_0,\pi) = 1. \tag{10}$$

So discriminants $D \equiv 5 \pmod 8$, corresponding to Property A conditions, are subset of OEIS sequence A031396 – all discriminants D such that ordinary negative Pell's equation (9) is soluble [4]. Therefore π is an even length palindrome and these discriminants must also satisfy necessary and sufficient conditions for ordinary negative Pell's equation:

- only Pythagorean primes in D factorization,
- non-square $D \equiv 1,2 \pmod 4$, and
- $K(a_0,\pi,2a_0,\pi) \equiv -1 \pmod{2D}$, see [5].

Remark. If instead of relations (6) we take

$$\begin{cases} K(\omega') &= 2K(\rho,\omega)\cdot K(\omega), \\ K(\rho',\omega') &= K^2(\rho,\omega) + D\cdot K^2(\omega), \end{cases}$$

then $N = 1$ and obtained are ordinary positive and negative Pell's equations.

From (4), (5) and (6) follows:

$$K(\rho',\omega') = \frac{1}{2}\cdot[K^2(\rho,\omega) + D\cdot K^2(\omega)] = \frac{1}{2}\cdot[2K^2(\rho,\omega)+4] = K^2(\rho,\omega)+2. \tag{11}$$

So $K(\rho',\omega') - 2 = K^2(\rho,\omega)$ and it is perfect square.

From (4) follows:

$$D\cdot K^2(\omega') = [K(\rho',\omega')+2]\cdot \underbrace{[K(\rho',\omega')-2]}_{\text{perfect square}}. \tag{12}$$

So $K(\rho',\omega') + 2$ is the product of D with perfect square.

In equations (4) and (10) discriminants are the same:

$$\frac{K^2(a_0,\pi,2a_0,\pi)-1}{K^2(\pi,2a_0,\pi)} = \frac{K^2(\rho',\omega')-4}{K^2(\omega')}. \tag{13}$$

In view of (8):

$$\frac{4[K^2(a_0,\pi,2a_0,\pi)-1]}{[K^2(\rho',\omega')-1]^2\cdot K^2(\omega')} = \frac{K^2(\rho',\omega')-4}{K^2(\omega')}, \tag{14}$$

$$K^2(a_0,\pi,2a_0,\pi) - 1 = \frac{1}{4}\cdot[K^2(\rho',\omega')-1]^2\cdot[K^2(\rho',\omega')-4], \tag{15}$$

$$\begin{aligned}&[K(a_0,\pi,2a_0,\pi)-1]\cdot[K(a_0,\pi,2a_0,\pi)+1] \\ &\quad = \frac{1}{4}\cdot[K(\rho',\omega')-1]^2\cdot[K(\rho',\omega')+1]^2\cdot[K(\rho',\omega')-2]\cdot[K(\rho',\omega')+2]. \end{aligned} \tag{16}$$

Now comments on obtained equation (16). As $D \equiv 5 \pmod 8$ or $D \equiv 1 \pmod 4$, then in equation (10) $K(a_0, \pi, 2a_0, \pi)$ must be odd, but $K(\pi, 2a_0, \pi)$ must be even number. Then both factors in the left side of (16) have single common factor 2, while their difference equals 2; such decomposition is single. In the right side of (16), as $K(\rho', \omega')$ is odd, both factors $[K(\rho', \omega') - 2]$ and $[K(\rho', \omega') + 2]$ are coprime odd numbers, but squared factors $K(\rho', \omega') - 1$ and $K(\rho', \omega') + 1$ have single common factor 2. Therefore we can also the right side of (16) split into two factors, whose difference is 2 and their single common factor is 2:

$$\begin{cases} K(a_0, \pi, 2a_0, \pi) + 1 &= \dfrac{1}{2} \cdot [K(\rho', \omega') - 1]^2 \cdot \underbrace{[K(\rho', \omega') + 2]}_{D\text{·perfect square}}, \\ K(a_0, \pi, 2a_0, \pi) - 1 &= \dfrac{1}{2} \cdot [K(\rho', \omega') + 1]^2 \cdot \underbrace{[K(\rho', \omega') - 2]}_{\text{perfect square}}. \end{cases} \tag{17}$$

Correctness of splitting can be easily tested by subtraction of the right sides of equations (17). The first of these equations shows part of sufficient conditions $K(a_0, \pi, 2a_0, \pi) \equiv -1 \pmod{2D}$ for the existence of roots to the negative Pell's equation (9).

Our result (17) means that numbers

$$2[K(a_0, \pi, 2a_0, \pi) - 1] = t^2 \tag{18}$$

and

$$\frac{2[K(a_0, \pi, 2a_0, \pi) + 1]}{D} = u^2 \tag{19}$$

are perfect squares, which satisfy generalized Pell's equation

$$t^2 - D \cdot u^2 = 2[K(a_0, \pi, 2a_0, \pi) - 1] - \frac{2D \cdot [K(a_0, \pi, 2a_0, \pi) + 1]}{D} = -4. \tag{20}$$

Numbers t and u correspond to Nagell's restrictions [6] for fundamental solutions of generalized Pell's equations:

- number t must be less or equal to $\sqrt{2[K(a_0, \pi, 2a_0, \pi) - 1]}$, we have equal;
- number u must be less or equal to $\sqrt{\dfrac{2[K(a_0, \pi, 2a_0, \pi) + 1] \cdot K(\pi, 2a_0, \pi)}{D}}$, we have less.

It can be easily checked that relation

$$\frac{t + u \cdot \sqrt{D}}{-t + u \cdot \sqrt{D}}$$

gives solution of the ordinary positive Pell's equation $K(a_0, \pi, 2a_0, \pi) + K(\pi, 2a_0, \pi) \cdot \sqrt{D}$. Thus numbers t and u are ambiguous fundamental roots of the generalized Pell's equation (20), but the pair of fundamental non-ambiguous roots for this equation are $K(\rho, \omega)/K(\omega)$; they are linked by relations $t = [K^2(\rho, \omega) + 3] \cdot K(\rho, \omega)$ and $u = [K^2(\rho, \omega) + 1] \cdot K(\omega)$, which can be easily checked by concerned reader.

2.2 Property A – main theorem and sequence

All conclusions on the equation system with Property A conditions can be united in

Theorem 2.1. *If natural non-square* $D = \dfrac{K(a_0, \pi, a_0)}{K(\pi)} \equiv 5 \pmod 8$ *is the discriminant of the ordinary negative Pell's equation* $x^2 - D \cdot y^2 = -1$*, whose fundamental roots are* $K(a_0, \pi)$ *and* $K(\pi)$*, if the fundamental root pairs* $K(\rho', \omega')/K(\omega')$ *and* $K(\rho, \omega)/K(\omega)$ *of the corresponding generalized Pell's equations* $x^2 - D \cdot y^2 = N$ *and* $x^2 - D \cdot y^2 = -N$ *are odd and are linked by relations*

$$\begin{cases} K(\omega') &= K(\rho, \omega) \cdot K(\omega), \\ K(\rho', \omega') &= \dfrac{1}{2} \cdot [K^2(\rho, \omega) + D \cdot K^2(\omega)], \end{cases}$$

then $N = 4$ *and there exists an unique pair* t/u *of ambiguous fundamental roots for the equation* $x^2 - D \cdot y^2 = -N$*:*

$$t = \sqrt{2[K(a_0, \pi, 2a_0, \pi) - 1]} = [K^2(\rho, \omega) + 3] \cdot K(\rho, \omega),$$

$$u = \sqrt{\frac{2[K(a_0, \pi, 2a_0, \pi) + 1]}{D}} = [K^2(\rho, \omega) + 1] \cdot K(\omega).$$

Here π *is an even length palindrome.*

The sequence of $D \equiv 5 \pmod 8$ values for equation system with Property A is the following: $D =$ 5, 13, 29, 53, 61, 85, 109, 125, 149, 157, 173, 181, 229, 277, 293, 317, 365, 397, 421, 445, 461, 493, 509, 533, 541, 565, 613, 629, 653, 661, 685, 733, 773, 797, 821, 845, 853, 941, 949, 965, 1013, 1021, ... They make a subset of OEIS sequence A031396, see [4].

Table 1 shows links with the corresponding ordinary positive Pell's equation (can it be named Pell's resolvent in this case?). Not-the-first D sequence values and factorized table records are selected for better illustration of obtained connections.

Table 1. Property A and it's links.

D	$K(a_0, \pi 2a_0, \pi) - 1$	$K(a_0, \pi 2a_0, \pi) + 1$
421	$2^3 \cdot 3^4 \cdot 7^2 \cdot 11^2 \cdot 67^2 \cdot 97^2 \cdot 139^2 \cdot 35176033^2$	$2 \cdot 5^2 \cdot 137^2 \cdot 421 \cdot 4337^2 \cdot 722520853^2$
541	$2^3 \cdot 3^4 \cdot 5^4 \cdot 7^2 \cdot 43^2 \cdot 433^2 \cdot 6133^2 \cdot 3785149^2$	$2 \cdot 29^2 \cdot 541 \cdot 60037^2 \cdot 33620737597^2$

D	$K(\rho', \omega') - 1$	$K(\rho', \omega') + 1$
421	$2 \cdot 137 \cdot 722520853$	$2^2 \cdot 3 \cdot 7 \cdot 67 \cdot 35176033$
541	$2 \cdot 29 \cdot 33620737597$	$2^2 \cdot 3 \cdot 7 \cdot 6133 \cdot 3785149$

D	$K(\rho', \omega') - 2$	$K(\rho', \omega') + 2$
421	$3^2 \cdot 11^2 \cdot 97^2 \cdot 139^2$	$5^2 \cdot 421 \cdot 4337^2$
541	$3^2 \cdot 5^4 \cdot 43^2 \cdot 433^2$	$541 \cdot 60037^2$

3 Further analysis of the equation $x^2 - Dy^2 = \pm 4$

3.1 Higher roots

We already know [1] that higher positive roots for the generalized Pell's equations are produced by "adding an increment" from the left side to the corresponding fundamental solutions $K(\rho,\omega)/K(\omega)$. For π even length an increment contains two π units, so these higher root pairs will be $K(a_0,\pi,2a_0,\pi,a_0+\rho,\omega)/K(\pi,2a_0,\pi,a_0+\rho,\omega)$; $K(a_0,\pi,2a_0,\pi,2a_0,\pi,2a_0,\pi,a_0+\rho,\omega)/K(\pi,2a_0,\pi,2a_0,\pi,2a_0,\pi,a_0+\rho,\omega)$; etc. In [1] we also introduced shortened notation for long repeating palindromic sequences, so $K(\pi,2a_0,\pi,2a_0,\pi,2a_0,\pi,a_0+\rho,\omega)$ can be labelled as $K(4\pi,a_0+\rho,\omega)$, but $K(a_0,\pi,2a_0,\pi,2a_0,\pi,2a_0,\pi,a_0+\rho,\omega)$ will be $K(a_0,4\pi,a_0+\rho,\omega)$. Thus sequence of positive root pairs can be labelled as $K(\rho,\omega)/K(\omega)$; $K(a_0,2\pi,a_0+\rho,\omega)/K(2\pi,a_0+\rho,\omega)$; $K(a_0,4\pi,a_0+\rho,\omega)/K(4\pi,a_0+\rho,\omega)$; $K(a_0,6\pi,a_0+\rho,\omega)/K(6\pi,a_0+\rho,\omega)$; etc.; in the same way from fundamental roots $K(\rho',\omega')/K(\omega')$.

Here we have Property A relations (6) as fundamentals and they result in many new interconnections between participating components of equations (4), (5), (9) and (10). At first we will discuss the negative branch – solutions $-K(\rho,\omega)+\sqrt{D}\cdot K(\omega)$ and $-K(\rho',\omega')+\sqrt{D}\cdot K(\omega')$.

Correctness of the relations

$$\left\{\begin{array}{rcl} K(\rho',\omega') & = & -K(\rho,\omega)\cdot K(a_0,\pi)+D\cdot K(\omega)\cdot K(\pi), \\ K(\omega') & = & -K(\rho,\omega)\cdot K(\pi)+K(\omega)\cdot K(a_0,\pi) \end{array}\right. \tag{21}$$

can be easily checked:

$$\begin{aligned} & K^2(\rho',\omega')-D\cdot K^2(\omega') \\ & = K^2(\rho,\omega)\cdot K^2(a_0,\pi)+D^2\cdot K^2(\omega)\cdot K^2(\pi)-2D\cdot K(\rho,\omega)\cdot K(\omega)\cdot K(a_0,\pi)\cdot K(\pi) \\ & \qquad - D\cdot[K^2(\rho,\omega)\cdot K^2(\pi)+K^2(\omega)\cdot K^2(a_0,\pi)-2K(\rho,\omega)\cdot K(\omega)\cdot K(a_0,\pi)\cdot K(\pi)] \\ & = K^2(\rho,\omega)\cdot[K^2(a_0,\pi)-D\cdot K^2(\pi)]-D\cdot K^2(\omega)\cdot[K^2(a_0,\pi)-D\cdot K^2(\pi)] \\ & = \underbrace{[K^2(\rho,\omega)-D\cdot K^2(\omega)]}_{-4}\cdot\underbrace{[K^2(a_0,\pi)-D\cdot K^2(\pi)]}_{-1}=4. \end{aligned}$$

An analogue of (21) is

$$\left\{\begin{array}{rcl} K(\rho,\omega) & = & K(\rho',\omega')\cdot K(a_0,\pi)-D\cdot K(\omega')\cdot K(\pi), \\ K(\omega) & = & K(\rho',\omega')\cdot K(\pi)-K(\omega')\cdot K(a_0,\pi). \end{array}\right. \tag{22}$$

The proof, very similar to previous one, is left to the concerned reader.

Now, in view of (21) and (22), in two steps we construct associated solutions of the negative branch.

$$\begin{aligned} & [-K(\rho,\omega)+\sqrt{D}\cdot K(\omega)]\cdot[K(a_0,\pi)+\sqrt{D}\cdot K(\pi)] \\ & = -K(\rho,\omega)\cdot K(a_0,\pi)+D\cdot K(\omega)\cdot K(\pi)+\sqrt{D}\cdot[-K(\rho,\omega)\cdot K(\pi)+K(\omega)\cdot K(a_0,\pi)] \\ & = K(\rho',\omega')+\sqrt{D}\cdot K(\omega'). \end{aligned}$$

We multiply the result with $K(a_0, \pi) + \sqrt{D} \cdot K(\pi)$ once more:

$$
\begin{aligned}
&= K(\rho', \omega') \cdot K(a_0, \pi) + D \cdot K(\omega') \cdot K(\pi) + \sqrt{D} \cdot [K(\rho', \omega') \cdot K(\pi) + K(\omega') \cdot K(a_0, \pi)] \\
&= K(a_0, \pi, a_0 + \rho', \omega') + \sqrt{D} \cdot K(\pi, a_0 + \rho', \omega').
\end{aligned}
$$

The reason – we multiplied twice with $K(a_0, \pi) + \sqrt{D} \cdot K(\pi)$, which is equivalent to multiplication with $K(a_0, \pi, 2a_0, \pi) + \sqrt{D} \cdot K(\pi, 2a_0, \pi)$. Notice the change of extensions in final result from ρ, ω to ρ', ω', all higher roots of this negative branch will have the extension ρ', ω'.

In an analogous two-step procedure we get:

$$
\begin{aligned}
&[-K(\rho', \omega') + \sqrt{D} \cdot K(\omega')] \cdot [K(a_0, \pi) + \sqrt{D} \cdot K(\pi)] \cdot [K(a_0, \pi) + \sqrt{D} \cdot K(\pi)] \\
&= (-1) \cdot [K(a_0, \pi, a_0 + \rho, \omega) + \sqrt{D} \cdot K(\pi, a_0 + \rho, \omega)].
\end{aligned}
$$

Again – change of extensions. The result explains structure of the root systems of equations (4) and (5), corresponding to Property A conditions.

Example 3.1. Equation $K^2(\rho', \omega') - 61 \cdot K^2(\omega') = 4$ has fundamental solution $\pm K(\rho', \omega') + \sqrt{61} \cdot K(\omega')$ with $K(\rho', \omega') = K(7, 1, 4, 3, 1, 2, 3) = 1523$, $K(\omega') = K(1, 4, 3, 1, 2, 3) = 195$. First root pair $K(a_0, \pi, 2a_0, \pi, a_0 + \rho', \omega')/K(\pi, 2a_0, \pi, a_0 + \rho', \omega')$ of the positive branch is

$$
\frac{K(7, 1, 4, 3, 1, 2, 2, 1, 3, 4, 1, 14, 1, 4, 3, 1, 2, 2, 1, 3, 4, 1, 14, 1, 4, 3, 1, 2, 3)}{K(1, 4, 3, 1, 2, 2, 1, 3, 4, 1, 14, 1, 4, 3, 1, 2, 2, 1, 3, 4, 1, 14, 1, 4, 3, 1, 2, 3)},
$$

but the first root pair $K(a_0, \pi, a_0 + \rho, \omega)/K(\pi, a_0 + \rho, \omega)$ of the negative branch is

$$
\frac{K(7, 1, 4, 3, 1, 2, 2, 1, 3, 4, 1, 14, 1, 4)}{K(1, 4, 3, 1, 2, 2, 1, 3, 4, 1, 14, 1, 4)}.
$$

Equation $K^2(\rho', \omega') - 61 \cdot K^2(\omega') = -4$ has fundamental roots $\pm K(\rho, \omega) + \sqrt{61} \cdot K(\omega)$ with $K(\rho, \omega) = K(7, 1, 4) = 39$, $K(\omega) = K(1, 4) = 5$. First root pair $K(a_0, \pi, 2a_0, \pi, a_0 + \rho, \omega)/K(\pi, 2a_0, \pi, a_0 + \rho, \omega)$ of the positive branch is

$$
\frac{K(7, 1, 4, 3, 1, 2, 2, 1, 3, 4, 1, 14, 1, 4, 3, 1, 2, 2, 1, 3, 4, 1, 14, 1, 4)}{K(1, 4, 3, 1, 2, 2, 1, 3, 4, 1, 14, 1, 4, 3, 1, 2, 2, 1, 3, 4, 1, 14, 1, 4)},
$$

but the first root pair $K(a_0, \pi, a_0 + \rho', \omega')/K(\pi, a_0 + \rho', \omega')$ of the negative branch is

$$
\frac{K(7, 1, 4, 3, 1, 2, 2, 1, 3, 4, 1, 14, 1, 4, 3, 1, 2, 3)}{K(1, 4, 3, 1, 2, 2, 1, 3, 4, 1, 14, 1, 4, 3, 1, 2, 3)}.
$$

Observe exchange of extensions.

Now several cross-relations between higher roots of both equations (4) and (5), as well as relations with their ordinary companions, having $N = \pm 1$ (Pell's resolvents –?). Not all of these small theorems, valid only under Property A conditions, will be confirmed by detailed proofs, which are generally very similar – by induction, but essential items will be discussed.

1. We split at the position of plus sign:

$$
K(\pi, 2a_0, \pi, a_0 + \rho', \omega') = K(\pi, 2a_0, \pi) \cdot K(\rho', \omega') + K(a_0, \pi, 2a_0, \pi) \cdot K(\omega') \tag{23}
$$

$$
= 2K(a_0, \pi) \cdot K(\pi) \cdot K(\rho', \omega') + K(\omega') \cdot [K^2(a_0, \pi) + K(a_0, \pi, a_0) \cdot K(\pi)]. \tag{24}
$$

Two more splittings:

$$K(a_0, \pi, a_0 + \rho, \omega) = K(a_0, \pi) \cdot K(\rho, \omega) + K(a_0, \pi, a_0) \cdot K(\omega), \tag{25}$$
$$K(\pi, a_0 + \rho, \omega) = K(\pi) \cdot K(\rho, \omega) + K(a_0, \pi) \cdot K(\omega). \tag{26}$$

We multiply right parts of (25) and (26) and the result coincides with (24):

$$\begin{aligned}
&= K^2(\omega) \cdot K(a_0, \pi) \cdot \underbrace{K(a_0, \pi, a_0)}_{D \cdot K(\pi)} + K(a_0, \pi, a_0) \cdot K(\pi) \cdot \underbrace{K(\rho, \omega) \cdot K(\omega)}_{K(\omega')} \\
&\quad + K^2(a_0, \pi) \cdot \underbrace{K(\rho, \omega) \cdot K(\omega)}_{K(\omega')} + K^2(\rho, \omega) \cdot K(a_0, \pi) \cdot K(\pi) \\
&= K(\omega') \cdot [K^2(a_0, \pi) + K(a_0, \pi, a_0) \cdot K(\pi)] + K(a_0, \pi) \cdot K(\pi) \cdot \underbrace{[K^2(\rho, \omega) + D \cdot K^2(\omega)]}_{2K(\rho', \omega'),} \\
&= K(\omega') \cdot [K^2(a_0, \pi) + K(a_0, \pi, a_0) \cdot K(\pi)] + 2K(\rho', \omega') \cdot K(a_0, \pi) \cdot K(\pi).
\end{aligned}$$

That means:

$$K(\pi, 2a_0, \pi, a_0 + \rho', \omega') = K(a_0, \pi, a_0 + \rho, \omega) \cdot K(\pi, a_0 + \rho, \omega).$$

Splittings (25) and (26) can be employed in confirmation of

$$\begin{aligned}
K(a_0, \pi, 2a_0, \pi, a_0 + \rho', \omega') &= \frac{1}{2} \cdot [K^2(a_0, \pi, a_0 + \rho, \omega) + D \cdot K^2(\pi, a_0 + \rho, \omega)] \\
&= D \cdot K^2(\pi, a_0 + \rho, \omega) + 2.
\end{aligned}$$

Inductively this can be generalized to longer palindrome sequences. For higher roots of equations (4) and (5) we declare that

Theorem 3.2. *Under Property A conditions the following relations exist:*

$$\left\{ \begin{aligned}
K(\pi, \underbrace{2a_0, \pi,}_{\text{(2k+1)-times}} a_0 + \rho', \omega') &= K(a_0, \pi, \underbrace{2a_0, \pi,}_{\text{k-times}} a_0 + \rho, \omega) \\
&\quad \cdot K(\pi, \underbrace{2a_0, \pi,}_{\text{k-times}} a_0 + \rho, \omega), \\
K(a_0, \pi, \underbrace{2a_0, \pi,}_{\text{(2k+1)-times}} a_0 + \rho', \omega') &= \frac{1}{2} \cdot [K^2(a_0, \pi, \underbrace{2a_0, \pi,}_{\text{k-times}} a_0 + \rho, \omega) \\
&\quad + D \cdot K^2(\pi, \underbrace{2a_0, \pi,}_{\text{k-times}} a_0 + \rho, \omega)] \\
&= D \cdot K^2(\pi, \underbrace{2a_0, \pi,}_{\text{k-times}} a_0 + \rho, \omega) \pm 2.
\end{aligned} \right. \tag{27}$$

Here $k = 0, 1, 2, 3, ...$; even/odd k values give $+2/-2$ in the last expression.

Relations (27) can be considered as analogues of (6).

We check the result for $k = 0$:

$$
\begin{aligned}
&K^2(a_0, 2\pi, a_0 + \rho', \omega') - D \cdot K^2(2\pi, a_0 + \rho', \omega') \\
&= \frac{1}{4} \cdot [K^2(a_0, \pi, a_0 + \rho, \omega) + D \cdot K^2(\pi, a_0 + \rho, \omega)]^2 - D \cdot K^2(\pi, a_0 + \rho, \omega) \cdot K^2(a_0, \pi, a_0 + \rho, \omega) \\
&= \frac{1}{4} \cdot [K^4(a_0, \pi, a_0 + \rho, \omega) + 2D \cdot K^2(a_0, \pi, a_0 + \rho, \omega) \cdot K^2(\pi, a_0 + \rho, \omega) + D^2 \cdot K^4(\pi, a_0 + \rho, \omega)] \\
&\quad - D \cdot K^2(\pi, a_0 + \rho, \omega) \cdot K^2(a_0, \pi, a_0 + \rho, \omega) \\
&= \frac{1}{4} \cdot [K^2(a_0, \pi, a_0 + \rho, \omega) - D \cdot K^2(\pi, a_0 + \rho, \omega)]^2 = 4.
\end{aligned}
$$

2. We take again splitting (26), but in view of (7):

$$
\begin{aligned}
&K(\pi, a_0 + \rho, \omega) = K(\pi) \cdot K(\rho, \omega) + K(a_0, \pi) \cdot K(\omega) \\
&= \frac{1}{2} \cdot [[K^2(\rho, \omega) + 3] \cdot \underbrace{K(\rho, \omega) \cdot K(\omega)}_{K(\omega')} + [K^2(\rho, \omega) + 1] \cdot \underbrace{K(\rho, \omega) \cdot K(\omega)}_{K(\omega')}] \\
&= \frac{1}{2} \cdot K(\omega') \cdot [2K^2(\rho, \omega) + 4] = \frac{1}{2} \cdot K(\omega') \cdot [K^2(\rho, \omega) + \underbrace{K^2(\rho, \omega) + 4}_{D \cdot K^2(\omega)}] \\
&= K(\omega') \cdot \frac{1}{2} \cdot [K^2(\rho, \omega) + D \cdot K^2(\omega)] = K(\omega') \cdot K(\rho', \omega').
\end{aligned}
$$

That means:

$$
K(\pi, a_0 + \rho, \omega) = K(\rho', \omega') \cdot K(\omega'). \tag{28}
$$

Analogously from (25) follows:

$$
K(a_0, \pi, a_0 + \rho, \omega) = \frac{1}{2} \cdot [K^2(\rho', \omega') + D \cdot K^2(\omega')]. \tag{29}
$$

Inductively this can be generalized to longer palindrome sequences. For higher roots of equations (4) and (5) we declare that

Theorem 3.3. *Under Property A conditions the following relations exist:*

$$
\left\{
\begin{aligned}
K(\pi, \underbrace{2a_0, \pi,}_{\text{2(k+1)-times}} a_0 + \rho, \omega) &= K(a_0, \pi, \underbrace{2a_0, \pi,}_{\text{k-times}} a_0 + \rho', \omega') \\
&\quad \cdot K(\pi, \underbrace{2a_0, \pi,}_{\text{k-times}} a_0 + \rho', \omega'), \\
K(a_0, \pi, \underbrace{2a_0, \pi,}_{\text{2(k+1)-times}} a_0 + \rho, \omega) &= \frac{1}{2} \cdot [K^2(a_0, \pi, \underbrace{2a_0, \pi,}_{\text{k-times}} a_0 + \rho', \omega') \\
&\quad + D \cdot K^2(\pi, \underbrace{2a_0, \pi,}_{\text{k-times}} a_0 + \rho', \omega')] \\
&= D \cdot K^2(\pi, \underbrace{2a_0, \pi,}_{\text{k-times}} a_0 + \rho', \omega') \pm 2.
\end{aligned}
\right. \tag{30}
$$

Here $k = 0, 1, 2, 3, ...$; even/odd k values give $-2/+2$ in the last expression.

Checking is analogous with that for Theorem 3.2.

3. Here we present higher analogues of already mentioned Cayley's relations (7) and (8).

Theorem 3.4. *Under Property A conditions the following relations exist:*

$$\begin{cases} K(5\pi\underbrace{,2a_0,\pi,2a_0,\pi}_{\text{3k-times}}) &= \dfrac{1}{2}\cdot[K^2(a_0,\pi,\underbrace{2a_0,\pi,2a_0,\pi,}_{\text{k-times}}a_0+\rho',\omega')+1] \\ & \quad \cdot K(\pi,\underbrace{2a_0,\pi,2a_0,\pi,}_{\text{k-times}}a_0+\rho',\omega'), \\ K(a_0,5\pi\underbrace{,2a_0,\pi,2a_0,\pi}_{\text{3k-times}}) &= \dfrac{1}{2}\cdot[K^2(a_0,\pi,\underbrace{2a_0,\pi,2a_0,\pi,}_{\text{k-times}}a_0+\rho',\omega')+3] \\ & \quad \cdot K(a_0,\pi,\underbrace{2a_0,\pi,2a_0,\pi,}_{\text{k-times}}a_0+\rho',\omega')]. \end{cases} \tag{31}$$

Here $k=0,1,2,3,\ldots$.

Proof. From the equation $K^2(a_0,\pi,a_0+\rho',\omega')-D\cdot K^2(\pi,a_0+\rho',\omega')=-4$ we obtain:

$$K^2(a_0,\pi,a_0+\rho',\omega')+3=D\cdot K^2(\pi,a_0+\rho',\omega')-1,$$
$$K^2(a_0,\pi,a_0+\rho',\omega')+1=D\cdot K^2(\pi,a_0+\rho',\omega')-3.$$

For $k=0$ Theorem 3.4 gives:

$$\begin{cases} K(5\pi) &= \dfrac{1}{2}\cdot[K^2(a_0,\pi,a_0+\rho',\omega')+1]\cdot K(\pi,a_0+\rho',\omega'), \\ K(a_0,5\pi) &= \dfrac{1}{2}\cdot[K^2(a_0,\pi,a_0+\rho',\omega')+3]\cdot K(a_0,\pi,a_0+\rho',\omega')]. \end{cases} \tag{32}$$

From (32) we obtain:

$$\begin{aligned}
&K^2(a_0,5\pi)-D\cdot K^2(5\pi) \\
&=\frac{1}{4}\cdot[D^2\cdot K^4(\pi,a_0+\rho',\omega')-2D\cdot K^2(\pi,a_0+\rho',\omega')+1]\cdot K^2(a_0,\pi,a_0+\rho',\omega') \\
&\qquad -\frac{1}{4}\cdot[D^2\cdot K^4(\pi,a_0+\rho',\omega')-6D\cdot K^2(\pi,a_0+\rho',\omega')+9]\cdot D\cdot K^2(\pi,a_0+\rho',\omega') \\
&=\frac{1}{4}\cdot[D^2\cdot K^4(\pi,a_0+\rho',\omega')\cdot K^2(a_0,\pi,a_0+\rho',\omega') \\
&\qquad -2D\cdot K^2(\pi,a_0+\rho',\omega')\cdot K^2(a_0,\pi,a_0+\rho',\omega')+K^2(a_0,\pi,a_0+\rho',\omega') \\
&\qquad -D^3\cdot K^6(\pi,a_0+\rho',\omega')+6D^2\cdot K^4(\pi,a_0+\rho',\omega')-9D\cdot K^2(\pi,a_0+\rho',\omega')] \\
&=\frac{1}{4}\cdot[D^2\cdot K^4(\pi,a_0+\rho',\omega')\cdot\underbrace{[K^2(a_0,\pi,a_0+\rho',\omega')-D\cdot K^2(\pi,a_0+\rho',\omega')]}_{-4} \\
&\qquad -2D\cdot K^2(\pi,a_0+\rho',\omega')\cdot\underbrace{[K^2(a_0,\pi,a_0+\rho',\omega')-D\cdot K^2(\pi,a_0+\rho',\omega')]}_{-4} \\
&\qquad +K^2(a_0,\pi,a_0+\rho',\omega')+4D^2\cdot K^4(\pi,a_0+\rho',\omega')-9D\cdot K^2(\pi,a_0+\rho',\omega')] \\
&=\frac{1}{4}\cdot[K^2(a_0,\pi,a_0+\rho',\omega')-D\cdot K^2(\pi,a_0+\rho',\omega')]=-1.
\end{aligned}$$

Analogous transformation for $k=1,2,3,\ldots$ values inductively confirms Theorem 3.4. □

Theorem 3.5. *Under Property A conditions the following relations exist:*

$$\left\{\begin{aligned}
K(4\pi\underbrace{,2a_0,\pi,2a_0,\pi}_{\text{3k-times}}) &= \frac{1}{2}\cdot[K^2(a_0,\pi,\underbrace{2a_0,\pi,2a_0,\pi,}_{\text{k-times}}a_0+\rho,\omega)-1] \\
&\quad\cdot K(\pi,\underbrace{2a_0,\pi,2a_0,\pi,}_{\text{k-times}}a_0+\rho,\omega), \\
K(a_0,4\pi\underbrace{,2a_0,\pi,2a_0,\pi}_{\text{3k-times}}) &= \frac{1}{2}\cdot[K^2(a_0,\pi,\underbrace{2a_0,\pi,2a_0,\pi,}_{\text{k-times}}a_0+\rho,\omega)-3] \\
&\quad\cdot K(a_0,\pi,\underbrace{2a_0,\pi,2a_0,\pi,}_{\text{k-times}}a_0+\rho,\omega)].
\end{aligned}\right. \tag{33}$$

Here $k=0,1,2,3,...$.

Theorem 3.6. *Under Property A conditions the following relations exist:*

$$\left\{\begin{aligned}
K(7\pi\underbrace{,2a_0,\pi,2a_0,\pi}_{\text{3k-times}}) &= \frac{1}{2}\cdot[K^2(a_0,\pi,2a_0,\pi,\underbrace{2a_0,\pi,2a_0,\pi,}_{\text{k-times}}a_0+\rho,\omega)+1] \\
&\quad\cdot K(\pi,2a_0,\pi,\underbrace{2a_0,\pi,2a_0,\pi,}_{\text{k-times}}a_0+\rho,\omega), \\
K(a_0,7\pi\underbrace{,2a_0,\pi,2a_0,\pi}_{\text{3k-times}}) &= \frac{1}{2}\cdot[K^2(a_0,\pi,2a_0,\pi,\underbrace{2a_0,\pi,2a_0,\pi,}_{\text{k-times}}a_0+\rho,\omega)+3] \\
&\quad\cdot K(a_0,\pi,2a_0,\pi,\underbrace{2a_0,\pi,2a_0,\pi,}_{\text{k-times}}a_0+\rho,\omega)].
\end{aligned}\right. \tag{34}$$

Here $k=0,1,2,3,...$.

Theorem 3.7. *Under Property A conditions the following relations exist:*

$$\left\{\begin{aligned}
K(8\pi\underbrace{,2a_0,\pi,2a_0,\pi}_{\text{3k-times}}) &= \frac{1}{2}\cdot[K^2(a_0,\pi,2a_0,\pi,\underbrace{2a_0,\pi,2a_0,\pi,}_{\text{k-times}}a_0+\rho',\omega')-1] \\
&\quad\cdot K(\pi,2a_0,\pi,\underbrace{2a_0,\pi,2a_0,\pi,}_{\text{k-times}}a_0+\rho',\omega'), \\
K(a_0,8\pi\underbrace{,2a_0,\pi,2a_0,\pi}_{\text{3k-times}}) &= \frac{1}{2}\cdot[K^2(a_0,\pi,2a_0,\pi,\underbrace{2a_0,\pi,2a_0,\pi,}_{\text{k-times}}a_0+\rho',\omega')-3] \\
&\quad\cdot K(a_0,\pi,2a_0,\pi,\underbrace{2a_0,\pi,2a_0,\pi,}_{\text{k-times}}a_0+\rho',\omega')].
\end{aligned}\right. \tag{35}$$

Here $k=0,1,2,3,...$.

Proofs of Theorems 3.5 – 3.7 are analogous to that for Theorem 3.4.

3.2 More about divisibility

In previous subsection we already proposed some divisibility connections between the limited number of participating components of equations (4), (5), (9) and (10). Here we will present twelve allied theorems describing one particular root as a divisor of unlimited number of other roots – dividends. Again – these theorems will be valid only under Property A conditions.

Theorem 3.8. *Under Property A conditions every continuant* $K(\pi, \underbrace{2a_0, \pi,}_{k\text{-times}} a_0+\rho', \omega')$ *is a divisor of all continuants* $K(\pi, \underbrace{2a_0, \pi,}_{l\text{-times}} a_0+\rho, \omega)$*. Here* $l = 2(k+1) + n(3k+5)$*;* $k = 0, 1, 2, ...$*;* $n = 0, 1, 2, ...$*.*

Proof. **1.** $k = 0, n = 0$ and $l = 2$. Theorem 3.3 gives $K(\pi, a_0+\rho', \omega')|K(3\pi, a_0+\rho, \omega)$.

$k = 0, n = 1$ and $l = 7$, so we must compare $K(\pi, a_0+\rho', \omega')$ with $K(8\pi, a_0+\rho, \omega)$.

$$\begin{aligned} K(8\pi, a_0+\rho, \omega) &= K(5\pi, 2a_0, 3\pi, a_0+\rho, \omega) \\ &= K(5\pi) \cdot K(a_0, 3\pi, a_0+\rho, \omega) + K(a_0, 5\pi) \cdot K(3\pi, a_0+\rho, \omega). \end{aligned}$$

For the second summand we already have $K(\pi, a_0+\rho', \omega')|K(3\pi, a_0+\rho, \omega)$, but for the first summand divisibility $K(\pi, a_0+\rho', \omega')|K(5\pi)$ is based on Theorem 3.4.

$k = 0, n = 2$ and $l = 12$, so we must compare $K(\pi, a_0+\rho', \omega')$ with $K(13\pi, a_0+\rho, \omega)$.

$$\begin{aligned} K(13\pi, a_0+\rho, \omega) &= K(10\pi, 2a_0, 3\pi, a_0+\rho, \omega) \\ &= K(10\pi) \cdot K(a_0, 3\pi, a_0+\rho, \omega) + K(a_0, 10\pi) \cdot K(3\pi, a_0+\rho, \omega). \end{aligned}$$

As $K(10\pi) = 2K(a_0, 5\pi) \cdot K(5\pi)$, divisibility $K(\pi, a_0+\rho', \omega')|K(13\pi, a_0+\rho, \omega)$ is confirmed.

For $n = 3, 4, ...$ we have $K(5\pi)|K(5n\pi)$, algorithm can be repeated. This confirms Theorem 3.8 for $k = 0$ and all n values.

2. Now $k = 1, n = 0$ and $l = 4$. Theorem 3.3 gives $K(2\pi, a_0+\rho', \omega')|K(5\pi, a_0+\rho, \omega)$.

$k = 1, n = 1$ and $l = 12$, so we must compare $K(2\pi, a_0+\rho', \omega')$ with $K(13\pi, a_0+\rho, \omega)$.

$$\begin{aligned} K(13\pi, a_0+\rho, \omega) &= K(8\pi, 2a_0, 5\pi, a_0+\rho, \omega) \\ &= K(8\pi) \cdot K(a_0, 5\pi, a_0+\rho, \omega) + K(a_0, 8\pi) \cdot K(5\pi, a_0+\rho, \omega). \end{aligned}$$

We have $K(8\pi) = 2K(4\pi) \cdot K(a_0, 4\pi)$ and, accordingly to Theorem 3.2,

$$K(2\pi, a_0+\rho', \omega') = K(\pi, a_0+\rho, \omega) \cdot K(a_0, \pi, a_0+\rho, \omega).$$

In view of Theorem 3.5 we have $K(\pi, a_0+\rho, \omega)|K(4\pi)$ and $K(a_0, \pi, a_0+\rho, \omega)|K(a_0, 4\pi)$, therefore $K(2\pi, a_0+\rho', \omega')|K(8\pi)$; this gives necessary $K(2\pi, a_0+\rho', \omega')|K(13\pi, a_0+\rho, \omega)$.

$k = 1, n = 2$ and $l = 20$, so we must compare $K(2\pi, a_0+\rho', \omega')$ with $K(21\pi, a_0+\rho, \omega)$.

$$\begin{aligned} K(21\pi, a_0+\rho, \omega) &= K(16\pi, 2a_0, 5\pi, a_0+\rho, \omega) \\ &= K(16\pi) \cdot K(a_0, 5\pi, a_0+\rho, \omega) + K(a_0, 16\pi) \cdot K(5\pi, a_0+\rho, \omega). \end{aligned}$$

We have $K(16\pi) = 2K(8\pi) \cdot K(a_0, 8\pi)$ and $K(8\pi)|K(8n\pi)$ for $n = 3, 4, ...$; repeating algorithm confirms Theorem 3.8 for $k = 1$ and all n values.

3. Now $k = 2, n = 0$ and $l = 6$. Theorem 3.3 gives $K(3\pi, a_0+\rho', \omega')|K(7\pi, a_0+\rho, \omega)$. For $n = 1$ and all greater n values we can split off $K(7\pi, a_0+\rho, \omega)$ fragment from the dividend and obtain repeating algorithm. In total, this confirms Theorem 3.8 for $k = 2$ and all greater k values. □

Theorem 3.8 suggests that each so defined continuant with the number of palindromes specified in the left section of the Table 2 is a divisor of all continuants of the defined type, specified in the right section of the Table 2.

Table 2.

Divisor	Dividends, π units							
1π	3π	8π	13π	18π	23π	28π	33π	...
2π	5π	13π	21π	29π	37π	45π	53π	...
3π	7π	18π	29π	40π	51π	(62π)	(73π)	...
4π	9π	23π	37π	51π	(65π)	(79π)	(93π)	...
5π	11π	28π	45π	(62π)	(79π)	(96π)	(113π)	...
6π	13π	33π	53π	(73π)	(93π)	(113π)	(133π)	...
...	...	...	...	...	...	...	...	...

Demonstrated values without brackets are confirmed experimentally and they are limited by my laptop's performance. Due to regularity proposed by Theorem 3.8, these table values can be easily extrapolated to items in brackets.

Theorem 3.9. *Under Property A conditions every continuant* $K(\pi, \underbrace{2a_0, \pi,}_{k\text{-times}} a_0+\rho', \omega')$ *is a divisor of all continuants* $K(\pi, \underbrace{2a_0, \pi,}_{l\text{-times}} a_0+\rho', \omega')$. *Here* $l = 4k + 5 + n(3k + 5)$; $k = 0, 1, 2, ...$; $n = 0, 1, 2, ...$

Theorem 3.9 suggests that each so defined continuant with the number of palindromes specified in the left section of the Table 3 is a divisor of all continuants of the defined type, specified in the right section of the Table 3.

Table 3.

Divisor	Dividends, π units							
1π	6π	11π	16π	21π	26π	31π	36π	...
2π	10π	18π	26π	34π	42π	50π	58π	...
3π	14π	25π	36π	47π	58π	(69π)	(80π)	...
4π	18π	32π	46π	60π	(74π)	(88π)	(102π)	...
5π	22π	39π	(56π)	(73π)	(90π)	(107π)	(124π)	...
6π	26π	46π	(66π)	(86π)	(106π)	(126π)	(146π)	...
...	...	...	...	...	...	...	...	...

Items without brackets are experimental, items in brackets – extrapolated.

Theorem 3.10. *Under Property A conditions every continuant* $K(\pi, \underbrace{2a_0, \pi,}_{k\text{-times}} a_0+\rho, \omega)$ *is a divisor of all continuants* $K(\pi, \underbrace{2a_0, \pi,}_{l\text{-times}} a_0+\rho', \omega')$. *Here* $l = 2k + 1 + n(3k + 4)$; $k = 0, 1, 2, ...$; $n = 0, 1, 2, ...$

Theorem 3.10 suggests that each so defined continuant with the number of palindromes specified in the left section of the Table 4 is a divisor of all continuants of the defined type, specified in the right section of the Table 4.

Table 4.

Divisor	Dividends, π units							
1π	2π	6π	10π	14π	18π	22π	26π	...
2π	4π	11π	18π	25π	32π	39π	46π	...
3π	6π	16π	26π	36π	46π	56π	(66π)	...
4π	8π	21π	34π	47π	60π	(73π)	(86π)	...
5π	10π	26π	42π	58π	(74π)	(90π)	(106π)	...
6π	12π	31π	50π	(69π)	(88π)	(107π)	(126π)	...
...	...	...	...	...	...	...	...	...

Items without brackets are experimental, items in brackets – extrapolated.

Theorem 3.11. *Under Property A conditions every continuant* $K(\pi, \underbrace{2a_0, \pi,}_{k\text{-times}} a_0 + \rho, \omega)$ *is a divisor of all continuants* $K(\pi, \underbrace{2a_0, \pi,}_{l\text{-times}} a_0 + \rho, \omega)$. *Here* $l = 4(k+1) + n(3k+4)$; $k = 0, 1, 2, ...$; $n = 0, 1, 2, ...$

Theorem 3.11 suggests that each so defined continuant with the number of palindromes specified in the left section of the Table 5 is a divisor of all continuants of the defined type, specified in the right section of the Table 5.

Table 5.

Divisor	Dividends, π units							
1π	5π	9π	13π	17π	21π	25π	29π	...
2π	9π	16π	23π	30π	37π	44π	51π	...
3π	13π	23π	33π	43π	53π	(63π)	(73π)	...
4π	17π	30π	43π	56π	(69π)	(82π)	(95π)	...
5π	21π	37π	53π	(69π)	(85π)	(101π)	(117π)	...
6π	25π	44π	(63π)	(82π)	(101π)	(120π)	(139π)	...
...	...	...	...	...	...	...	...	...

Items without brackets are experimental, items in brackets – extrapolated.

Theorem 3.12. *Under Property A conditions every continuant* $K(a_0, \pi, \underbrace{2a_0, \pi,}_{k\text{-times}} a_0 + \rho', \omega')$ *is a divisor of all continuants* $K(\pi, \underbrace{2a_0, \pi,}_{l\text{-times}} a_0 + \rho', \omega')$. *Here* $l = 4k + 5 + 2n(3k+5)$; $k = 0, 1, 2, ...$; $n = 0, 1, 2, ...$

Theorem 3.12 suggests that each so defined continuant with the number of palindromes specified in the left section of the Table 6 is a divisor of all continuants of the defined type, specified in the right section of the Table 6.

Table 6.

Divisor	Dividends, π units							
1π	6π	16π	26π	36π	46π	56π	...	
2π	10π	26π	42π	58π	(74π)	(90π)	...	
3π	14π	36π	58π	(80π)	(102π)	(124π)	...	
4π	18π	46π	(74π)	(102π)	(130π)	(158π)	...	
5π	22π	56π	(90π)	(124π)	(158π)	(192π)	...	
6π	26π	(66π)	(106π)	(146π)	(186π)	(226π)	...	
...	...	...	...	...	...	...	...	...

Items without brackets are experimental, items in brackets – extrapolated.

Theorem 3.13. *Under Property A conditions every continuant* $K(a_0, \pi, \underbrace{2a_0, \pi,}_{k\text{-times}} a_0 + \rho', \omega')$ *is a divisor of all continuants* $K(\pi, \underbrace{2a_0, \pi,}_{l\text{-times}} a_0 + \rho, \omega)$. *Here* $l = 2(k+1) + 2n(3k+5)$; $k = 0, 1, 2, ...$; $n = 0, 1, 2, ...$

Theorem 3.13 suggests that each so defined continuant with the number of palindromes specified in the left section of the Table 7 is a divisor of all continuants of the defined type, specified in the right section of the Table 7.

Table 7.

Divisor	Dividends, π units							
1π	3π	13π	23π	33π	43π	53π	...	
2π	5π	21π	37π	53π	(69π)	(85π)	...	
3π	7π	29π	51π	(73π)	(95π)	(117π)	...	
4π	9π	37π	(65π)	(93π)	(121π)	(149π)	...	
5π	11π	45π	(79π)	(113π)	(147π)	(181π)	...	
6π	13π	53π	(93π)	(133π)	(173π)	(213π)	...	
...	...	...	...	...	...	...	...	...

Items without brackets are experimental, items in brackets – extrapolated.

Theorem 3.14. *Under Property A conditions every continuant* $K(a_0, \pi, \underbrace{2a_0, \pi,}_{k\text{-times}} a_0 + \rho, \omega)$ *is a divisor of all continuants* $K(\pi, \underbrace{2a_0, \pi,}_{l\text{-times}} a_0 + \rho', \omega')$. *Here* $l = 2k + 1 + 2n(3k+4)$; $k = 0, 1, 2, ...$; $n = 0, 1, 2, ...$

Theorem 3.14 suggests that each so defined continuant with the number of palindromes specified in the left section of the Table 8 is a divisor of all continuants of the defined type, specified in the right section of the Table 8.

Table 8.

Divisor	Dividends, π units						
1π	2π	10π	18π	26π	34π	42π	...
2π	4π	18π	32π	46π	60π	(74π)	...
3π	6π	26π	46π	(66π)	(86π)	(106π)	...
4π	8π	34π	60π	(86π)	(112π)	(138π)	...
5π	10π	42π	(74π)	(106π)	(138π)	(170π)	...
6π	12π	50π	(88π)	(126π)	(164π)	(202π)	...
...	...	...	...	...	...	...	... ...

Items without brackets are experimental, items in brackets – extrapolated.

Theorem 3.15. *Under Property A conditions every continuant* $K(a_0, \pi, \underbrace{2a_0, \pi,}_{k\text{-times}} a_0 + \rho, \omega)$ *is a divisor of all continuants* $K(\pi, \underbrace{2a_0, \pi,}_{l\text{-times}} a_0 + \rho, \omega)$. *Here* $l = 4(k+1) + 2n(3k+4)$; $k = 0, 1, 2, ...$; $n = 0, 1, 2, ...$

Theorem 3.15 suggests that each so defined continuant with the number of palindromes specified in the left section of the Table 9 is a divisor of all continuants of the defined type, specified in the right section of the Table 9.

Table 9.

Divisor	Dividends, π units						
1π	5π	13π	21π	29π	37π	45π	...
2π	9π	23π	37π	51π	(65π)	(79π)	...
3π	13π	33π	53π	(73π)	(93π)	(113π)	...
4π	17π	43π	(69π)	(95π)	(121π)	(147π)	...
5π	21π	53π	(85π)	(117π)	(149π)	(181π)	...
6π	25π	(63π)	(101π)	(139π)	(177π)	(215π)	...
...	...	...	...	...	...	...	... ...

Items without brackets are experimental, items in brackets – extrapolated.

Theorem 3.16. *Under Property A conditions every continuant* $K(a_0, \pi, \underbrace{2a_0, \pi,}_{k\text{-times}} a_0 + \rho', \omega')$ *is a divisor of all continuants* $K(a_0, \pi, \underbrace{2a_0, \pi,}_{l\text{-times}} a_0 + \rho', \omega')$. *Here* $l = 7k + 10 + 2n(3k + 5)$; $k = 0, 1, 2, ...$; $n = 0, 1, 2, ...$

Theorem 3.16 suggests that each so defined continuant with the number of palindromes specified in the left section of the Table 10 is a divisor of all continuants of the defined type, specified in the right section of the Table 10.

Table 10.

Divisor	Dividends, π units							
1π	11π	21π	31π	41π	51π	(61π)	...	
2π	18π	34π	50π	(66π)	(82π)	(98π)	...	
3π	25π	47π	(69π)	(91π)	(113π)	(135π)	...	
4π	32π	60π	(88π)	(116π)	(144π)	(172π)	...	
5π	39π	(73π)	(107π)	(141π)	(175π)	(209π)	...	
6π	46π	(86π)	(126π)	(166π)	(206π)	(246π)	...	
...	...	...	...	...	...	...	...	...

Items without brackets are experimental, items in brackets – extrapolated.

Theorem 3.17. *Under Property A conditions every continuant* $K(a_0, \pi, \underbrace{2a_0, \pi,}_{k\text{-times}} a_0 + \rho', \omega')$ *is a divisor of all continuants* $K(a_0, \pi, \underbrace{2a_0, \pi,}_{l\text{-times}} a_0 + \rho, \omega)$. *Here* $l = 5k+7+2n(3k+5)$; $k = 0, 1, 2, ...$; $n = 0, 1, 2, ...$

Theorem 3.17 suggests that each so defined continuant with the number of palindromes specified in the left section of the Table 11 is a divisor of all continuants of the defined type, specified in the right section of the Table 11.

Table 11.

Divisor	Dividends, π units							
1π	8π	18π	28π	38π	48π	58π	...	
2π	13π	29π	45π	(61π)	(77π)	(93π)	...	
3π	18π	40π	(62π)	(84π)	(106π)	(128π)	...	
4π	23π	51π	(79π)	(107π)	(135π)	(163π)	...	
5π	28π	(62π)	(96π)	(130π)	(164π)	(198π)	...	
6π	33π	(73π)	(113π)	(153π)	(193π)	(233π)	...	
...	...	...	...	...	...	...	...	...

Items without brackets are experimental, items in brackets – extrapolated.

Theorem 3.18. *Under Property A conditions every continuant* $K(a_0, \pi, \underbrace{2a_0, \pi,}_{k\text{-times}} a_0 + \rho, \omega)$ *is a divisor of all continuants* $K(a_0, \pi, \underbrace{2a_0, \pi,}_{l\text{-times}} a_0 + \rho', \omega')$. *Here* $l = 5(k+1) + 2n(3k+4)$; $k = 0, 1, 2, ...$; $n = 0, 1, 2, ...$

Theorem 3.18 suggests that each so defined continuant with the number of palindromes specified in the left section of the Table 12 is a divisor of all continuants of the defined type, specified in the right section of the Table 12.

Table 12.

Divisor	Dividends, π units						
1π	6π	14π	22π	30π	38π	46π	...
2π	11π	25π	39π	53π	(67π)	(81π)	...
3π	16π	36π	56π	(76π)	(96π)	(116π)	...
4π	21π	47π	(73π)	(99π)	(125π)	(151π)	...
5π	26π	58π	(90π)	(122π)	(154π)	(186π)	...
6π	31π	(69π)	(107π)	(145π)	(183π)	(221π)	...
...	...	...	...	...	...	...	... ...

Items without brackets are experimental, items in brackets – extrapolated.

Theorem 3.19. *Under Property A conditions every continuant* $K(a_0, \pi, \underbrace{2a_0, \pi,}_{k\text{-times}} a_0 + \rho, \omega)$ *is a divisor of all continuants* $K(a_0, \pi, \underbrace{2a_0, \pi,}_{l\text{-times}} a_0 + \rho, \omega)$*. Here* $l = 7k + 8 + 2n(3k + 4)$; $k = 0, 1, 2, ...$; $n = 0, 1, 2, ...$

Theorem 3.19 suggests that each so defined continuant with the number of palindromes specified in the left section of the Table 13 is a divisor of all continuants of the defined type, specified in the right section of the Table 13.

Table 13.

Divisor	Dividends, π units						
1π	9π	17π	25π	33π	41π	49π	...
2π	16π	30π	44π	58π	(72π)	(86π)	...
3π	23π	43π	(63π)	(83π)	(103π)	(123π)	...
4π	30π	56π	(82π)	(108π)	(134π)	(160π)	...
5π	37π	(69π)	(101π)	(133π)	(165π)	(197π)	...
6π	44π	(82π)	(120π)	(158π)	(196π)	(234π)	...
...	...	...	...	...	...	...	... ...

Items without brackets are experimental, items in brackets – extrapolated.

Proofs of Theorems 3.9 – 3.19 are very similar to that for Theorem 3.8.

3.3 Property A and *abc*-triples

Experimental calculations revealed that *abc*-triples $(4, Dy_i^2, x_i^2)$ and $(4, x_i^2, Dy_i^2)$ arise according to some rules between the root pairs of equations, corresponding to Property A.

Theorem 3.20. *Under Property A conditions the following phenomenons take place.*

1. *If roots* $K(a_0, \pi, \underbrace{2a_0, \pi,}_{k\text{-times}} a_0 + \rho', \omega')$/ $K(\pi, \underbrace{2a_0, \pi,}_{k\text{-times}} a_0 + \rho', \omega')$ $(k = 0, 1, 2, ...)$ *of the generalized Pell's equation* $x^2 - D \cdot y^2 = \pm 4$ *produce an abc-triple, then abc-triples are produced by:*

1.1. *all root pairs* $K(a_0, \pi, \underbrace{2a_0, \pi,}_{l\text{-times}} a_0 + \rho, \omega)$/ $K(\pi, \underbrace{2a_0, \pi,}_{l\text{-times}} a_0 + \rho, \omega)$, *where* $l = 2(k+1) + n \cdot (3k+5); n = 0, 1, 2, ...,$

1.2. *and all root pairs* $K(a_0, \pi, \underbrace{2a_0, \pi,}_{l\text{-times}} a_0 + \rho', \omega')$/ $K(\pi, \underbrace{2a_0, \pi,}_{l\text{-times}} a_0 + \rho', \omega')$,

where $l = 4k + 5 + n \cdot (3k+5); n = 0, 1, 2, ...$

2. *If roots* $K(a_0, \pi, \underbrace{2a_0, \pi,}_{k\text{-times}} a_0 + \rho, \omega)$/ $K(\pi, \underbrace{2a_0, \pi,}_{k\text{-times}} a_0 + \rho, \omega)$ $(k = 0, 1, 2, ...)$ *of the generalized Pell's equation* $x^2 - D \cdot y^2 = \pm 4$ *produce an abc-triple, then abc-triples are produced by:*

2.1. *all root pairs* $K(a_0, \pi, \underbrace{2a_0, \pi,}_{l\text{-times}} a_0 + \rho', \omega')$/ $K(\pi, \underbrace{2a_0, \pi,}_{l\text{-times}} a_0 + \rho', \omega')$,

where $l = 2k + 1 + n \cdot (3k+4); n = 0, 1, 2, ...,$

2.2. *and all root pairs* $K(a_0, \pi, \underbrace{2a_0, \pi,}_{l\text{-times}} a_0 + \rho, \omega)$/ $K(\pi, \underbrace{2a_0, \pi,}_{l\text{-times}} a_0 + \rho, \omega)$,

where $l = 4(k+1) + n \cdot (3k+4); n = 0, 1, 2, ...$

Proof. The proof slightly differs for equations resulting in +4 and –4, so we begin with generalized Pell's equations $x^2 - D \cdot y^2 = 4$ as producents of *abc*-triples. These are items 1.1 and 1.2 with $k = 1, 3, 5, ...$ as well as items 2.1 and 2.2 with $k = 0, 2, 4, ...$

1.1. Now $k = 1$ and our point of departure is the equation

$$K^2(a_0, 2\pi, a_0 + \rho', \omega') - D \cdot K^2(2\pi, a_0 + \rho', \omega') = 4, \tag{36}$$

which is *abc*-triple, therefore

$$R[K^2(a_0, 2\pi, a_0 + \rho', \omega')] \cdot 2R[D \cdot K^2(2\pi, a_0 + \rho', \omega')] < K^2(a_0, 2\pi, a_0 + \rho', \omega'). \tag{37}$$

Excepting factor 2, the left side of (37) contains two odd coprime radicals, so the right side of (37) also can be splitted in two factors. As $R[K^2(a_0, 2\pi, a_0 + \rho', \omega')] \le K(a_0, 2\pi, a_0 + \rho', \omega')$, the maximal value for one of these factors in the right side of (37) is $K(a_0, 2\pi, a_0 + \rho', \omega')$. But, for inequality (37) to be satisfied, the maximal value of the other factor cannot exceed $K(a_0, 2\pi, a_0 + \rho', \omega') - 1$, because this is the greatest natural number coprime to $K(a_0, 2\pi, a_0 + \rho', \omega')$ and not exceeding it. Therefore:

$$\begin{aligned} R[K^2(a_0, 2\pi, a_0 + \rho', \omega')] \cdot 2R[D \cdot K^2(2\pi, a_0 + \rho', \omega')] \\ \le K(a_0, 2\pi, a_0 + \rho', \omega') \cdot [K(a_0, 2\pi, a_0 + \rho', \omega') - 1]. \end{aligned} \tag{38}$$

For odd n values we must show that for $l = 2(k+1) + n \cdot (3k+5) = 4 + 8n = 12, 28, 44, 60, 76, ...$ we also get *abc*-triples.

For $n = 1$ the corresponding equation is

$$K^2(a_0, 13\pi, a_0 + \rho, \omega) - D \cdot K^2(13\pi, a_0 + \rho, \omega) = 4, \tag{39}$$

so we must confirm

$$2 \cdot R[K^2(a_0, 13\pi, a_0 + \rho, \omega)] \cdot R[D \cdot K^2(13\pi, a_0 + \rho, \omega)] < K^2(a_0, 13\pi, a_0 + \rho, \omega). \tag{40}$$

From (36) and (39) we have $D = \dfrac{K^2(a_0, 2\pi, a_0 + \rho', \omega') - 4}{K^2(2\pi, a_0 + \rho', \omega')} = \dfrac{K^2(a_0, 13\pi, a_0 + \rho, \omega) - 4}{K^2(13\pi, a_0 + \rho, \omega)}$, so

$$\frac{K^2(a_0, 13\pi, a_0 + \rho, \omega) - 4}{K^2(a_0, 2\pi, a_0 + \rho', \omega') - 4} = \frac{K^2(13\pi, a_0 + \rho, \omega)}{K^2(2\pi, a_0 + \rho', \omega')}. \tag{41}$$

In view of Theorem 3.8 and Table 2, $K(2\pi, a_0 + \rho', \omega')|K(13\pi, a_0 + \rho, \omega)$, so fraction (41) is natural square and

$$R[\frac{K^2(a_0, 13\pi, a_0 + \rho, \omega) - 4}{K^2(a_0, 2\pi, a_0 + \rho', \omega') - 4}] \leq \frac{K(13\pi, a_0 + \rho, \omega)}{K(2\pi, a_0 + \rho', \omega').} \tag{42}$$

In view of Theorem 3.17 and Table 11, $K(a_0, 2\pi, a_0+\rho', \omega')|K(a_0, 13\pi, a_0+\rho, \omega)$, so analogously

$$R[\frac{K^2(a_0, 13\pi, a_0 + \rho, \omega)}{K^2(a_0, 2\pi, a_0 + \rho', \omega')}] \leq \frac{K(a_0, 13\pi, a_0 + \rho, \omega)}{K(a_0, 2\pi, a_0 + \rho', \omega').} \tag{43}$$

We multiply (38), (42) and (43):

$$\begin{aligned} &2 \cdot R[K^2(a_0, 13\pi, a_0 + \rho, \omega)] \cdot R[D \cdot K^2(13\pi, a_0 + \rho, \omega)] \\ &\qquad \leq K(a_0, 13\pi, a_0 + \rho, \omega) \cdot \frac{K(13\pi, a_0 + \rho, \omega)}{K(2\pi, a_0 + \rho', \omega')} \cdot [K(a_0, 2\pi, a_0 + \rho', \omega') - 1]. \end{aligned} \tag{44}$$

Now we compare

$$K(a_0, 13\pi, a_0 + \rho, \omega) \quad \text{and} \quad \frac{K(13\pi, a_0 + \rho, \omega)}{K(2\pi, a_0 + \rho', \omega')} \cdot [K(a_0, 2\pi, a_0 + \rho', \omega') - 1].$$

We square both sides:

$$K^2(a_0, 13\pi, a_0 + \rho, \omega) \quad \text{and} \quad \frac{K^2(13\pi, a_0 + \rho, \omega)}{K^2(2\pi, a_0 + \rho', \omega')} \cdot [K(a_0, 2\pi, a_0 + \rho', \omega') - 1]^2.$$

In view of (41) we compare

$$\begin{aligned} &K^2(a_0, 13\pi, a_0 + \rho, \omega) \\ &\text{and} \quad [K^2(a_0, 13\pi, a_0 + \rho, \omega) - 4] \cdot \frac{K^2(a_0, 2\pi, a_0 + \rho', \omega') + 1 - 2K(a_0, 2\pi, a_0 + \rho', \omega')}{K^2(a_0, 2\pi, a_0 + \rho', \omega') - 4}. \end{aligned}$$

In the right side of obtained comparison there are two factors, the first of which is less than the square in the left side, but the second factor is a fraction less than 1, therefore clearly sign $>$ must be used in this comparison. As the result we obtain from (44) the necessary confirmation of (40).

For $n = 3$ the corresponding equation is

$$K^2(a_0, 29\pi, a_0 + \rho, \omega) - D \cdot K^2(29\pi, a_0 + \rho, \omega) = 4, \tag{45}$$

so we must confirm

$$2 \cdot R[K^2(a_0, 29\pi, a_0 + \rho, \omega)] \cdot R[D \cdot K^2(29\pi, a_0 + \rho, \omega)] < K^2(a_0, 29\pi, a_0 + \rho, \omega). \tag{46}$$

In Table 2 and Table 11 (Theorems 3.8 and 3.17) we find

$$K(2\pi, a_0 + \rho', \omega')|K(29\pi, a_0 + \rho, \omega)$$

and

$$K(a_0, 2\pi, a_0 + \rho', \omega')|K(a_0, 29\pi, a_0 + \rho, \omega),$$

so we can work fully analogously to the case with $n = 1$. As algorithm works for all odd n values, this confirms Theorem 3.20, case 1.1 for $k = 1$ and all odd n values. Ultimately here the ruling factor is divisibility, illustrated by Tables 2 and 11.

For even n values we must show that for $l = 2(k + 1) + n \cdot (3k + 5) = 4 + 8n = 4, 20, 36, 52, 68, ...$ we also get *abc*-triples.

For $n = 0$ the corresponding equation is

$$K^2(a_0, 5\pi, a_0 + \rho, \omega) - D \cdot K^2(5\pi, a_0 + \rho, \omega) = 4, \tag{47}$$

so we must confirm

$$2 \cdot R[K^2(a_0, 5\pi, a_0 + \rho, \omega)] \cdot R[D \cdot K^2(5\pi, a_0 + \rho, \omega)] < K^2(a_0, 5\pi, a_0 + \rho, \omega). \tag{48}$$

From radical properties

$$R[K^2(a_0, 5\pi, a_0 + \rho, \omega)] \leq K(a_0, 5\pi, a_0 + \rho, \omega). \tag{49}$$

From Theorem 3.3:

$$D \cdot K^2(5\pi, a_0 + \rho, \omega) = D \cdot K^2(a_0, 2\pi, a_0 + \rho', \omega') \cdot K^2(2\pi, a_0 + \rho', \omega'),$$

therefore, in view of (38)

$$2 \cdot R[D \cdot K^2(5\pi, a_0 + \rho, \omega)] \leq K(a_0, 2\pi, a_0 + \rho', \omega') \cdot [K(a_0, 2\pi, a_0 + \rho', \omega') - 1]. \tag{50}$$

We multiply (49) and (50):

$$\begin{aligned} 2 \cdot R[K^2(a_0, 5\pi, a_0 + \rho, \omega)] &\cdot R[D \cdot K^2(5\pi, a_0 + \rho, \omega)] \\ &\leq K(a_0, 5\pi, a_0 + \rho, \omega) \cdot K(a_0, 2\pi, a_0 + \rho', \omega') \cdot [K(a_0, 2\pi, a_0 + \rho', \omega') - 1]. \end{aligned} \tag{51}$$

Now we compare

$$K(a_0, 5\pi, a_0 + \rho, \omega) \quad \text{and} \quad K(a_0, 2\pi, a_0 + \rho', \omega') \cdot [K(a_0, 2\pi, a_0 + \rho', \omega') - 1],$$

that is

$$K^2(a_0, 2\pi, a_0 + \rho', \omega') - 2 \quad \text{and} \quad K^2(a_0, 2\pi, a_0 + \rho', \omega') - K(a_0, 2\pi, a_0 + \rho', \omega').$$

As $K(a_0, 2\pi, a_0 + \rho', \omega') > 2$, then clearly $>$ sign must be used and from (51) we get necessary relation (48).

Then $n = 2$ and for $l = 20$ the corresponding equation is

$$K^2(a_0, 21\pi, a_0 + \rho, \omega) - D \cdot K^2(21\pi, a_0 + \rho, \omega) = 4, \tag{52}$$

so we must confirm

$$2 \cdot R[K^2(a_0, 21\pi, a_0 + \rho, \omega)] \cdot R[D \cdot K^2(21\pi, a_0 + \rho, \omega)] < K^2(a_0, 21\pi, a_0 + \rho, \omega). \tag{53}$$

From radical properties

$$R[K^2(a_0, 21\pi, a_0 + \rho, \omega)] \le K(a_0, 21\pi, a_0 + \rho, \omega). \tag{54}$$

From Theorems 3.3 and 3.2:

$$\begin{aligned} K(21\pi, a_0 + \rho, \omega) &= K(a_0, 10\pi, a_0 + \rho', \omega') \cdot K(10\pi, a_0 + \rho', \omega') \\ &= K(a_0, 10\pi, a_0 + \rho', \omega') \cdot K(a_0, 5\pi, a_0 + \rho, \omega) \cdot K(5\pi, a_0 + \rho, \omega). \end{aligned} \tag{55}$$

From recently proved (48), by the same argumentation which was used for shifting from (37) to (38), we have:

$$\begin{aligned} &2 \cdot R[K^2(a_0, 5\pi, a_0 + \rho, \omega)] \cdot R[D \cdot K^2(5\pi, a_0 + \rho, \omega)] \\ &\qquad \le K(a_0, 5\pi, a_0 + \rho, \omega) \cdot [K(a_0, 5\pi, a_0 + \rho, \omega) - 1]. \end{aligned} \tag{56}$$

Again from radical properties:

$$R[K^2(a_0, 10\pi, a_0 + \rho', \omega')] \le K(a_0, 10\pi, a_0 + \rho', \omega'). \tag{57}$$

From (55), (56) and (57) we get

$$\begin{aligned} &2 \cdot R[D \cdot K^2(21\pi, a_0 + \rho, \omega)] \\ &\qquad \le K(a_0, 10\pi, a_0 + \rho', \omega') \cdot K(a_0, 5\pi, a_0 + \rho, \omega) \cdot [K(a_0, 5\pi, a_0 + \rho, \omega) - 1]. \end{aligned} \tag{58}$$

We multiply this with (54) and get

$$\begin{aligned} &2 \cdot R[K^2(a_0, 21\pi, a_0 + \rho, \omega)] \cdot R[D \cdot K^2(21\pi, a_0 + \rho, \omega)] \\ &\qquad \le K(a_0, 21\pi, a_0 + \rho, \omega) \cdot K(a_0, 10\pi, a_0 + \rho', \omega') \\ &\qquad\qquad \cdot K(a_0, 5\pi, a_0 + \rho, \omega) \cdot [K(a_0, 5\pi, a_0 + \rho, \omega) - 1]. \end{aligned} \tag{59}$$

Now we must compare

$$K(a_0, 21\pi, a_0 + \rho, \omega) = K^2(a_0, 10\pi, a_0 + \rho', \omega') - 2$$

and

$$K(a_0, 10\pi, a_0 + \rho', \omega') \cdot K(a_0, 5\pi, a_0 + \rho, \omega) \cdot [K(a_0, 5\pi, a_0 + \rho, \omega) - 1],$$

or, equivalently:

$$K^2(a_0, 10\pi, a_0 + \rho', \omega')$$

and

$$K(a_0, 10\pi, a_0 + \rho', \omega') \cdot [K^2(a_0, 5\pi, a_0 + \rho, \omega) - K(a_0, 5\pi, a_0 + \rho, \omega)] + 2. \tag{60}$$

From Theorem 3.2 we have

$$K(a_0, 10\pi, a_0 + \rho', \omega') = K^2(a_0, 5\pi, a_0 + \rho, \omega) - 2,$$

therefore

$$\begin{aligned}(60) &= K(a_0,10\pi,a_0+\rho',\omega')\cdot[K(a_0,10\pi,a_0+\rho',\omega')+2-K(a_0,5\pi,a_0+\rho,\omega)]+2\\ &= K^2(a_0,10\pi,a_0+\rho',\omega')+2+K(a_0,10\pi,a_0+\rho',\omega')\cdot[2-K(a_0,5\pi,a_0+\rho,\omega)]. \quad (61)\end{aligned}$$

As $K(a_0, 5\pi, a_0 + \rho, \omega) > 2$, the term in square brackets in (61) is negative, so clearly $>$ sign must be used in our comparison and from (59) we get necessary (53). Argumentation, analogous to that for shifting from (37) to (38), gives

$$\begin{aligned}2\cdot R[K^2(a_0,21\pi,a_0+\rho,\omega)]\cdot R[D\cdot K^2(21\pi,a_0+\rho,\omega)]&\\ \le K(a_0,21\pi,a_0+\rho,\omega)\cdot[K(a_0,21\pi,a_0+\rho,\omega)-1]&. \quad (62)\end{aligned}$$

Now $n = 4$ and for $l = 36$ the corresponding equation is

$$K^2(a_0,37\pi,a_0+\rho,\omega)-D\cdot K^2(37\pi,a_0+\rho,\omega)=4, \quad (63)$$

so we must confirm

$$2\cdot R[K^2(a_0,37\pi,a_0+\rho,\omega)]\cdot R[D\cdot K^2(37\pi,a_0+\rho,\omega)]<K^2(a_0,37\pi,a_0+\rho,\omega). \quad (64)$$

From radical properties

$$R[K^2(a_0,37\pi,a_0+\rho,\omega)]\le K(a_0,37\pi,a_0+\rho,\omega). \quad (65)$$

The already known splitting gives

$$K(37\pi,a_0+\rho,\omega)=K(a_0,18\pi,a_0+\rho',\omega')\cdot K(18\pi,a_0+\rho',\omega'). \quad (66)$$

From Table 3 we have $K(2\pi, a_0 + \rho', \omega')|K(18\pi, a_0 + \rho', \omega')$, while from Table 10 we have $K(a_0, 2\pi, a_0 + \rho', \omega')|K(a_0, 18\pi, a_0 + \rho', \omega')$, therefore we discuss an equation

$$K^2(a_0,18\pi,a_0+\rho',\omega')-D\cdot K^2(18\pi,a_0+\rho',\omega')=4. \quad (67)$$

From (36) and (67) we have $D = \dfrac{K^2(a_0,2\pi,a_0+\rho',\omega')-4}{K^2(2\pi,a_0+\rho',\omega')} = \dfrac{K^2(a_0,18\pi,a_0+\rho',\omega')-4}{K^2(18\pi,a_0+\rho',\omega')}$, so

$$\frac{K^2(a_0,18\pi,a_0+\rho',\omega')-4}{K^2(a_0,2\pi,a_0+\rho',\omega')-4}=\frac{K^2(18\pi,a_0+\rho',\omega')}{K^2(2\pi,a_0+\rho',\omega')}. \quad (68)$$

Divisibility data (Tables 3 and 10) gives natural squares and radical evaluations, analogous to (42) and (43), so by multiplication with (38) we get

$$\begin{aligned}2\cdot R[K^2(a_0,18\pi,a_0+\rho',\omega')]\cdot R[D\cdot K^2(18\pi,a_0+\rho',\omega')]&\\ \le K(a_0,18\pi,a_0+\rho',\omega')\cdot\frac{K(18\pi,a_0+\rho',\omega')}{K(2\pi,a_0+\rho',\omega')}\cdot[K(a_0,2\pi,a_0+\rho',\omega')-1]&. \quad (69)\end{aligned}$$

Obtained equation (69) is an analogue of (44), therefore comparison stage can be processed by the same method, which confirms

$$2\cdot R[K^2(a_0,18\pi,a_0+\rho',\omega')]\cdot R[D\cdot K^2(18\pi,a_0+\rho',\omega')]<K^2(a_0,18\pi,a_0+\rho',\omega'), \quad (70)$$

and, by already known argumentation (see shifting from (37) to (38)):

$$2 \cdot R[K^2(a_0, 18\pi, a_0 + \rho', \omega')] \cdot R[D \cdot K^2(18\pi, a_0 + \rho', \omega')] \\ \leq K(a_0, 18\pi, a_0 + \rho', \omega') \cdot [K(a_0, 18\pi, a_0 + \rho', \omega') - 1]. \quad (71)$$

But our ultimate target was equation (63). From (66) we can assert

$$2 \cdot R[D \cdot K^2(37\pi, a_0 + \rho, \omega)] = 2 \cdot R[K^2(a_0, 18\pi, a_0 + \rho', \omega')] \cdot R[D \cdot K^2(18\pi, a_0 + \rho', \omega')], \quad (72)$$

which, in view of (71), is less or equal to $K(a_0, 18\pi, a_0 + \rho', \omega') \cdot [K(a_0, 18\pi, a_0 + \rho', \omega') - 1]$. We multiply this result by (65):

$$2 \cdot R[K^2(a_0, 37\pi, a_0 + \rho, \omega)] \cdot R[D \cdot K^2(37\pi, a_0 + \rho, \omega)] \\ \leq K(a_0, 37\pi, a_0 + \rho, \omega) \cdot K(a_0, 18\pi, a_0 + \rho', \omega') \cdot [K(a_0, 18\pi, a_0 + \rho', \omega') - 1]. \quad (73)$$

Now we compare

$$K(a_0, 37\pi, a_0 + \rho, \omega) \quad \text{and} \quad K(a_0, 18\pi, a_0 + \rho', \omega') \cdot [K(a_0, 18\pi, a_0 + \rho', \omega') - 1],$$

that is

$$K^2(a_0, 18\pi, a_0 + \rho', \omega') - 2 \quad \text{and} \quad K^2(a_0, 18\pi, a_0 + \rho', \omega') - K(a_0, 18\pi, a_0 + \rho', \omega').$$

Clearly $>$ sign must be used and from (73) we get necessary (64), simultaneously confirming

$$2 \cdot R[K^2(a_0, 37\pi, a_0 + \rho, \omega)] \cdot R[D \cdot K^2(37\pi, a_0 + \rho, \omega)] \\ \leq K(a_0, 37\pi, a_0 + \rho, \omega) \cdot [K(a_0, 37\pi, a_0 + \rho, \omega) - 1]. \quad (74)$$

For $n = 6, 10, 14, ...$ we can act analogously with the method for $n = 2$, while for $n = 8, 12, 16, ...$ we must use the method for $n = 4$. Ultimately here the ruling factor is divisibility

$$K(2\pi, a_0 + \rho', \omega') | K(y\pi, a_0 + \rho', \omega'), \quad y = 10, 18, 26, 34, 42, 50, ... \quad \text{(Table 3)},$$
$$K(a_0, 2\pi, a_0 + \rho', \omega') | K(a_0, y\pi, a_0 + \rho', \omega'), \quad y = 18, 34, 50, 66, 82, 98, ... \quad \text{(Table 10)},$$
$$K(a_0, 2\pi, a_0 + \rho', \omega') | K(y\pi, a_0 + \rho', \omega'), \quad y = 10, 26, 42, 58, 74, 90, ... \quad \text{(Table 6)}.$$

In total this inductively confirms Theorem 3.20, case 1.1 for $k = 1$ and all n values.

For $k = 3$ our point of departure is the equation

$$K^2(a_0, 4\pi, a_0 + \rho', \omega') - D \cdot K^2(4\pi, a_0 + \rho', \omega') = 4, \quad (75)$$

which is *abc*-triple, therefore

$$R[K^2(a_0, 4\pi, a_0 + \rho', \omega')] \cdot 2R[D \cdot K^2(4\pi, a_0 + \rho', \omega')] < K^2(a_0, 4\pi, a_0 + \rho', \omega'), \quad (76)$$

or, more precisely:

$$R[K^2(a_0, 4\pi, a_0 + \rho', \omega')] \cdot 2R[D \cdot K^2(4\pi, a_0 + \rho', \omega')] \\ \leq K(a_0, 4\pi, a_0 + \rho', \omega') \cdot [K(a_0, 4\pi, a_0 + \rho', \omega') - 1]. \quad (77)$$

Again we will treat cases with odd and even n values separately.

For $n = 1$ we have $l = 2(k+1) + n(3k+5) = 22$ and the corresponding equation is

$$K^2(a_0, 23\pi, a_0 + \rho, \omega) - D \cdot K^2(23\pi, a_0 + \rho, \omega) = 4, \tag{78}$$

so we must confirm

$$2 \cdot R[K^2(a_0, 23\pi, a_0 + \rho, \omega)] \cdot R[D \cdot K^2(23\pi, a_0 + \rho, \omega)] < K^2(a_0, 23\pi, a_0 + \rho, \omega). \tag{79}$$

As $K(4\pi, a_0 + \rho', \omega')|K(23\pi, a_0 + \rho, \omega)$ and $K(a_0, 4\pi, a_0 + \rho', \omega')|K(a_0, 23\pi, a_0 + \rho, \omega)$ (see Tables 2 and 11), we can proceed analogously to that for $k = 1, n = 1$ (equations (39)–(44)) and confirm necessary (79). Also:

$$\begin{aligned} 2 \cdot R[K^2(a_0, 23\pi, a_0 + \rho, \omega)] \cdot R[D \cdot K^2(23\pi, a_0 + \rho, \omega)] \\ \leq K(a_0, 23\pi, a_0 + \rho, \omega) \cdot [K(a_0, 23\pi, a_0 + \rho, \omega) - 1]. \end{aligned} \tag{80}$$

The situation repeats for further odd values $n = 3, 5, ...$, because ultimately it depends from the divisibility in Tables 2 and 11.

Cases with even values $n = 0, 2, 4, ...$ we can treat analogously with that for $k = 1$ due to their dependency from Tables 3, 6 and 10.

As situation repeats for further odd k values, this confirms Theorem 3.20, case 1.1 for all odd k values $k = 1, 3, 5, ...$

1.2. Now we have the same initial equations, which are *abc*-triples in case 1.1, but new *abc*-triples are produced by root pairs $K(a_0, \pi, \underbrace{2a_0, \pi,}_{l\text{-times}} a_0 + \rho', \omega')$/ $K(\pi, \underbrace{2a_0, \pi,}_{l\text{-times}} a_0 + \rho', \omega')$, differs $l = 4k + 5 + n \cdot (3k + 5), n = 0, 1, 2, ...$

For $k = 1$ initial *abc*-triple is from equation (36), we repeat it here:

$$K^2(a_0, 2\pi, a_0 + \rho', \omega') - D \cdot K^2(2\pi, a_0 + \rho', \omega') = 4.$$

For odd n values we must show that corresponding equations with $l = 17, 33, 49, ...$ also produce *abc*-triples. As initial equations are the same, we can refer to the case 1.1, $l = 36, 68, 100, ...$, where necessary results were obtained as intermediaries (see (71)). Combination of useful splittings (equations, analogous to (55) and (66), based on Theorems 3.2 and 3.3) with methods, discussed in case 1.1, allows inductive confirmations for odd and even n values for $k = 1, 3, 5, ...$, thus confirming Theorem 3.20, case 1.2 for all values $k = 1, 3, 5, ...$

2.1. and **2.2.** Here initial producers of *abc*-triples are root pairs $K(a_0, \pi, \underbrace{2a_0, \pi,}_{l\text{-times}} a_0 + \rho, \omega)$/ $K(\pi, \underbrace{2a_0, \pi,}_{l\text{-times}} a_0 + \rho, \omega)$, but $k = 0, 2, 4, 6, ...$ Verification of Theorem 3.20 for these cases can be processed inductively by the same methods as for cases 1.1 and 1.2.

Nevertheless we are only halfway down in our proof, remaining ones are generalized Pell's equations $x^2 - D \cdot y^2 = -4$ as producents of *abc*-triples. These are items 1.1 and 1.2 with $k = 0, 2, 4, ...$, as well as items 2.1 and 2.2 with $k = 1, 3, 5, ...$ Apart from one crucial moment, verification methods will be the same, therefore only one example will be presented – case 1.1 with $k = 0$ and $n = 0$.

1.1. Now $k = 0$ and our point of departure is the equation

$$K^2(a_0, \pi, a_0 + \rho', \omega') - D \cdot K^2(\pi, a_0 + \rho', \omega') = -4, \tag{81}$$

which is *abc*-triple, therefore

$$R[K^2(a_0, \pi, a_0 + \rho', \omega')] \cdot 2R[D \cdot K^2(\pi, a_0 + \rho', \omega')] < D \cdot K^2(\pi, a_0 + \rho', \omega'). \tag{82}$$

Excepting factor 2, the left side of (82) contains two odd coprime radicals, therefore we split the right side of it into two factors $K(\pi, a_0 + \rho', \omega')$ and $D \cdot K(\pi, a_0 + \rho', \omega')$. If we put all primes from $R[D \cdot K^2(\pi, a_0 + \rho', \omega')]$ into right side factor $D \cdot K(\pi, a_0 + \rho', \omega')$, then from the other radical $R[K^2(a_0, \pi, a_0 + \rho', \omega')]$ we will get only coprime factors, which, for inequality (82) to be satisfied, can compose the other right side factor maximally as $K(\pi, a_0 + \rho', \omega') - 1$. Then this other factor $K(\pi, a_0 + \rho', \omega') - 1$ will be coprime with $K(\pi, a_0 + \rho', \omega')$, but it's coprimality with D is not guaranteed. Therefore we must do it conversely and predict the other (and coprime) factor in the right side of (82) maximally as $D \cdot K(\pi, a_0 + \rho', \omega') - 1$, which confirms

$$\begin{aligned} R[K^2(a_0, \pi, a_0 + \rho', \omega')] \cdot 2R[D \cdot K^2(\pi, a_0 + \rho', \omega')] \\ \leq K(\pi, a_0 + \rho', \omega') \cdot [D \cdot K(\pi, a_0 + \rho', \omega') - 1]. \end{aligned} \tag{83}$$

For $n = 0$ and $l = 2(k + 1) + n \cdot (3k + 5) = 2$ the corresponding equation is

$$K^2(a_0, 3\pi, a_0 + \rho, \omega) - D \cdot K^2(3\pi, a_0 + \rho, \omega) = 4, \tag{84}$$

so we must confirm

$$2 \cdot R[K^2(a_0, 3\pi, a_0 + \rho, \omega)] \cdot R[D \cdot K^2(3\pi, a_0 + \rho, \omega)] < K^2(a_0, 3\pi, a_0 + \rho, \omega). \tag{85}$$

From Theorem 3.3:

$$K(3\pi, a_0 + \rho, \omega) = K(a_0, \pi, a_0 + \rho', \omega') \cdot K(\pi, a_0 + \rho', \omega'). \tag{86}$$

From (83) and (86) we get

$$2 \cdot R[D \cdot K^2(3\pi, a_0 + \rho, \omega)] \leq K(\pi, a_0 + \rho', \omega') \cdot [D \cdot K(\pi, a_0 + \rho', \omega') - 1]. \tag{87}$$

As $R[K^2(a_0, 3\pi, a_0 + \rho, \omega)] \leq K(a_0, 3\pi, a_0 + \rho, \omega)$,

$$\begin{aligned} 2 \cdot R[K^2(a_0, 3\pi, a_0 + \rho, \omega)] \cdot R[D \cdot K^2(3\pi, a_0 + \rho, \omega)] \\ \leq K(a_0, 3\pi, a_0 + \rho, \omega) \cdot K(\pi, a_0 + \rho', \omega') \cdot [D \cdot K(\pi, a_0 + \rho', \omega') - 1]. \end{aligned} \tag{88}$$

We compare

$$K(a_0, 3\pi, a_0 + \rho, \omega) \quad \text{and} \quad K(\pi, a_0 + \rho', \omega') \cdot [D \cdot K(\pi, a_0 + \rho', \omega') - 1],$$

or, equivalently

$$D \cdot K^2(\pi, a_0 + \rho', \omega') - 2 \quad \text{and} \quad D \cdot K^2(\pi, a_0 + \rho', \omega') - K(\pi, a_0 + \rho', \omega').$$

Clearly $>$ sign must be used, which confirms the necessary equation (85).

The shift from (82) to (83) is this crucial analogue to that from (37) to (38), but the remaining approaches are similar to discussed ones. This completes the proof of Theorem 3.20. □

3.4 We include fundamental roots

Sometimes just fundamental roots $K(\rho,\omega)/K(\omega)$ or $K(\rho',\omega')/K(\omega')$ gives rise to *abc*-triples, but these cases are not covered by Theorem 3.20. Formulation of corresponding rules is possible, but at first we must discuss some divisibility relations. In the following proofs (Theorems 3.21–3.26) we will frequently use some already known facts:

- $K(\omega)|K(\pi)$ and $K(\rho,\omega)|K(a_0,\pi)$,
  $K(\omega')|K(\pi,2a_0,\pi)$ and $K(\rho',\omega')|K(a_0,2\pi)$, see (7) and (8);
- $K(\pi,a_0+\rho,\omega) = K(\rho',\omega')\cdot K(\omega')$, see (28);
- $K(\pi)|K(y\pi)$ for $y=(1),2,3,4,...$,
  $K(a_0,\pi)|K(a_0,y\pi)$ for $y=(1),3,5,7,...$,
  $K(a_0,2\pi)|K(a_0,y\pi)$ for $y=(2),6,10,14,...$,
  – these are properties of ordinary positive/negative Pell's equations $x^2-D\cdot y^2=\pm 1$.

Theorem 3.21. *Under Property A conditions every continuant $K(\omega')$ is a divisor of*
1. *all continuants $K(n\pi,a_0+\rho,\omega)$, where $n=1,3,5,...$, and*
2. *all continuants $K(n\pi,a_0+\rho',\omega')$, where $n=2,4,6,....$*

Proof. **1.** For $n=1$ the necessary relation already comes from (28), but it can be proved independently. We split $K(\pi,a_0+\rho,\omega)=K(a_0,\pi)\cdot K(\omega)+K(\pi)\cdot K(\rho,\omega)$. As $K(\omega)|K(\pi)$ and $K(\rho,\omega)|K(a_0,\pi)$, then both summands are divisible by $K(\rho,\omega)\cdot K(\omega)=K(\omega')$.

For $n=3$ we must show that $K(\omega')|K(3\pi,a_0+\rho,\omega)$. Again splitting:

$$K(3\pi,a_0+\rho,\omega)=K(a_0,3\pi)\cdot K(\omega)+K(3\pi)\cdot K(\rho,\omega).$$

As $K(\omega)|K(\pi)|K(3\pi)$ and $K(\rho,\omega)|K(a_0,\pi)|K(a_0,3\pi)$, then both summands are divisible by $K(\omega')$. Similar splittings for $n=5,7,9,...$ inductively confirm Theorem 3.21, case 1 for all values $n=1,3,5,....$

2. For $n=2$ we must show that $K(\omega')|K(2\pi,a_0+\rho',\omega')$. From Theorem 3.2:

$$K(2\pi,a_0+\rho',\omega')=K(\pi,a_0+\rho,\omega)\cdot K(a_0,\pi,a_0+\rho,\omega),$$

but $K(\omega')|K(\pi,a_0+\rho,\omega)$. The same splitting is suitable for $n=6,10,14,...$, with confirmation based on case 1.

For $n=4$ we must show that $K(\omega')|K(4\pi,a_0+\rho',\omega')$. From Theorem 3.2:

$$K(4\pi,a_0+\rho',\omega')=K(2\pi,a_0+\rho,\omega)\cdot K(a_0,2\pi,a_0+\rho,\omega). \tag{89}$$

$K(2\pi,a_0+\rho,\omega)=K(a_0,2\pi)\cdot K(\omega)+K(2\pi)\cdot K(\rho,\omega)$ and $K(\omega)|K(\pi)|K(2\pi)$, therefore $K(\omega)|K(2\pi,a_0+\rho,\omega)$.

$$\begin{aligned}K(a_0,2\pi,a_0+\rho,\omega) &= K(a_0,2\pi)\cdot K(\rho,\omega)+K(a_0,2\pi,a_0)\cdot K(\omega)\\ &= K(a_0,2\pi)\cdot K(\rho,\omega)+2K(a_0,\pi)\cdot K(a_0,\pi,a_0)\cdot K(\omega).\end{aligned} \tag{90}$$

As $K(\rho,\omega)|K(a_0,\pi)$, so is the sum in (90), therefore $K(\rho,\omega)|K(a_0,2\pi,a_0+\rho,\omega)$. In view of (89), this confirms $K(\omega)\cdot K(\rho,\omega)=K(\omega')|K(4\pi,a_0+\rho',\omega')$.

Similar approach for $n=8,12,16,...$ inductively confirms theorem 3.21. □

Theorem 3.22. *Under Property A conditions every continuant $K(\omega)$ is a divisor of*
1. *all continuants $K(n\pi, a_0 + \rho, \omega)$, where $n = 1, 2, 3, ...$, and*
2. *all continuants $K(n\pi, a_0 + \rho', \omega')$, where $n = 1, 2, 3, ...$*

Theorem 3.23. *Under Property A conditions every continuant $K(\rho', \omega')$ is a divisor of*
1. *all continuants $K(n\pi, a_0 + \rho, \omega)$, where $n = 1, 5, 9, ...$, and*
2. *all continuants $K(n\pi, a_0 + \rho', \omega')$, where $n = 2, 6, 10, ...$*

Theorem 3.24. *Under Property A conditions every continuant $K(\rho, \omega)$ is a divisor of*
1. *all continuants $K(n\pi, a_0 + \rho, \omega)$, where $n = 1, 3, 5, ...$, and*
2. *all continuants $K(n\pi, a_0 + \rho', \omega')$, where $n = 2, 4, 6, ...$*

Theorem 3.25. *Under Property A conditions every continuant $K(\rho', \omega')$ is a divisor of*
1. *all continuants $K(a_0, n\pi, a_0 + \rho, \omega)$, where $n = 3, 7, 11, ...$, and*
2. *all continuants $K(a_0, n\pi, a_0 + \rho', \omega')$, where $n = 4, 8, 12, ...$*

Theorem 3.26. *Under Property A conditions every continuant $K(\rho, \omega)$ is a divisor of*
1. *all continuants $K(a_0, n\pi, a_0 + \rho', \omega')$, where $n = 1, 3, 5, ...$, and*
2. *all continuants $K(a_0, n\pi, a_0 + \rho, \omega)$, where $n = 2, 4, 6, ...$*

Proofs of Theorems 3.22 – 3.26 are very similar to that for Theorem 3.21.

Now we are in a suitable position to formulate the corresponding rules for *abc*-triples.

Theorem 3.27. *Under Property A conditions the following phenomenons take place. If fundamental roots $K(\rho', \omega')/K(\omega')$ of the generalized Pell's equation $x^2 - D \cdot y^2 = 4$ produce an abc-triple, then:*
1. *all root pairs $K(a_0, \pi, \underbrace{2a_0, \pi,}_{l\text{-times}} a_0 + \rho, \omega)$/ $K(\pi, \underbrace{2a_0, \pi,}_{l\text{-times}} a_0 + \rho, \omega)$, where $l = 0, 2, 4, ...$, produce abc-triples, and*
2. *all root pairs $K(a_0, \pi, \underbrace{2a_0, \pi,}_{l\text{-times}} a_0 + \rho', \omega')$/ $K(\pi, \underbrace{2a_0, \pi,}_{l\text{-times}} a_0 + \rho', \omega')$, where $l = 1, 3, 5, ...$, produce abc-triples.*

Proof. As initial condition, equation

$$K^2(\rho', \omega') - D \cdot K^2(\omega') = 4 \tag{91}$$

is an *abc*-trio, that is

$$2R[K^2(\rho', \omega')] \cdot R[D \cdot K^2(\omega')] < K^2(\rho', \omega'), \tag{92}$$

or, by the same argumentation, as for shifting from (37) to (38),

$$2R[K^2(\rho', \omega')] \cdot R[D \cdot K^2(\omega')] \le K(\rho', \omega') \cdot [K(\rho', \omega') - 1]. \tag{93}$$

1. Now $l = 0$ and we must confirm that

$$2R[K^2(a_0, \pi, a_0 + \rho, \omega)] \cdot R[D \cdot K^2(\pi, a_0 + \rho, \omega)] < K^2(a_0, \pi, a_0 + \rho, \omega). \tag{94}$$

We already know that

$$R[K^2(a_0,\pi,a_0+\rho,\omega)] \le K(a_0,\pi,a_0+\rho,\omega). \tag{95}$$

In view of (28) and (93):

$$2R[D\cdot K^2(\pi,a_0+\rho,\omega)] = 2R[K^2(\rho',\omega')]\cdot R[D\cdot K^2(\omega')] \le K(\rho',\omega')\cdot[K(\rho',\omega')-1]. \tag{96}$$

We multiply (95) and (96):

$$\begin{aligned}2R[K^2(a_0,\pi,a_0+\rho,\omega)]\cdot R[D\cdot K^2(\pi,a_0+\rho,\omega)]&\\ \le K(a_0,\pi,a_0+\rho,\omega)\cdot K(\rho',\omega')\cdot[K(\rho',\omega')-1].&\end{aligned} \tag{97}$$

We compare $K(a_0,\pi,a_0+\rho,\omega)$ and $K(\rho',\omega')\cdot[K(\rho',\omega')-1]$. In view of (29), objects for comparison are $K^2(\rho',\omega')-2$ and $K^2(\rho',\omega')-K(\rho',\omega')$. As minimal $K(\rho',\omega')$ equals 3, clearly sign $>$ can be used in this comparison, which confirms necessary (94). From this also

$$\begin{aligned}2R[K^2(a_0,\pi,a_0+\rho,\omega)]\cdot R[D\cdot K^2(\pi,a_0+\rho,\omega)]&\\ \le K(a_0,\pi,a_0+\rho,\omega)\cdot[K(a_0,\pi,a_0+\rho,\omega)-1].&\end{aligned} \tag{98}$$

Now $l=2$ and for the equation

$$K^2(a_0,3\pi,a_0+\rho,\omega)-D\cdot K^2(3\pi,a_0+\rho,\omega)=4 \tag{99}$$

we must confirm

$$2R[K^2(a_0,3\pi,a_0+\rho,\omega)]\cdot R[D\cdot K^2(3\pi,a_0+\rho,\omega)] < K^2(a_0,3\pi,a_0+\rho,\omega). \tag{100}$$

We use the same method, as for Theorem 3.20, case 1.1 with $n=1$, see equations (41) – (44). As $K(\omega')|K(3\pi,a_0+\rho,\omega)$ and $K(\rho',\omega')|K(a_0,3\pi,a_0+\rho,\omega)$ (see Theorems 3.21 and 3.25), we get radicals of two natural squares, and, finally:

$$\begin{aligned}2\cdot R[K^2(a_0,3\pi,a_0+\rho,\omega)]\cdot R[D\cdot K^2(3\pi,a_0+\rho,\omega)]&\\ \le K(a_0,3\pi,a_0+\rho,\omega)\cdot\frac{K(3\pi,a_0+\rho,\omega)}{K(\omega')}\cdot[K(\rho',\omega')-1].&\end{aligned} \tag{101}$$

We compare $K(a_0,3\pi,a_0+\rho,\omega)$ and $\dfrac{K(3\pi,a_0+\rho,\omega)}{K(\omega')}\cdot[K(\rho',\omega')-1]$;

$$K^2(a_0,3\pi,a_0+\rho,\omega) \quad\text{and}\quad K^2(3\pi,a_0+\rho,\omega)\cdot\frac{K^2(\rho',\omega')-2K(\rho',\omega')+1}{K^2(\omega')};$$

$$D\cdot K^2(3\pi,a_0+\rho,\omega)+4 \quad\text{and}\quad K^2(3\pi,a_0+\rho,\omega)\cdot\frac{K^2(\rho',\omega')-2K(\rho',\omega')+1}{K^2(\omega')};$$

$$K^2(3\pi,a_0+\rho,\omega) \quad\text{and}\quad K^2(3\pi,a_0+\rho,\omega)\cdot\frac{K^2(\rho',\omega')-2K(\rho',\omega')+1}{D\cdot K^2(\omega')}-\frac{4}{D};$$

$$K^2(3\pi, a_0+\rho,\omega) \quad \text{and} \quad K^2(3\pi, a_0+\rho,\omega)\cdot\frac{K^2(\rho',\omega')-2K(\rho',\omega')+1}{K^2(\rho',\omega')-4}-\frac{4}{D}.$$

As minimal $K(\rho',\omega')$ equals 3, sign $>$ can be used in this comparison, which confirms necessary (100). The same processing can be used for $l = 6, 10, 14, ...$

Now $l = 4$ and for the equation

$$K^2(a_0, 5\pi, a_0+\rho,\omega) - D\cdot K^2(5\pi, a_0+\rho,\omega) = 4 \tag{102}$$

we must confirm

$$2R[K^2(a_0, 5\pi, a_0+\rho,\omega)]\cdot R[D\cdot K^2(5\pi, a_0+\rho,\omega)] < K^2(a_0, 5\pi, a_0+\rho,\omega). \tag{103}$$

Splitting gives

$$K(5\pi, a_0+\rho,\omega) = K(a_0, 2\pi, a_0+\rho',\omega')\cdot K(a_0,\pi, a_0+\rho,\omega)\cdot K(\pi, a_0+\rho,\omega),$$

so we discuss an equation

$$K^2(a_0, 2\pi, a_0+\rho',\omega') - D\cdot K^2(2\pi, a_0+\rho',\omega') = 4. \tag{104}$$

As $K(2\pi, a_0+\rho',\omega') = K(a_0,\pi, a_0+\rho,\omega)\cdot K(\pi, a_0+\rho,\omega)$, then, in analogy with proved result for $l = 0$, equation (104) also gives *abc*-triple. This is an additional result. But then

$$2R[D\cdot K^2(5\pi, a_0+\rho,\omega)] = 2R[D\cdot K^2(2\pi, a_0+\rho',\omega')]\cdot R[K^2(a_0, 2\pi, a_0+\rho',\omega')],$$

which, in view that (104) is an *abc*-triple, is $\leq K(a_0, 2\pi, a_0+\rho',\omega')\cdot[K(a_0, 2\pi, a_0+\rho',\omega')-1]$. We multiply this with our standard $R[K^2(a_0, 5\pi, a_0+\rho,\omega)] \leq K(a_0, 5\pi, a_0+\rho,\omega)$ and obtain

$$\begin{aligned} &2R[K^2(a_0, 5\pi, a_0+\rho,\omega)]\cdot R[D\cdot K^2(5\pi, a_0+\rho,\omega)] \\ &\qquad \leq K(a_0, 5\pi, a_0+\rho,\omega)\cdot K(a_0, 2\pi, a_0+\rho',\omega')\cdot[K(a_0, 2\pi, a_0+\rho',\omega')-1]. \end{aligned} \tag{105}$$

Now we compare $K(a_0, 5\pi, a_0+\rho,\omega) = K^2(a_0, 2\pi, a_0+\rho',\omega')-2$ and $K^2(a_0, 2\pi, a_0+\rho',\omega') - K(a_0, 2\pi, a_0+\rho',\omega')$. Clearly $>$ sign can be used, which confirms necessary (103), as well as

$$\begin{aligned} &2R[K^2(a_0, 5\pi, a_0+\rho,\omega)]\cdot R[D\cdot K^2(5\pi, a_0+\rho,\omega)] \\ &\qquad \leq K(a_0, 5\pi, a_0+\rho,\omega)\cdot[K(a_0, 5\pi, a_0+\rho,\omega)-1]. \end{aligned} \tag{106}$$

The same processing can be used for $l = 8, 12, 16, ...,$ which confirms Theorem 3.27, case 1. Values $l = 4, 8, 12, ...$ have additional results – equation (104) and it's analogues

$$K^2(a_0, 4\pi, a_0+\rho',\omega') - D\cdot K^2(4\pi, a_0+\rho',\omega') = 4,$$

$$K^2(a_0, 6\pi, a_0+\rho',\omega') - D\cdot K^2(6\pi, a_0+\rho',\omega') = 4, \text{etc.},$$

also are *abc*-triples. Sequence of these additional results proves case 2 of Theorem 3.27. □

Theorem 3.28. *Under Property A conditions the following phenomenons take place. If fundamental roots $K(\rho,\omega)/K(\omega)$ of the generalized Pell's equation $x^2 - D \cdot y^2 = -4$ produce an abc-triple, then:*
***1.** root pair $K(\rho',\omega')/K(\omega')$ also produces an abc-triple;*
***2.** all root pairs $K(a_0, \pi, \underbrace{2a_0, \pi,}_{l\text{-times}} a_0 + \rho, \omega)$/ $K(\pi, \underbrace{2a_0, \pi,}_{l\text{-times}} a_0 + \rho, \omega)$, where $l = 0, 1, 2, ...,$ produce abc-triples, and*
***3.** all root pairs $K(a_0, \pi, \underbrace{2a_0, \pi,}_{l\text{-times}} a_0 + \rho', \omega')$/ $K(\pi, \underbrace{2a_0, \pi,}_{l\text{-times}} a_0 + \rho', \omega')$, where $l = 0, 1, 2, ...,$ produce abc-triples.*

Proof. As initial condition, equation

$$K^2(\rho,\omega) - D \cdot K^2(\omega) = -4 \tag{107}$$

is an *abc*-triple, that is

$$2R[K^2(\rho,\omega)] \cdot R[D \cdot K^2(\omega)] < D \cdot K^2(\omega), \tag{108}$$

or

$$2R[K^2(\rho,\omega)] \cdot R[D \cdot K^2(\omega)] \le D \cdot K^2(\omega) - 1. \tag{109}$$

As $D \cdot K^2(\omega) - 1 \equiv 0 \pmod 4$, we divide both sides of (109) by 2:

$$R[K^2(\rho,\omega)] \cdot R[D \cdot K^2(\omega)] \le \frac{D \cdot K^2(\omega) - 1}{2}. \tag{110}$$

In the right side of (110) there is an even number, while the left side is a product of two odd numbers. Therefore:

$$R[K^2(\rho,\omega)] \cdot R[D \cdot K^2(\omega)] \le \frac{D \cdot K^2(\omega) - 1}{2} - 1 = \frac{D \cdot K^2(\omega) - 3}{2}. \tag{111}$$

From (111):

$$2R[K^2(\rho,\omega)] \cdot R[D \cdot K^2(\omega)] \le D \cdot K^2(\omega) - 3. \tag{112}$$

1. We must show that

$$2R[K^2(\rho',\omega')] \cdot R[D \cdot K^2(\omega')] < K^2(\rho',\omega'). \tag{113}$$

From $K(\omega') = K(\rho,\omega) \cdot K(\omega)$ and (112):

$$2R[D \cdot K^2(\omega')] = 2R[K^2(\rho,\omega)] \cdot R[D \cdot K^2(\omega)] \le D \cdot K^2(\omega) - 3. \tag{114}$$

From radical properties

$$R[K^2(\rho',\omega')] \le K(\rho',\omega'). \tag{115}$$

We multiply (114) and (115):

$$2R[K^2(\rho',\omega')] \cdot R[D \cdot K^2(\omega')] \le K(\rho',\omega') \cdot [D \cdot K^2(\omega) - 3]. \tag{116}$$

Now we compare $K(\rho', \omega') = D \cdot K^2(\omega) - 2$ and $D \cdot K^2(\omega) - 3$. Clearly $>$ sign must be used, so from (116) we get necessary (113), which proves Theorem 3.28, case 1.

2. From initial conditions (equations (107) – (112)) we must show that all root pairs $K(a_0, \pi, \underbrace{2a_0, \pi,}_{l\text{-times}} a_0 + \rho, \omega)$/ $K(\pi, \underbrace{2a_0, \pi,}_{l\text{-times}} a_0 + \rho, \omega)$, where $l = 0, 1, 2, ...$, also produce *abc*-triples. In view of already proved Theorem 3.28, case 1 and Theorem 3.27, this is true for all even values $l = 0, 2, 4, ...$

Now $l = 1$ and for the equation

$$K^2(a_0, 2\pi, a_0 + \rho, \omega) - D \cdot K^2(2\pi, a_0 + \rho, \omega) = -4 \tag{117}$$

we must confirm

$$2R[K^2(a_0, 2\pi, a_0 + \rho, \omega)] \cdot R[D \cdot K^2(2\pi, a_0 + \rho, \omega)] < D \cdot K^2(2\pi, a_0 + \rho, \omega). \tag{118}$$

From (107) and (117) we have $D = \dfrac{K^2(\rho, \omega) + 4}{K^2(\omega)} = \dfrac{K^2(a_0, 2\pi, a_0 + \rho, \omega) + 4}{K^2(2\pi, a_0 + \rho, \omega)}$, so

$$\frac{K^2(a_0, 2\pi, a_0 + \rho, \omega) + 4}{K^2(\rho, \omega) + 4} = \frac{K^2(2\pi, a_0 + \rho, \omega)}{K^2(\omega).} \tag{119}$$

Again we get fractions which are natural squares (see Theorems 3.22 and 3.26), then we take radicals and multiply inequalities, and finally:

$$2R[K^2(a_0, 2\pi, a_0 + \rho, \omega)] \cdot R[D \cdot K^2(2\pi, a_0 + \rho, \omega)] \\ \leq \frac{K(2\pi, a_0 + \rho, \omega)}{K(\omega)} \cdot \frac{K(a_0, 2\pi, a_0 + \rho, \omega)}{K(\rho, \omega)} \cdot [D \cdot K^2(\omega) - 3], \tag{120}$$

or, equivalently:

$$2R[K^2(a_0, 2\pi, a_0 + \rho, \omega)] \cdot R[D \cdot K^2(2\pi, a_0 + \rho, \omega)] \\ \leq K(2\pi, a_0 + \rho, \omega) \cdot \frac{K(a_0, 2\pi, a_0 + \rho, \omega)}{K(\omega')} \cdot [K(\rho', \omega') - 1]. \tag{121}$$

We compare $D \cdot K(2\pi, a_0 + \rho, \omega)$ and $\dfrac{K(a_0, 2\pi, a_0 + \rho, \omega)}{K(\omega')} \cdot [K(\rho', \omega') - 1]$. We square both sides of comparison and divide by D :

$$D \cdot K^2(2\pi, a_0 + \rho, \omega) \quad \text{and} \quad K^2(a_0, 2\pi, a_0 + \rho, \omega) \cdot \frac{K^2(\rho', \omega') - 2K(\rho', \omega') + 1}{D \cdot K^2(\omega')},$$

or, equivalently:

$$K^2(a_0, 2\pi, a_0 + \rho, \omega) + 4 \quad \text{and} \quad K^2(a_0, 2\pi, a_0 + \rho, \omega) \cdot \frac{K^2(\rho', \omega') - 2K(\rho', \omega') + 1}{K^2(\rho', \omega') - 4}.$$

As $K(\rho', \omega')$ minimally equals 3, the fraction in the right side is less than 1, which gives $>$ sign in comparison and confirms relation (118).

For $l = 3, 5, 7, ...$ we get the same divisibility with $K(\rho, \omega)$ and $K(\omega)$, therefore we can work analogously. This confirms Theorem 3.28, case 2 for all values $l = 0, 1, 2, ...$

3. From initial conditions (equations (107) – (112)) we must show that all root pairs $K(a_0, \pi, \underbrace{2a_0, \pi,}_{l\text{-times}} a_0 + \rho', \omega')$/ $K(\pi, \underbrace{2a_0, \pi,}_{l\text{-times}} a_0 + \rho', \omega')$, where $l = 0, 1, 2, ...$, also produce *abc*-triples. In view of already proved Theorem 3.28, case 1 and Theorem 3.27, this is true for all odd values $l = 1, 3, 5, ...$

Now $l = 0$ and for the equation

$$K^2(a_0, \pi, a_0 + \rho', \omega') - D \cdot K^2(\pi, a_0 + \rho', \omega') = -4 \tag{122}$$

we must confirm

$$2R[K^2(a_0, \pi, a_0 + \rho', \omega')] \cdot R[D \cdot K^2(\pi, a_0 + \rho', \omega')] < D \cdot K^2(\pi, a_0 + \rho', \omega'). \tag{123}$$

Again we get the divisibility $K(\rho, \omega)|K(a_0, \pi, a_0 + \rho', \omega')$ and $K(\omega)|K(\pi, a_0 + \rho', \omega')$, so the further processing is analogous with formation of natural squares. This confirms relation (123).

For higher even values $l = 2, 4, 6, ...$ we can work analogously, which completes the proof of Theorem 3.28. $\square$

Remark. Under Property A conditions cases of *abc*-triples with $K(\omega) = 1$ are rare, experimental testing reveals this for $D = 125, 22667125, 217238125, 6640131173$, etc.

3.5 Connection with ordinary positive/negative Pell's equations

Under Property A conditions roots $K(\rho, \omega)/K(\omega,)$ $K(\rho', \omega')/K(\omega')$, as well as their higher analogues $K(a_0, \pi, \underbrace{2a_0, \pi,}_{k\text{-times}} a_0 + \rho, \omega)$/ $K(\pi, \underbrace{2a_0, \pi,}_{k\text{-times}} a_0 + \rho, \omega)$ and $K(a_0, \pi, \underbrace{2a_0, \pi,}_{k\text{-times}} a_0 + \rho', \omega')$/ $K(\pi, \underbrace{2a_0, \pi,}_{k\text{-times}} a_0 + \rho', \omega')$ ($k = 0, 1, 2, ...$ for both cases) emerge as divisors of fundamental or higher roots of ordinary positive/negative Pell's equations $x^2 - Dy^2 = \pm 1$, affecting occurence of *abc-triples* here. Corresponding rules are the content of given subsection, beginning with divisibility properties.

Theorem 3.29. *Under Property A conditions every continuant $K(\omega)$ is a divisor of all continuants $K(\pi \underbrace{, 2a_0, \pi}_{k\text{-times}})$, where $k = 0, 1, 2, ...$*

Proof. This follows from equations (7) and (8), as well as from divisibility properties of higher roots for ordinary positive/negative Pell's equations (see the beginning of the previous subsection). $\square$

Theorem 3.30. *Under Property A conditions every continuant $K(\rho, \omega)$ is a divisor of*

***1.** all continuants $K(a_0, \pi \underbrace{, 2a_0, \pi}_{k\text{-times}})$, where $k = 0, 2, 4, ...$, and*

***2.** all continuants $K(\pi \underbrace{, 2a_0, \pi}_{k\text{-times}})$, where $k = 1, 3, 5, ...$*

Proof. Again – this follows from equations (7) and (8), as well as from divisibility properties of higher roots for ordinary positive/negative Pell's equations. □

Theorem 3.31. *Under Property A conditions every continuant $K(\omega')$ is a divisor of all continuants $K(\pi\underbrace{,2a_0,\pi}_{k\text{-times}})$, where $k = 1, 3, 5, ...$*

Proof. This follows from $K(\omega') = K(\rho, \omega) \cdot K(\omega)$ and from two previous Theorems. □

Theorem 3.32. *Under Property A conditions every continuant $K(\rho', \omega')$ is a divisor of*
1. *all continuants $K(a_0, \pi\underbrace{,2a_0,\pi}_{k\text{-times}})$, where $k = 1, 5, 9, ...,$ and*
2. *all continuants $K(\pi\underbrace{,2a_0,\pi}_{k\text{-times}})$, where $k = 3, 7, 11, ...$*

Proof. The proof is wery similar to that for Theorems 3.29 – 3.31. □

Theorem 3.33. *Now π is an even length palindromic unit and the remaining conditions of Property A are taken into account. Then:*
1. *Every continuant $K(\pi,\underbrace{2a_0,\pi,}_{k\text{-times}}a_0+\rho,\omega)$ is a divisor of all continuants $K(\pi\underbrace{,2a_0,\pi}_{l\text{-times}})$. Here $l = 3(k + 1) + n(3k + 4)$; $k = 0, 1, 2, ...$; $n = 0, 1, 2, ...$; and*
2. *Every continuant $K(\pi,\underbrace{2a_0,\pi,}_{k\text{-times}}a_0+\rho',\omega')$ is a divisor of all continuants $K(\pi\underbrace{,2a_0,\pi}_{l\text{-times}})$. Here $l = 3k + 4 + n(3k + 5)$; $k = 0, 1, 2, ...$; $n = 0, 1, 2, ...$*

Proof. **1.** Now $k = 0$ and we must show that $K(\pi, a_0+\rho, \omega)$ is a divisor of all $K(4\pi)$, $K(8\pi), ...$ Divisibility $K(\pi, a_0 + \rho, \omega)|K(4\pi)$ follows directly from Theorem 3.5. As $K(4\pi)$ is a divisor of $K(8\pi)$, $K(12\pi), ...$ (see Table 1 from [1]), then $K(\pi, a_0 + \rho, \omega)$ is also their divisor.

Then $k = 1$ and we must show that $K(\pi, 2a_0, \pi, a_0 + \rho, \omega)$ is a divisor of all $K(7\pi)$, $K(14\pi), ...$ Divisibility $K(\pi, 2a_0, \pi, a_0 + \rho, \omega)|K(7\pi)$ follows directly from Theorem 3.6. As $K(7\pi)$ is a divisor of $K(14\pi)$, $K(21\pi), ...$ (see Table 1 from [1]), then $K(\pi, 2a_0, \pi, a_0 + \rho, \omega)$ is also their divisor.

Further we can work analogously, alternating these two approaches.
2. The only differences from case 1 are usage of Theorems 3.4 and 3.7. This inductively confirms Theorem 3.33. □

Theorem 3.33 suggests that each so defined continuant with the number of palindromes specified in the two left sections of the Table 14 is a divisor of all continuants of the defined type, specified in the right section of the Table 14. Initial extra-lines for $0\pi = K(\omega)$ and $0\pi = K(\omega')$ are added from Theorems 3.29 and 3.31. All Table 14 values are experimental.

Table 14.

Divisor $K(\pi, ..., \pi, a_0 + ...)$	Divisor extension	Dividends, π units $K(\pi, ..., \pi)$					
$0\pi = K(\omega)$		1	2	3	4	5	...
$0\pi = K(\omega')$		2	4	6	8	10	...
1π	ρ, ω	4	8	12	16	20	...
	ρ', ω'	5	10	15	20	25	...
2π	ρ, ω	7	14	21	28	35	...
	ρ', ω'	8	16	24	32	40	...
3π	ρ, ω	10	20	30	40	50	...
	ρ', ω'	11	22	33	44	55	...
4π	ρ, ω	13	26	39	52	65	...
	ρ', ω'	14	28	42	56	70	...
5π	ρ, ω	16	32	48	64	80	...
	ρ', ω'	17	34	51	68	85	...
6π	ρ, ω	19	38	57	76	95	...
	ρ', ω'	20	40	60	80	100	...
...	...	...	...	...	...	...	...

Theorem 3.34. *Again π is an even length palindromic unit and the remaining conditions of Property A are taken into account. Then:*
***1.** Every continuant $K(a_0, \pi, \underbrace{2a_0, \pi,}_{k\text{-times}} a_0 + \rho, \omega)$ is a divisor of all continuants $K(\pi \underbrace{, 2a_0, \pi}_{l\text{-times}})$. Here $l = 6k + 7 + n(6k + 8)$; $k = 0, 1, 2, ...$; $n = 0, 1, 2, ...$; and*
***2.** Every continuant $K(a_0, \pi, \underbrace{2a_0, \pi,}_{k\text{-times}} a_0 + \rho', \omega')$ is a divisor of all continuants $K(\pi \underbrace{, 2a_0, \pi,}_{l\text{-times}})$. Here $l = 6k + 9 + n(6k + 10)$; $k = 0, 1, 2, ...$; $n = 0, 1, 2, ...$*

Proof is analogous to that for Theorem 3.33, so details are left to concerned reader.

Theorem 3.34 suggests that each so defined continuant with the number of palindromes specified in the two left sections of the Table 15 is a divisor of all continuants of the defined type, specified in the right section of the Table 15. Initial extra-lines for $0\pi = K(\rho, \omega)$ and $0\pi = K(\rho', \omega')$ are added from Theorems 3.30 and 3.32.

Table 15.

Divisor $K(a_0, \pi, ..., \pi, a_0 + ...)$	Divisor extension	Dividends, π units $K(\pi, ..., \pi)$					
$0\pi = K(\rho, \omega)$		2	4	6	8	10	...
$0\pi = K(\rho', \omega')$		4	8	12	16	20	...
1π	ρ, ω	8	16	24	32	40	...
	ρ', ω'	10	20	30	40	(50)	...
2π	ρ, ω	14	28	42	56	70	...
	ρ', ω'	16	32	48	64	80	...
3π	ρ, ω	20	40	60	80	100	...
	ρ', ω'	22	44	66	88	(110)	...
4π	ρ, ω	26	52	78	(104)	(130)	...
	ρ', ω'	28	56	84	(112)	(140)	...
5π	ρ, ω	32	64	96	(128)	(160)	...
	ρ', ω'	34	68	102	(136)	(170)	...
6π	ρ, ω	38	76	(114)	(152)	(190)	...
	ρ', ω'	40	80	(120)	(160)	(200)	...
...	...	...	...	...	...	...	...

Demonstrated values without brackets are confirmed experimentally and they are limited by my laptop's performance. Due to regularity proposed by Theorem 3.34, these table values can be easily extrapolated to items in brackets.

Theorem 3.35. *Again π is an even length palindromic unit and the remaining conditions of Property A are taken into account. Then:*
1. *Every continuant $K(a_0, \pi, \underbrace{2a_0, \pi,}_{k\text{-times}} a_0 + \rho, \omega)$ is a divisor of all continuants $K(a_0, \pi \underbrace{, 2a_0, \pi}_{l\text{-times}})$. Here $l = 3(k + 1) + n(6k + 8)$; $k = 0, 1, 2, ...$; $n = 0, 1, 2, ...$; and*
2. *Every continuant $K(a_0, \pi, \underbrace{2a_0, \pi,}_{k\text{-times}} a_0 + \rho', \omega')$ is a divisor of all continuants $K(a_0, \pi \underbrace{, 2a_0, \pi}_{l\text{-times}})$. Here $l = 3k + 4 + n(6k + 10)$; $k = 0, 1, 2, ...$; $n = 0, 1, 2, ...$*

Proof is analogous to that for Theorem 3.33, so details are left to concerned reader.

Theorem 3.35 suggests that each so defined continuant with the number of palindromes specified in the two left sections of the Table 16 is a divisor of all continuants of the defined type, specified in the right section of the Table 16. Initial extra-lines for $0\pi = K(\rho, \omega)$ and $0\pi = K(\rho', \omega')$ are added from Theorems 3.30 and 3.32.

Table 16.

Divisor $K(a_0, \pi, ..., \pi, a_0 + ...)$	Divisor extension	Dividends, π units $K(\pi, ..., \pi)$					
$0\pi = K(\rho, \omega)$		1	3	5	7	9	...
$0\pi = K(\rho', \omega')$		2	6	10	14	18	...
1π	ρ, ω	4	12	20	28	36	...
	ρ', ω'	5	15	25	35	(45)	...
2π	ρ, ω	7	21	35	49	63	...
	ρ', ω'	8	24	40	56	72	...
3π	ρ, ω	10	30	50	70	90	...
	ρ', ω'	11	33	55	77	(99)	...
4π	ρ, ω	13	39	65	(91)	(117)	...
	ρ', ω'	14	42	70	98	(126)	...
5π	ρ, ω	16	48	80	(112)	(144)	...
	ρ', ω'	17	51	(85)	(119)	(153)	...
6π	ρ, ω	19	57	(95)	(133)	(171)	...
	ρ', ω'	20	60	100	(140)	(180)	...
...	...	...	...	...	...	...	...

Items without brackets are experimental, items in brackets – extrapolated.

Currently we can formulate rules of *abc*-triples formation by roots of ordinary Pell's equations $x^2 - Dy^2 = \pm 1,$ directed by occurence of *abc*-triples between fundamental or higher roots of generalized Pell's equations $x^2 - Dy^2 = \pm 4.$

Theorem 3.36. *Under Property A conditions the following phenomenons take place.*
1. *If fundamental roots $K(\rho, \omega)/K(\omega)$ of the generalized Pell's equation $x^2 - D \cdot y^2 = -4$ produce an abc-triple, then abc-triple is produced by fundamental root pair $K(a_0, \pi)/K(\pi)$ of the ordinary negative Pell's equation $x^2 - D \cdot y^2 = -1$.*
2. *If fundamental roots $K(\rho', \omega')/K(\omega')$ of the generalized Pell's equation $x^2 - D \cdot y^2 = 4$ produce an abc-triple, then abc-triple is produced by fundamental root pair $K(a_0, \pi, 2a_0, \pi)/K(\pi, 2a_0, \pi)$ of the ordinary positive Pell's equation $x^2 - D \cdot y^2 = 1$.*
3. *If roots $K(a_0, \pi, \underbrace{2a_0, \pi,}_{k\text{-times}} a_0 + \rho, \omega)$/ $K(\pi, \underbrace{2a_0, \pi,}_{k\text{-times}} a_0 + \rho, \omega)$ $(k = 0, 1, 2, ...)$ of the generalized Pell's equation $x^2 - D \cdot y^2 = \pm 4$ produce an abc-triple, then abc-triples are produced by all root pairs $K(a_0, \pi \underbrace{, 2a_0, \pi}_{l\text{-times}})$/ $K(\pi \underbrace{, 2a_0, \pi}_{l\text{-times}})$ of the positive/negative Pell's equations $x^2 - D \cdot y^2 = \pm 1$ with $l = 3 \cdot (k + 1) + n \cdot (3k + 4); n = 0, 1, 2, ...$*
4. *If roots $K(a_0, \pi, \underbrace{2a_0, \pi,}_{k\text{-times}} a_0 + \rho', \omega')$/ $K(\pi, \underbrace{2a_0, \pi,}_{k\text{-times}} a_0 + \rho', \omega')$ $(k = 0, 1, 2, ...)$ of the generalized Pell's equation $x^2 - D \cdot y^2 = \pm 4$ produce an abc-triple, then abc-triples are produced by all root pairs $K(a_0, \pi \underbrace{, 2a_0, \pi}_{l\text{-times}})$/ $K(\pi \underbrace{, 2a_0, \pi}_{l\text{-times}})$ of the positive/negative Pell's equations $x^2 - D \cdot y^2 = \pm 1$ with $l = 3k + 4 + n \cdot (3k + 5); n = 0, 1, 2, ...$*

Proof. **1.** As initial condition, fundamental roots of an equation

$$x^2 - D \cdot y^2 = -4 \tag{124}$$

produce an *abc*-triple, that is

$$2R[K^2(\rho,\omega)] \cdot R[D \cdot K^2(\omega)] < D \cdot K^2(\omega). \tag{125}$$

We must prove that for fundamental roots $K(a_0,\pi)/K(\pi)$ of an equation

$$x^2 - D \cdot y^2 = -1 \tag{126}$$

accomplishes inequality

$$R[K^2(a_0,\pi)] \cdot R[D \cdot K^2(\pi)] < D \cdot K^2(\pi). \tag{127}$$

From (125), by argumentation, used for shift from (82) to (83), we get

$$2R[K^2(\rho,\omega)] \cdot R[D \cdot K^2(\omega)] \leq K(\omega) \cdot [D \cdot K(\omega) - 1]. \tag{128}$$

Fundamental roots of equations (124) and (126) give $D = \dfrac{K^2(\rho,\omega)+4}{K^2(\omega)} = \dfrac{K^2(a_0,\pi)+1}{K^2(\pi)}$, so

$$\frac{K^2(a_0,\pi)+1}{K^2(\rho,\omega)+4} = \frac{K^2(\pi)}{K^2(\omega)} \tag{129}$$

is a natural square, because $K(\omega)|K(\pi)$. As also $K(\rho,\omega)|K(a_0,\pi)$, we have

$$R[\frac{K^2(a_0,\pi)+1}{K^2(\rho,\omega)+4}] \leq \frac{K(\pi)}{K(\omega)} \tag{130}$$

and

$$R[\frac{K^2(a_0,\pi)}{K^2(\rho,\omega)}] \leq \frac{K(a_0,\pi)}{K(\rho,\omega).} \tag{131}$$

We multiply (128), (130) and (131), and obtain

$$2R[K^2(a_0,\pi)] \cdot R[D \cdot K^2(\pi)] \leq K(\pi) \cdot \frac{K(a_0,\pi)}{K(\rho,\omega)} \cdot [D \cdot K(\omega) - 1]. \tag{132}$$

Now we compare

$$D \cdot K(\pi) \quad \text{and} \quad \frac{K(a_0,\pi)}{K(\rho,\omega)} \cdot \frac{D \cdot K(\omega) - 1}{2};$$

$$[K^2(\rho,\omega)+1] \cdot D \cdot K(\omega) \quad \text{and} \quad [K^2(\rho,\omega)+3] \cdot [\frac{D \cdot K(\omega) - 1}{2}];$$

$$[2K^2(\rho,\omega)+2] \cdot D \cdot K(\omega) \quad \text{and} \quad [K^2(\rho,\omega)+3] \cdot [D \cdot K(\omega) - 1].$$

Clearly $>$ sign can be used, so from (132) we obtain necessary (127), which proves case 1 of Theorem 3.36.

2. Now an *abc*-triple is $K^2(\rho', \omega') - D \cdot K^2(\omega') = 4$, so from

$$2R[K^2(\rho', \omega')] \cdot R[D \cdot K^2(\omega')] \leq K(\rho', \omega') \cdot [K(\rho', \omega') - 1] \tag{133}$$

we must prove that

$$R[K^2(a_0, \pi, 2a_0, \pi)] \cdot R[D \cdot K^2(\pi, 2a_0, \pi)] < K^2(a_0, \pi, 2a_0, \pi). \tag{134}$$

From (8) we have $K(\omega')|K(2\pi)$ and $K(\rho', \omega')|K(a_0, 2\pi)$, so we can proceed analogously to previous case, thus obtaining necessary (134).
3. For $k = 0$, that is, for *abc*-triple $K^2(a_0, \pi, a_0 + \rho, \omega) - D \cdot K^2(\pi, a_0 + \rho, \omega) = 4$ we have

$$2R[K^2(a_0, \pi, a_0 + \rho, \omega)] \cdot R[D \cdot K^2(\pi, a_0 + \rho, \omega)] < K^2(a_0, \pi, a_0 + \rho, \omega), \tag{135}$$

that is

$$2R[K^2(a_0, \pi, a_0+\rho, \omega)] \cdot R[D \cdot K^2(\pi, a_0+\rho, \omega)] \leq K(\pi, a_0+\rho, \omega) \cdot [K(\pi, a_0+\rho, \omega) - 1]. \tag{136}$$

$$l = 3(k + 1) + n \cdot (3k + 4) = 3 + 4n = 3, 7, 11, ...$$

If $n = 0$, we must confirm that equation with $l = 3$ or $K^2(a_0, 4\pi) - D \cdot K^2(4\pi) = 1$ also makes *abc*-triple, that is

$$R[K^2(a_0, 4\pi)] \cdot R[D \cdot K^2(4\pi)] < K^2(a_0, 4\pi). \tag{137}$$

In view of Theorem 3.5, we get $K(\pi, a_0 + \rho, \omega)|K(4\pi)$ and $K(a_0, \pi, a_0 + \rho, \omega)|K(a_0, 4\pi)$, so the further processing is analogous with formation of two natural squares. This confirms relation (137). From Table 9 of [1] this means formation of *abc*-triples for remaining values $l = 7, 11, 15, ...$, thus confirming Theorem 3.36, case 3 for $k = 0$.

Proof for $k = 2$ is analogous, because we get the divisibility $K(3\pi, a_0 + \rho, \omega)|K(10\pi)$ and $K(a_0, 3\pi, a_0 + \rho, \omega)|K(a_0, 10\pi)$, similarly for all other even k values.

For odd k values we can proceed analogously, using Theorem 3.6, thus confirming Theorem 3.36, case 3.
4. Proof of this case is analogous with case 3, only Theorem 3.4 is necessary for even k values and Theorem 3.7 for odd k values. □

3.6 Finally – primary/secondary tables

Theorems 3.20, 3.27, 3.28 and 3.36 allow construction of corresponding primary/secondary tables. Thus each primary *abc*-triple with extension ρ', ω' and the number of palindromes specified in the left column of Table 17, induces three infinite sequences of secondary *abc*-triples, specified in the right columns of Table 17. These sequences differ for ordinary Pell's equations $x^2 - Dy^2 = \pm 1$, whose roots have continuant expressions ending in π, and generalized Pell's equations $x^2 - Dy^2 = \pm 4$, whose roots have continuant expressions ending in ρ, ω or ρ', ω'.

Table 17. Primary *abc*-triple has extensions ρ', ω'.

Primary	Secondary, extensions	Secondary, π units					
0π	ρ, ω	1	3	5	7	9	...
	π	2	4	6	8	10	...
	ρ', ω'	2	4	6	8	10	...
1π	ρ, ω	3	8	13	18	23	...
	π	5	10	15	20	25	...
	ρ', ω'	6	11	16	21	26	...
2π	ρ, ω	5	13	21	29	37	...
	π	8	16	24	32	40	...
	ρ', ω'	10	18	26	34	42	...
3π	ρ, ω	7	18	29	40	51	...
	π	11	22	33	44	55	...
	ρ', ω'	14	25	36	47	58	...
4π	ρ, ω	9	23	37	51	65	...
	π	14	28	42	56	70	...
	ρ', ω'	18	32	46	60	74	...
5π	ρ, ω	11	28	45	62	79	...
	π	17	34	51	68	85	...
	ρ', ω'	22	39	56	73	90	...
6π	ρ, ω	13	33	53	73	93	...
	π	20	40	60	80	100	...
	ρ', ω'	26	46	66	86	106	...
...	...	...	...	...	...	...	...

Similar table for primary *abc*-triples with extensions ρ, ω is Table 18.

Table 18. Primary *abc*-triple has extensions ρ, ω.

Primary	Secondary, extensions	Secondary, π units					
0π	ρ, ω	1	2	3	4	5	...
	π	1	2	3	4	5	...
	ρ', ω'	0	1	2	3	4	...
1π	ρ, ω	5	9	13	17	21	...
	π	4	8	12	16	20	...
	ρ', ω'	2	6	10	14	18	...
2π	ρ, ω	9	16	23	30	37	...
	π	7	14	21	28	35	...
	ρ', ω'	4	11	18	25	32	...
3π	ρ, ω	13	23	33	43	53	...
	π	10	20	30	40	50	...
	ρ', ω'	6	16	26	36	46	...
4π	ρ, ω	17	30	43	56	69	...
	π	13	26	39	52	65	...
	ρ', ω'	8	21	34	47	60	...
5π	ρ, ω	21	37	53	69	85	...
	π	16	32	48	64	80	...
	ρ', ω'	10	26	42	58	74	...
6π	ρ, ω	25	44	63	82	101	...
	π	19	38	57	76	95	...
	ρ', ω'	12	31	50	69	88	...
...	...	...	...	...	...	...	...

The following experimental Table 19 shows emerging of *abc*-triples from higher roots of generalized Pell's equations $x^2 - 5y^2 = \pm 4$ as well as from higher roots of ordinary Pell's equations $x^2 - 5y^2 = \pm 1$; sequences are limited by my laptop's performance. T means "True" – we get an *abc*-triple; F means "False".

Table 19. Equations $x^2 - 5y^2 = \pm N$ and abc-triples.

		Number of π units															
Extension	N	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
ρ', ω'	$+4$		F		F		T		F		F		F		F		T
ρ, ω	$+4$	F		T		F		F		T		F		T		F	
ρ, ω	-4		F		F		F		T		F		F		F		F
ρ', ω'	-4	T		F		F		F		F		T		F		F	
	$+1$		T		T		T		T		T		T		T		T
	-1	F		F		T		F		F		F		F		T	

		17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
ρ', ω'	$+4$		T		F		F		F		T		F		F		F
ρ, ω	$+4$	F		F		F		T		F		F		F		F	
ρ, ω	-4		T		F		F		F		F		T		T		F
ρ', ω'	-4	F		F		T		F		F		F		F		T	
	$+1$		T		T		T		T		T		T		T		T
	-1	F		F		F		F		T		F		F		F	

		33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48
ρ', ω'	$+4$		F		T		F		F		F		F		T		F
ρ, ω	$+4$	T		F		T		F		F		T		F		F	
ρ, ω	-4		F		F		T		F		F		F		F		T
ρ', ω'	-4	F		F		F		F		T		F		F		F	
	$+1$		T		T		T		T		T		T		T		T
	-1	F		T		F		F		F		F		T		F	

		49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64
ρ', ω'	$+4$		F		F		F		T		F		T		F		F
ρ, ω	$+4$	F		F		T		F		F		F		F		T	
ρ, ω	-4		F		F		F		F		T		F				
ρ', ω'	-4	F		T		F		F		F		F		T			
	$+1$		T		T		T		T		T		T		T		T
	-1	F		T		F		T		T		F		F		F	

		65	66	67	68	69	70	71	72	73	74	75	76	77	78	79	80
ρ', ω'	$+4$		T		F		F		F		T		T		F		
ρ, ω	$+4$	T		F		F		F		T							
ρ, ω	-4																
ρ', ω'	-4																
	$+1$		T		T		T		T		T		T		T		T
	-1	T		F		F		F		F		T		F		F	

In Table 19 we see the first appearance of primary *abc*-triple at 1π and extension ρ', ω'. That means secondary *abc*-triples for ordinary negative Pell's equations at 5π, 15π, 25π, ... and for ordinary positive Pell's equations at 10π, 20π, 30π, ... In the case of ordinary positive Pell's equation these secondary *abc*-triples are overlapping with ones, generated by primary *abc*-triple for fundamental roots of it (see Table 9 from [1]). Next portions of secondary *abc*-triples emerge for equations with extensions ρ, ω at 3, 8, 13, 18, ... π units, but for equations with extensions ρ', ω' they are at 6, 11, 16, 21, ... π units.

The next primary has 9 π units and ρ, ω extension, it will generate secondary *abc*-triples for ordinary positive Pell's equations at 28, 56, 84, ... π units, all of which are overlapping with already induced ones. For equations with extensions ρ, ω this primary gives (in the limits of Table 19) new secondary triples at 37π and 65π, while for equations with extensions ρ', ω' we have secondary triples at 18π (new), 46π (overlapping) and 74π (new).

There is one more primary *abc*-triple at 30π with extension ρ, ω, which induces secondary triple at 60π and ρ', ω' extension, but all other secondary triples, induced by this primary, are out of the Table 19 limits. Remains two primary *abc*-triples with 51 and 57 π units, which are ordinary negative Pell's equations.

4 Some applications

In the following subsections we will touch on some possible starting points for new investigations and the reader is welcomed to extend and generalize them.

4.1 More or less composite?

Here we look at the divisibility of continuants $K(\pi, \underbrace{2a_0, \pi,}_{k\text{-times}} a_0 + \rho, \omega)$ $(k = 0, 1, 2, ..., 20)$, of course, under the conditions of Property A and in the framework of previous exposition. Such continuants are mentioned as dividends in Theorems 3.8, 3.11, 3.13, 3.15, 3.21–3.24, but with different frequency. We come across on continuants with 13 and 17 π units very often, but continuants with 10, 14 and 20 π units as dividends are rare. The result is more or less composite number, which is illustrated by Table 20.

Table 20.

	Number of prime divisors				
D	10π	14π	20π	13π	17π
5	2	1	2	6	4
13	4	3	1	9	8
29	3	3	No data	8	7
53	3	2	No data	12	8
125	5	3	No data	13	12

Similar differences are recorded also for other higher roots of equations $x^2 - D \cdot y^2 = \pm 4$.

4.2 Lucas sequences

A well-known equation

$$L_n^2 - 5F_n^2 = 4 \cdot (-1)^n, \tag{138}$$

connecting Fibonacci numbers F_n and Lucas numbers L_n, is closely related to our exposition. We need only odd terms, therefore from Theorem 3.20 we compile

Theorem 4.1. *Now L_i are Lucas numbers ($L_0 = 2$, $L_1 = 1$, $L_2 = 3$, ..., $L_n = L_{n-1} + L_{n-2}$ for all $n > 1$) and F_i are Fibonacci numbers ($F_0 = 0$, $F_1 = 1$, $F_2 = 1$, ..., $F_n = F_{n-1} + F_{n-2}$ for all $n > 1$). If the terms of the equation $L_i^2 - 5F_i^2 = 4 \cdot (-1)^i$, where L_i and F_i are odd numbers, produce an abc-triple, then abc-triples are produced by the terms of all such equations $L_k^2 - 5F_k^2 = 4 \cdot (-1)^k$, where*
***1.** $k = 2i + 3i \cdot n$, $(n = 0, 1, 2, ...)$, and*
***2.** $k = i + 3i \cdot n$, $(n = 1, 2, 3, ...)$.*

The reader is invited to test these relations and compare them with Table 19 entries.

Equation (138) is specific case of a more general relation between two Lucas sequences of the first kind $U_n(P, Q)$ and of the second kind $V_n(P, Q)$:

$$V_n^2 - D \cdot U_n^2 = 4 \cdot Q^n. \tag{139}$$

Sequences $U_n(P, Q)$ and $V_n(P, Q)$ are defined recurrently (P and Q are fixed integers).

$$U_0(P, Q) = 0$$
$$U_1(P, Q) = 1$$
$$U_n(P, Q) = P \cdot U_{n-1}(P, Q) - Q \cdot U_{n-2}(P, Q) \text{ for all } n > 1.$$

Case $U_n(1, -1)$ are Fibonacci numbers 0, 1, 1, 2, 3, 5, 8, ...

$$V_0(P, Q) = 2$$
$$V_1(P, Q) = P$$
$$V_n(P, Q) = P \cdot V_{n-1}(P, Q) - Q \cdot V_{n-2}(P, Q) \text{ for all } n > 1.$$

Case $V_n(1, -1)$ are Lucas numbers 2, 1, 3, 4, 7, 11, 18, ...

In equation (139) discriminant $D = P^2 - 4Q$, so only $Q = \pm 1$ gives necessary generalized Pell's equations $x^2 - D \cdot y^2 = \pm 4$. As we need odd D values, parameter P also is odd and we get suitable discriminant values for following P/Q pairs (Table 21).

Table 21.

P	Q	D	P	Q	D	P	Q	D
1	-1	5	9	-1	85	19	-1	365
3	-1	13	11	-1	125	21	-1	445
3	+1	5	13	-1	173	23	-1	533
5	-1	29	15	-1	229	25	-1	629
7	-1	53	17	-1	293	...	...	...

Apart from combination $P = 3, Q = +1$, giving $D = 5$, all other suitable P/Q pairs have $Q = -1$ combined with $P = 1, 3, 5, ...$ Acquired discriminant sequence is $D = 5$, 13, 29, 53, 85, 125, 173, ..., see [7]. These D values are subset of our Property A discriminant sequence, see subsection 2.2. As the result, obtained primary/secondary tables (Tables 17 and 18) cover all these P/Q combinations from Table 21 and theorems, similar to Theorem 4.1, can be derived for them.

References

[1] Kuzmanis J., *On the origin of abc-triples. 2023. hal-04044029*. `https://hal.science/hal-04044029`, 24 Mar. 2023, 28 pages.

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[6] Nagell T., *Introduction to Number Theory*. 2nd ed., reprint, AMS Chelsea Publishing, 2010.

[7] `https://oeis.org/A078370`.

IMPORTANT REMARK

Reference [1] is now available at https://vixra.org/abs/2411.0052. It was available in HAL/science archive under HAL Id: hal-04044029 (submitted 24. Mar. 2023), but was kicked out due to non-existence of academic affiliation for the author. There were lots of downloads and readings, so do not confuse at copies with earlier priority date.