SYMMETRY IN FORMAL CALCULATION

PENG JI[1]

CONTENTS

ABSTRACT

Formal Calculation uses an auxiliary form to calculate various nested sums and provides results in three forms. It is also a powerful tool for analysis. This article studies the symmetry of the coefficients in Formal Calculation. Three types of extended numbers were introduced, and many formulas for symmetric functions were obtained.

* *Department of Electronic Information, Nanjing University, Nanjing, China. mcfroo@sina.com*

1 INTRODUCTION

The notion of formal computation was introduced in [1].

Definition 1. The definition of ∇^p is recursive. $p \in \mathbb{Z}$,
$\nabla^0 f(n) = f(n),\ \sum_{n=0}^{N-1} \nabla^1 f(n+1) = f(N),\ \sum_{n=0}^{N-1} f(n+1) = \nabla^{-1} f(N), \nabla^1 = \nabla.$

Definition 2. The definition of $SUM(N) = SUM(N, PS, PT)$ is recursive. $K_i, D_i \in \mathbb{C}; T_i \in \mathbb{N}$.
$SUM(N, [K_1 : D_1], [T_1 = 1]) = \sum_{n=0}^{N-1} (K_1 + nD_1).$

$SUM(N, [K_1 : D_1, K_2 : D_2], [T_1, T_2 = T_1 + 2 - p]) = \sum_{n=0}^{N-1} (K_2 + nD_2) \nabla^p SUM(n + 1, [K_1 : D_1], [T_1]).$

$[K_1 : D, K_2 : D...K_M : D] = [K_1, K_2...K_M] : D,\ [K_1, K_2...K_M] : 1 = [K_1, K_2...K_M]$
Use $\mathbb{K}$ to represent the set $\{K_1...K_M\}$, $\mathbb{T}$ to represent the set $\{T_1...T_M\}$.
Use the auxiliary form: $(K_1 + T_1)(K_2 + T_2)...(K_M + T_M) = \sum \prod_{i=1}^{M} X_i, X_i = T_i\ or\ K_i$.

Definition 3. $X(T)$=Number of $\{X_1, X_2...X_M\} \in \mathbb{T}$.

Definition 4. X_{T-1}=Number of $\{X_1, X_2...X_{i-1}\} \in \mathbb{T}$, X_{K-1}=Number of $\{X_1, X_2...X_{i-1}\} \in \mathbb{K}$.

Obviously:$X_{T-1} + X_{K-1} = i - 1$.
Use the auxiliary form and each X_i cannot be exchanged, [1] draws conclusions:
$SUM(N, PS, PT) =$
$Form_1 \to \sum_{g=0}^{M} H_1(g) \binom{N+T_M-M}{N-1-g} = \sum_{g=0}^{M} H_1(g) \binom{N+T_M-M}{T_M-M+1+g},\ B_i = \begin{cases} (T_i - X_{K-1})D_i,\ X_i = T_i \\ K_i + X_{T-1}D_i,\ X_i = K_i \end{cases},$
$Form_2 \to \sum_{g=0}^{M} H_2(g) \binom{N+T_M-M+g}{N-1} = \sum_{g=0}^{M} H_2(g) \binom{N+T_M-M+g}{T_M-M+1+g},\ B_i = \begin{cases} (T_i - X_{K-1})D_i,\ X_i = T_i \\ K_i + (X_{K-1} - T_i)D_i,\ X_i = K_i \end{cases},$
$Form_3 \to \sum_{g=0}^{M} H_3(g) \binom{N+T_M-g}{N-1-g} = \sum_{g=0}^{M} H_3(g) \binom{N+T_M-g}{T_M+1},\ B_i = \begin{cases} -K_i + (T_i - X_{T-1})D_i,\ X_i = T_i \\ K_i + X_{T-1}D_i,\ X_i = K_i \end{cases}.$
$H_i(g) = H_i(g, PS, PT) = H_i(g, M)$, is defined as $\sum_{X(T)=g} \prod_{i=1}^{M} B_i$.

For example:
$SUM(N, PS, [1, 2, 3...M]) = \sum_{n=0}^{N-1} \prod_{i=1}^{M} (K_i + nD_i).$
$SUM(N, PS, [1, 3...2M - 1]) = \sum_{n_M=0}^{N-1} (K_M + n_M D_M)... \sum_{n_2=0}^{n_3} (K_2 + n_2 D_2) \sum_{n_1=0}^{n_2} (K_1 + n_1 D_1).$
$SUM(N, PS, [1, 2, 4]) = \sum_{n_3=0}^{N-1} (K_3 + n_3 D_3) \sum_{n=0}^{n_3} (K_1 + nD_1)(K_2 + nD_2).$
$SUM(N, PS, [1, 3, 4]) = \sum_{n_3=0}^{N-1} (K_3 + n_3 D_3)(K_2 + n_3 D_2) \sum_{n=0}^{n_3} (K_1 + nD_1).$

$PS = [K_1 : D_1, K_2 : D_2...K_M : D_M], PT = [T_1, T_2...T_M]$. There are recursive relationships:

- $H_1(g, M) = (A_M + gD_M)H_1(g, M-1) + (B_M + (g-1)D_M)H_1(g-1, M-1),$
  $A_M = K_M, B_M = D_M(T_M - ((i-1) - (g-1))) - (g-1)D_M = T_M D_M - (i-1)D_M.$

- $H_2(g, M) = (A_M - gD_M)H_2(g, M-1) + (B_M + (g-1)D_M)H_2(g-1, M-1),$
  $A_M = K_M + (i - 1 - g - T_M)D_M + gD_M = K_M + (i - 1 - T_M)D_M, B_M = T_M D_M - (i-1)D_M.$

- $H_3(g, M) = (A_M + gD_M)H_3(g, M-1) + (B_M - (g-1)D_M)H_3(g-1, M-1),$
  $A_M = K_M, B_M = -K_M + (T_M - (g-1))D_M + (g-1)D_M = -K_M + T_M D_M.$

Consider the general two-dimensional second-order linear recursive equations:
$R(M, g) = (A_M + gD_M)R(M-1, g) + (B_M + (g-1)E_M)R(M-1, g-1).$
It can also be calculated in a similar way to $H(g)$, which is easier to understand. $H(g)$ itself requires $|D_i| = |E_i|$ and can't change the sign, so that three forms exist. Many conclusions have been drawn [1]:

1. $H_1(g) = \sum_{k=g}^{M} H_2(k) \binom{k}{g} = \sum_{k=0}^{g} H_3(k) \binom{M-k}{M-g}.$
2. $H_2(g) = \sum_{k=g}^{M} (-1)^{k+g} H_1(k) \binom{k}{g}.\ H_3(g) = \sum_{k=0}^{g} (-1)^{k+g} H_1(k) \binom{M-k}{M-g}.$
3. $\sum_{g=0}^{M} H_1(g) q^g (1-q)^{M-g} = \sum_{g=0}^{M} H_2(g)(1-q)^{M-g} = \sum_{k=0}^{M} H_3(g) q^g$

4. $\sum_{g=0}^{M} H_1(g) = \sum_{g=0}^{M} H_2(g)2^g = \sum_{g=0}^{M} H_3(g)2^{M-g}$
5. $\sum_{g=0}^{M} H_1(g)\binom{X}{Y-g} = \sum_{g=0}^{M} H_2(g)\binom{X+g}{Y} = \sum_{g=0}^{M} H_3(g)\binom{X+M-g}{Y-g}, Y \in \mathbb{N}, X \in \mathbb{C}$.
6. $SUM(N,[L_1,L_2...L_q,PS],[L_1,L_2...L_q,PT]) = \prod_{i=1}^{q} L_i \times SUM(N,PS,PT)$.

Recurrence relations $\rightarrow$ 1), inversion $\rightarrow$ 2). 1) $\rightarrow$ 3)4)5). 5) is the basis of $SUM(N)$.
6) show that T_1 can be great than 1. Regardless of the practical implications, we can make the definition domain of PT extend to $\mathbb{C}$.
When $D_i \neq 0, K_i + D_i n = D_i(\frac{K_i}{D_i} + 1)$, so only the case $D_i = 1$ needs to be dealt with.
In this paper, if not specified, the default is $D_i = 1$. $PS = [K_1, K_2...K_M], PT = [T_1, T_2...T_M]$.

Definition 5. $F_g^{\mathbb{K}} = \sum_{1 \leqslant \lambda_1 < \lambda_2 < ... < \lambda_g \leqslant M} K_{\lambda_1} K_{\lambda_2} ... K_{\lambda_g}, F_0^{\mathbb{K}} = 1, F_g^N = F_g^{\{1,2...N\}}$.

Definition 6. $E_g^{\mathbb{K}} = \sum_{1 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant ... \leqslant \lambda_g \leqslant M} K_{\lambda_1} K_{\lambda_2} ... K_{\lambda_g}, E_0^{\mathbb{K}} = 1, E_g^N = E_g^{\{1,2...N\}}$.

$F_M^{N+M-1} = S_1(N+M,N)$, S_1 is unsigned stirling number of the first kind.
$E_M^N = S_2(N+M,N)$, S_2 is stirling number of the second kind.

2 PROPERTIES OF H(G)

In the calculation of $H(g)$, $\prod X_i = (\prod X_i, X_i \in \mathbb{T})(\prod X_i, X_i \in \mathbb{K})$.

Definition 7. $H(g,T) = H(g,T,PS,PT) = \prod_{X_i \in \mathbb{T}} B_i$, $H(g, \sum T) = \sum \prod_{X_i \in \mathbb{T}} B_i$.
Also define $H(g,K), H(g, \sum K)$.

Theorem 2.1.

1. $H_1(g, \sum K) = H_3(g, \sum K) = F_{M-g}^{\mathbb{K}} E_0^g + F_{M-g-1}^{\mathbb{K}} E_1^g + ... + F_0^{\mathbb{K}} E_{M-g}^g$.
2. $H_1(g, \sum T) = H_2(g, \sum T) = F_g^{\mathbb{T}} E_0^{M-g} - F_{g-1}^{\mathbb{T}} E_1^{M-g} + ... + (-1)^g F_0^{\mathbb{T}} E_g^{M-g}$.
3. $H_2(g, \sum K) = (-1)^{M-g}(F_{M-g}^{\mathbb{S}} E_0^g + ... + F_0^{\mathbb{S}} E_{M-g}^g), \mathbb{S} = \{T_i - K_i - i + 1\}$.
4. $H_3(g, \sum T) = (-1)^g F_g^{\mathbb{S}} E_0^{M-g} + (-1)^{g-1} F_{g-1}^{\mathbb{S}} E_1^{M-g} ... + F_0^{\mathbb{S}} E_g^{M-g}, \mathbb{S} = \{-(T_i - K_i - i + 1)\}$.
5. $H_1(g, \sum K) = \sum \prod_{i=1}^{M-g} (K_{i+\lambda_1} + \lambda_i), 0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant ... \leqslant \lambda_{M-g} \leqslant g$.
6. $H_1(g, \sum T) = \sum \prod_{i=1}^{g} (T_{i+\lambda_1} - \lambda_i), 0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant ... \leqslant \lambda_g \leqslant M-g$.

Proof.
$PS1 = [PS, K_{M+1}], PT1 = [PT, T_{M+1}]$. Using induction to prove 2).
$H_1(g, \sum T, PS1, PT1) = H_1(g, \sum T) + H_1(g-1, \sum T)(T_{M+1} - (M-g+1))$.
$F_{g-x}^{\{PT1\}}$ in $H_1(g, \sum T, PS1, PT1)$ has three sources.
$= (-1)^x F_{g-x}^{\mathbb{T}} E_x^{M-g} + (-1)^x F_{(g-1)-x}^{\mathbb{T}} E_x^{M-(g-1)} T_{M+1}$
$+ (-1)^{x-1} F_{(g-1)-(x-1)}^{\mathbb{T}} E_{x-1}^{M-(g-1)} (-(M-g+1))$
$= (-1)^x F_{g-x}^{\mathbb{T}} (E_x^{M-g} + E_{x-1}^{M+1-g}(M+1-g)) + (-1)^x F_{g-x-1}^{\mathbb{T}} E_x^{M+1-g} T_{M+1}$ (*)
$= (-1)^x F_{g-x}^{\mathbb{T}} E_x^{M+1-g} + (-1)^x F_{g-x-1}^{\mathbb{T}} E_x^{M+1-g} T_{M+1} = (-1)^x E_x^{M+1-g} F_{g-x}^{\{PT1\}}$.
$E_x^{M-g} + E_{x-1}^{M+1-g}(M+1-g)$
$= S_2(M-g+x, M-g) + (M+1-g) S_2(M-g+x, M+1-g)$
$= S_2(M-g+x+1, M+1-g) = E_x^{M+1-g} \rightarrow$ (*).
5) and 6) are definitions.

□

Theorem 2.2. $PS = [K_i : D_i], D_i \neq 0$,
$H_1(g) = (-1)^{M-g} H_2(g, [-K_i + T_i - (i-1) : D_i], PT) = (-1)^g H_3(g, PS, [\frac{K_i}{D_i} - T_i + i - 1])$.
$H_2(g) = (-1)^{M-g} H_1(g, [-K_i + T_i - (i-1) : D_i], PT), H_3(g) = (-1)^g H_1(g, PS, [\frac{K_i}{D_i} - T_i + i - 1])$.
$H_2(g) = (-1)^M H_3(g, [-T_i : D_i], [\frac{K_i}{D_i} - T_i + i - 1])$.
$H_3(g) = (-1)^M H_2(g, [-K_i + T_i - (i-1) : D_i], [-\frac{K_i}{D_i}])$.

3 SYMMETRIC EXPRESSIONS IN H(G)

$PT = [T, T+1...T+M-1]$. It can be inferred from the definition of $SUM(N)$ that K_i can exchange orders. $H_1(g) = [T+g-1]_g H_1(g, \sum K)$. It is clearly a symmetric function of $\mathbb{K}$, we also reached the same conclusion. $H_2(g), H_3(g)$ are also symmetric functions, and there's no K_i^2 factors.

Definition 8. $F_T(N+M, N) = F_M^{\{T,T+1...T+N+M-1\}}, E_T(N+M, N) = E_M^{\{T,T+1...T+N-1\}}$.

Obviously:

- $F_T(0,0) = E_T(0,0) = 1, F_T(1,0) = E_T(1,0) = 0, F_T(1,1) = E_T(1,1) = 1$.
- $F_0(N+M, N) = S_1(N+M, N), E_1(N+M, N) = S_2(N+M, N)$.
- $F_T(M+1, g+1) = F_{M-g}^{\{T,T+1...T+M\}} = (T+M)F_T(M, g+1) + F_T(M, g)$.
- $E_T(M+1, g+1) = E_{M-g}^{\{T,T+1...T+g\}} = (T+g)E_T(M, g+1) + E_T(M, g)$.
- $F_T(N+M, N) = SUM(N, [T, T+1...T+M-1], [1, 3...2M-1])$.
- $E_T(N+M, N) = SUM(N, [T, T...T], [1, 3...2M-1])$.

Theorem 3.1. $PT = [T...T+M-1]$, $H_2(g, \sum K) = \sum_{j=0}^{M-g} (-1)^{M-g-j} F_j^{\mathbb{K}} E_{M-g-j}^{\{T,T+1...T+g\}}$.

Proof.
$H_2(0,1) = K_1 - T, H_2(1,1) = T$. It's holds when $M = 1$.
$H_2(g, M) = (K_M - T - g)H_2(g, M-1) + (T+g-1)H_2(g-1, M-1)$.
$H_2(g)$ is a symmetric function $= \sum_{j=0}^{M} A_M^g(j) F_j^{\mathbb{K}}$

$A_1^0(0) = -T, A_1^1(0) = T$.
$A_M^g(0) = -(T+g)A_{M-1}^g(0) + (T+g-1)A_{M-1}^{g-1}(0) \rightarrow$
$A_M^g(0) = (-1)^{M-g}[T+g-1]_g E_T(M+1, g+1) = (-1)^{M-g}[T+g-1]_g E_{M-g}^{\{T,T+1...T+g\}}$.

The rest only need to consider terms that multiply with K_M.
$A_M^g(j) = A_{M-1}^g(j-1) = A_{M-j}^g(0) = (-1)^{M-g-j}[T+g-1]_g E_T(M+1-j, g+1)$.
$H_2(g) = [T+g-1]_g \sum_{j=0}^{M-g} (-1)^{M-g-j} F_j^{\mathbb{K}} E_{M-g-j}^{\{T,T+1...T+g\}}$.
$H_2(g, T) = [T+g-1]_g \rightarrow$ the conclusion. □

Definition 9. $\left\langle {n \atop j} \right\rangle_T = (T-1+n-j)\left\langle {n-1 \atop j-1} \right\rangle_T + (j+1)\left\langle {n-1 \atop j} \right\rangle_T$, $\langle {1 \atop 0} \rangle_T = T$, $\langle {1 \atop 1} \rangle_T = 0$, $\left\langle {n \atop j>n, j<0} \right\rangle_T = 0$.

Obviously: $\langle {n \atop 0} \rangle_1 = T$, $\langle {n \atop n} \rangle_T = 0$. $n > 0$, $\left\langle {n \atop j} \right\rangle_1 = \left\langle {n \atop j} \right\rangle$ is Eulerian number.

Definition 10. $\left\langle {M \atop g} \right\rangle_T^j = \sum_{i=0}^{j} (-1)^i \left\langle {M-j \atop g-1-i} \right\rangle_T \binom{j}{i}, 0 \leqslant j < g, 0 < g < M$.

Theorem 3.2. $PT = [T, T+1...T+M-1]$, $H_3(0) = \prod_{i=1}^{M} K_i, H_3(M) = \prod_{i=1}^{M} (T - K_i)$,
$H_3(0 < g < M) = \sum_{j=0}^{M-1} \left\langle {M \atop g} \right\rangle_T^j F_j^{\mathbb{K}} + (-1)^g \binom{M}{g} F_M^{\mathbb{K}}$.

Proof.
The coefficient before $F_M^{\mathbb{K}}$ is obvious, so $H_3(g)$ can be written in that form.
$H_3(0 < g < M) = \sum_{j=0}^{M-1} A_M^g(j) F_j^{\mathbb{K}} + (-1)^g \binom{M}{g} F_M^{\mathbb{K}}$.
$H_3(0,1) = K_1, H_3(1,1) = T - K_1 \rightarrow A_1^0(0) = 0, A_1^1(0) = T$.
$H_3(g, M) = (K_M + g)H_3(g, M-1) + (T+M-g-K_M)H_3(g-1, M-1) \rightarrow$
$A_M^g(0) = gA_{M-1}^g(0) + (T+M-g)A_{M-1}^{g-1}(0) \rightarrow A_M^g(0) = \left\langle {M \atop g-1} \right\rangle_T$.
$A_M^g(j) = A_{M-1}^g(j-1) - A_{M-1}^{g-1}(j-1) \rightarrow A_M^g(j) = \langle {M \atop g} \rangle_T^j$.
□

Similarly, $PS = [K, K-1...K-M+1]$, then $H(g)$ are symmetric functions of $\mathbb{T}$.

Theorem 3.3. $PS = [K...K-M+1]$, $H_3(g, \sum T) = \sum_{j=0}^{g} (-1)^{g-j} F_j^{\mathbb{T}} E_{g-j}^{\{K,K-1...K-M+g\}}$.

Theorem 3.4. $PS = [K, K-1...K-M+1]$, $H_2(M) = \prod_{i=1}^{M} T_i$, $H_2(0) = \prod_{i=1}^{M}(K-T_i)$,
$H_2(0<g<M) = \sum_{j=0}^{M-1} - \left\langle {M \atop M-g} \right\rangle_{-K}^{j} F_j^{\mathbb{T}} + (-1)^{M-g}\binom{M}{g} F_M^{\mathbb{T}}$.

Proof.
$H_2(0<g<M) = \sum_{j=0}^{M-1} A_M^g(j) F_j^{\mathbb{T}} + (-1)^{M-g}\binom{M}{g} F_M^{\mathbb{T}}$.
$H_2(0,1) = K - T_1, H_2(1,1) = T_1 \rightarrow A_1^0(0) = K, A_1^1(0) = 0$.
$H_2(g,M) = (K-g-T_M)H_2(g,M-1) + (T_M - M + g)H_2(g-1,M-1) \rightarrow$
$A_M^g(0) = (K-g)A_{M-1}^g(0) - (M-g)A_{M-1}^{g-1}(0) \rightarrow A_M^g(0) = -\left\langle {M \atop M-g-1} \right\rangle_{-K}$.
$A_M^g(j) = -A_{M-1}^g(j-1) + A_{M-1}^{g-1}(j-1) \rightarrow A_M^g(j) = -\left\langle {M \atop M-g} \right\rangle_{-K}^{j}$.

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4 PROPERTIES OF EXTENDED NUMBERS

Theorem 4.1.
$(x+T)(x+T+1)...(x+T+M-1) = \sum_{k=0}^{M} F_T(M,k)x^k$.
$(x+T)^M = \sum_{k=0}^{M} E_T(M+1,k+1)[x]_k$.

Proof.
$\nabla SUM(N,[T,T...T],[1,2...M]) = (T+n)^M = \sum_{g=0}^{M} H_1(g)\binom{n}{g}$
$= \sum_{g=0}^{M} g! E_{M-g}^{\{T,T+1...T+g\}}\binom{n}{g} = \sum_{g=0}^{M} g! E_T(M+1,g+1)\binom{n}{g} \rightarrow 2)$.

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$x^M = \sum_{k=0}^{M} S_2(M,k)x^k$ because $E_0(M+1,k+1) = S_2(M,k)$.

Theorem 4.2. $E_T(M+1,g+1)$
$= \frac{1}{g!}\sum_{k=0}^{g}(-1)^{g+k}\binom{g}{k}(T+k)^M = \sum_{k=0}^{M-g} T^k \binom{M}{k} E_{M-k}^g$.

Proof.
$H_1(g,[T,T...T],[1,2...M]) = g!E_T(M+1,g+1)$.
Based on Crammer's law, $H_1(g) = \sum_{k=1}^{g+1}(-1)^{g+1+k}\binom{T_M-M+g}{T_M-M+k-1}\nabla SUM(k)$. [1]
$E_T(M+1,g+1) = \frac{1}{g!}\sum_{k=1}^{g+1}(-1)^{g+1+k}\binom{g}{k-1}(T+k-1)^M \rightarrow 1)$, 2.1 $\rightarrow 2)$.

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Theorem 4.3. $\left\langle {n \atop j} \right\rangle_{-1} = (-1)^{j-1}\binom{n-1}{j}$. $\left\langle {n \atop j} \right\rangle_{-2} = (-2)^{j-1}(\binom{n-2}{j} + 2^{n-1}\binom{n-2}{j})$.

Proof.
1) is to prove: $\binom{n-1}{j} = -(n-j-2)\binom{n-2}{j-1} + (j+1)\binom{n-2}{j}$.
$Right = -(n-1)\binom{n-2}{j-1} + (j+1)\binom{n-1}{j} = -j\binom{n-1}{j} + (j+1)\binom{n-1}{j} = Left$.
Prove 2) in a similar way.

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By recurrence relation:

Theorem 4.4. $\left\langle {M \atop g} \right\rangle_T = H_3(g,[T,1...1],[T...T+M-1]) = T \times H_3(g,[1...1],[T+1...T+M-1])$.

Theorem 4.5. $T \in \mathbb{N}$, $\left\langle {M \atop g} \right\rangle_T = \frac{1}{(T-1)!}\sum_{k=0}^{g}(-1)^{g+k}\binom{T+M}{g-k}[T+k]_T(k+1)^{M-1}$
$= T \times (\sum_{k=0}^{M-2}\left\langle {M-1 \atop g} \right\rangle_{T+1}^{k}\binom{M-1}{k} + (-1)^g\binom{M-1}{g}), 0 < g < M-1$.

Proof.
Based on Crammer's law, $H_3(g) = \sum_{n=1}^{g+1}(-1)^{g+n+1}\binom{T_M+1}{g+1-n}\nabla SUM(n)$. [1]
$PS = [1...1], PT = [T+1...T+M-1], PS1 = [1,2...T,1...1], PT1 = [1,2...T,T+1...T+M-1]$.

$\left\langle {M \atop g} \right\rangle_T = T \times H_3(g) = \frac{1}{(T-1)!} H_3(g, PS1, PT1)$.
$= \frac{1}{(T-1)!} \sum_{n=1}^{g+1} (-1)^{g+n+1} \binom{T+M}{g+1-n} [T+n-1]_T \times n^{M-1} \to 1)$, 3.2 $\to 2)$.

□

$\left\langle {M \atop 1} \right\rangle_T = T(T+1)2^{M-1} - T(T+M)$.

Theorem 4.6. $\sum_{g=0}^{M} (-1)^g g! E_T(M+1, g+1) = (T-1)^M$. $\sum_{g=0}^{M} \left\langle {M \atop g} \right\rangle_T = [T]^M$.

Proof.
$H_2(0) = (T-1)^M = \sum_{k=0}^{M} (-1)^k H_1(k) = \sum_{k=0}^{M} (-1)^k k! E_{M-k}^{\{T...T+k\}} \to 1)$.
$PS = [1...1], PT = [T+1...T+M-1], T \times H_1(M) = T \times \sum_{k=0}^{M} H_3(k) \to 2)$.

□

K_1 is a constant, $K_1 K_2 ... K_M, K_{i+1} - K_i = 1$ or 2, there are M-1 intervals between factors. $K_{i+1} - K_i = 1$ is defined as continuity, $K_{i+1} - K_i = 2$ is defined as discontinuity.

Definition 11. $MIN_g^K(M) = K \sum K_2 ... K_M, K = K_1, MIN_g^1(M) = MIN_g(M)$, count of discontinuities=g.

Obviously, $0 \leqslant g \leqslant M-1$, $MIN_g^K(M)$ have $\binom{M-1}{g}$ items.
$PS = [T+1, T+2...T+M-1], PT = [T+2, T+4...T+2M-2], T \times H_1(g) = MIN_g^T(M)$
$PS = [T, T+1...T+M-1], PT = [T, T+2...T+2M-2], H_1(g) = MIN_g^T(M) + MIN_{g-1}^T(M)$
By definition:

Theorem 4.7. $\lambda_1 + \lambda_2 + ... + \lambda_{g+1} = M - g, \lambda_i \geqslant 0$,
$\left\langle {M \atop g} \right\rangle_T = T \sum 1^{\lambda_1} 2^{\lambda_2} ... (g+1)^{\lambda_{g+1}} (T+\lambda_1)(T+\lambda_1+\lambda_2)...(T+\lambda_1+...+\lambda_g)$.
$PS = [T, T...T], PT = [1, 3...2M-1]$.
$H_1(g) = \sum T^{\lambda_1} (T+1)^{\lambda_2} ... (T+g)^{\lambda_{g+1}} (1+\lambda_1)(3+\lambda_1+\lambda_2)...(2g-1+\lambda_1+...+\lambda_g)$.
$H_2(g)$=Changed $MIN_{g-1}(M) + MIN_g(M)$. Select M-g from M factors, change i to (T-i).
$PS = [T, T+1...T+M-1], PT = [1, 3...2M-1]$.
$H_1(g)$=Changed $MIN_{g-1}(M) + MIN_g(M)$. Select M-g from M factors, change i to (T+i-1).
$H_2(g) = \sum (T-1)^{\lambda_1} (T-2)^{\lambda_2} ... (T-g-1)^{\lambda_{g+1}} (1+\lambda_1)(3+\lambda_1+\lambda_2)...(2g-1+\lambda_1+...+\lambda_g)$.

For example, express the product in terms of ():
$MIN_0(3) + MIN_1(3) = (123) + (124) + (134)$,
$H_2(1) = (T-1, T-2, 3) + (T-1, 2, T-4) + (1, T-3, T-4)$.
$MIN_1(3) + MIN_2(3) = (124) + (134) + (135)$,
$H_1(2) = (T, 2, 4) + (1, T+2, 3) + (1, 3, T+4)$.

5 THE FORMULAS FOR SYMMETRIC FUNCTIONS

If $1 + K_i = K_{i+1}$ and $1 + T_i = T_{i+1}$, $H_1(g) = \binom{M}{g} \prod_{i=1}^{g} T_i \prod_{i=g+1}^{M} K_i$. 2.1 $\to$ [1]:
$\sum_{j=0}^{M-g} F_j^{\{K, K+1...K+M-1\}} E_{M-g-j}^g = \prod_{i=g+1}^{M} (K+i-1) \binom{M}{g}$.
$\sum_{j=0}^{g} (-1)^j F_{g-j}^{\{T, T+1...T+M-1\}} E_j^{M-g} = \prod_{i=1}^{g} (T+i-1) \binom{M}{g}$.

2.1 $\to H_2(g, \sum K) = (-1)^{M-g} \sum_{j=0}^{M-g} F_j^{\{T-K_i\}} E_{M-g-j}^g$, 3.1 $\to$

Theorem 5.1. $\sum_{j=0}^{M-g} F_j^{\{T-K_i\}} E_{M-g-j}^g = \sum_{j=0}^{M-g} (-1)^j F_j^{\mathbb{K}} E_{M-g-j}^{\{T, T+1...T+g\}}$.

$K_i = T \to S_2(M, g) = \sum_{j=0}^{M-g} (-T)^j \binom{M}{j} E_T(M+1-j, g+1)$.
$K_i = T = 1 \to S_2(M, g) = \sum_{j=0}^{M-g} (-1)^j \binom{M}{j} S_2(M+1-j, g+1)$.

$K_i = -q \rightarrow \sum_{j=0}^{M-g} (T+q)^j \binom{M}{j} S_2(M-j,g) = \sum_{j=0}^{M-g} q^j \binom{M}{j} E_T(M+1-j,g+1)$.
$K_i = q, T = 1 \rightarrow \sum_{j=0}^{M-g} (1-q)^j \binom{M}{j} S_2(M-j,g) = \sum_{j=0}^{M-g} (-q)^j \binom{M}{j} S_2(M+1-j,g+1)$.

2.1 $\rightarrow PS = [K...K-M+1], H_3(g, \sum T) = \sum_{j=0}^{g} F_j^{\{T_i-K\}} E_{g-j}^{M-g}$, 3.3 $\rightarrow$

Theorem 5.2. $\sum_{j=0}^{M-g} F_j^{\{K_i-T\}} E_{M-g-j}^{g} = (-1)^{M-g} \sum_{j=0}^{M-g} (-1)^j F_j^{\mathbb{K}} E_{M-g-j}^{\{T,T-1...T-g\}}$.

Compared to 5.1, this is a little different.

Theorem 5.3. $PS1 = [0,-1...-(p-1), K_{p+1}...K_M], PT1 = [L_1, L_2...L_p, T_{p+1}...T_M]$,
$SUM(N, PS1, PT1) = \prod L_i \times SUM(N-p, [K_{p+1}+p...K_M+p], [T_{p+1}...T_M])$.
$\sum_{j=0}^{M-g} F_j^{\{0,-1...-(p-1),K_{p+1}...K_M\}} E_{M-g-j}^{g} = 0, g < p$.
$\sum_{j=0}^{M-g-p} F_j^{\{0...-(p-1),K_{p+1}...K_M\}} E_{M-g-p-j}^{g+p} = \sum_{j=0}^{M-p-g} F_j^{\{K_{p+1}+p...K_M+p\}} E_{M-p-g-j}^{g}, 0 \leqslant g \leqslant M-p$.

Proof.
By the definition of $H_1(g)$, there clearly is:
$H_1(g+p, PS1, PT1) = \prod L_i \times H_1(g, [K_{p+1}+p...K_M+p], [T_{p+1}...T_M]) \rightarrow SUM(N)$.
Let $PT = [1, 2...M]$, 2.1 $\rightarrow$ the rest.

□

Special:
$H_1(0, \sum K, [K_i+p], PT) \rightarrow \prod_{i=1}^{M} (p+K_i) = \sum_{j=0}^{M} F_j^{\{0,-1...-(p-1),K_1...K_M\}} E_{M-j}^{p}$.
$p^M = \sum_{j=0}^{p-1} (-1)^j F_j^{p-1} E_{M-j}^{p}, p > 1$.

From $\prod_{i=1}^{M} (x+K_i) = \sum_{j=0}^{M} F_j^{\mathbb{K}} x^{M-j}$, it's easy to get:
$\sum_{j=0}^{M} k^{M-j} F_j^M = \prod_{i=1}^{M} (k+i)$. $\sum_{j=0}^{M} (-1)^j k^{M-j} F_j^M = 0, M \geqslant k \geqslant 1$.
The extension can be obtained with $H_3(g)$.
$PS = [0,-1...-(M-1)], PT = [T, T+1...T+M-1], H_3(g < M) = 0$,
$PS1 = PT1 = [T, T+1...T+M-1], H_3(g > 0) = 0 \rightarrow$

Theorem 5.4.
$\sum_{j=0}^{M-1} (-1)^j \left\langle {M \atop g} \right\rangle_T^j F_j^{M-1} = 0, 0 < g < M$.
$\sum_{j=0}^{M-1} \left\langle {M \atop g} \right\rangle_T^j F_j^{\mathbb{K}} + (-1)^g \binom{M}{g} F_M^{\mathbb{K}} = 0, 0 < g < M, \mathbb{K} = \{T, T+1...T+M-1\}$.
$\left\langle {M \atop g} \right\rangle_T^j = \frac{1}{(T-1)!} \sum_{k=0}^{g} (-1)^{g+k} \binom{T+M}{g-k} [T+k]_T (k+1)^{M-1-j}, T \in \mathbb{N}$.

Proof.
Proof of the third equation. From $\sum_{j=0}^{M} (-1)^j k^{M-j} F_j^M = 0$ and the first equation, it can be seen that $\left\langle {M \atop g} \right\rangle_T^j$ and $\left\langle {M \atop g} \right\rangle_T^0 = \left\langle {M \atop g-1} \right\rangle_T$ have the same thing:
4.5 $\rightarrow \left\langle {M \atop g} \right\rangle_T^0 = a_1 1^x + a_2 2^y + ... \rightarrow \left\langle {M \atop g} \right\rangle_T^j = a_1 1^{x-j} + a_2 2^{y-j} + ... \rightarrow$ the expression.
The same conclusion can be obtained by combining definition of $\left\langle {M \atop g} \right\rangle_T^j$ and 4.5.

□

$PS = [T, T+1...T+p-1, K_{p+1}...K_M], PT = [T, T+1...T+M-1]$,
$H_3(g) = \prod_{i=1}^{p} (T+i-1) H_3(g, [K_{p+1}...K_M], [T+p...T+M-1]) \rightarrow$

Theorem 5.5. $\mathbb{K} = \{T, T+1...T+p-1, K_{p+1}...K_M\}, \mathbb{K}_1 = \{K_{p+1}...K_M\}$,
$\sum_{j=0}^{M-1} \left\langle {M \atop g} \right\rangle_T^j F_j^{\mathbb{K}} + (-1)^g \binom{M}{g} F_M^{\mathbb{K}} = 0, M > g > M-p$.
$\sum_{j=0}^{M-1} \left\langle {M \atop g} \right\rangle_T^j F_j^{\mathbb{K}} + (-1)^g \binom{M}{g} F_M^{\mathbb{K}} =$
$\prod_{i=1}^{p} (T+i-1) (\sum_{j=0}^{M-p-1} \left\langle {M-p \atop g} \right\rangle_{T+p}^j F_j^{\mathbb{K}_1} + (-1)^g \binom{M-p}{g} F_{M-p}^{\mathbb{K}_1}), 0 < g < M-p$.

$PS = [K, K-1..K-M+1], PT = [0, 1...M-1], H_2(g>0) = 0,$
$PS1 = PT1 = [T, T+1...T+M-1], H_2(g<M) = 0 \rightarrow$

Theorem 5.6.
$\sum_{j=0}^{M-1} - \left\langle {M \atop M-g} \right\rangle_{-K}^{j} F_j^{M-1} = 0, 0 < g < M.$
$\sum_{j=0}^{M-1} - \left\langle {M \atop M-g} \right\rangle_{1-M-T}^{j} F_j^{\mathbb{K}} + (-1)^{M-g} \binom{M}{g} F_M^{\mathbb{K}} = 0, 0 < g < M, \mathbb{K} = \{T, T+1...T+M-1\}.$

$PS = [K, K-1...K-M+1], PT = [K, K-1...K-p+1, T_{p+1}...T_M],$
$H_2(g+p) = \prod_{i=1}^{p} (K-i+1) H_2(g, [K-p...K-M+1], [T_{p+1}...T_M]) \rightarrow$

Theorem 5.7. $\mathbb{T} = \{K, K-1...K-p+1, T_{p+1}...T_M\}, \mathbb{T}_{\mathbb{1}} = \{T_{p+1}...T_M\},$
$\sum_{j=0}^{M-1} - \left\langle {M \atop M-g} \right\rangle_{-K}^{j} F_j^{\mathbb{T}} + (-1)^{M-g} \binom{M}{g} F_M^{\mathbb{T}} = 0, 0 < g < p.$
$\sum_{j=0}^{M-1} - \left\langle {M \atop M-p-g} \right\rangle_{-K}^{j} F_j^{\mathbb{T}} + (-1)^{M-p-g} \binom{M}{g+p} F_M^{\mathbb{T}} =$
$\prod_{i=1}^{p} (K-i+1) (\sum_{j=0}^{M-p-1} - \left\langle {M-p \atop M-p-g} \right\rangle_{p-K}^{j} F_j^{\mathbb{T}_{\mathbb{1}}} + (-1)^{g} \binom{M-p}{g} F_{M-p}^{\mathbb{T}_{\mathbb{1}}}), 0 < g < M-p.$

Theorem 5.8. $T \in \mathbb{Z}, p \in \mathbb{N}, 0 \leqslant g \leqslant M, p \leqslant M,$
$\sum_{j=0}^{M-g} F_j^{\{T, T+1...T+p-1, K_{p+1}...K_M\}} E_{M-g-j}^{g} =$
$\sum_{i=0}^{p} \binom{p}{i} (T+p+g-i-1)_{p-i} \sum_{j=0}^{M-p-g+i} F_j^{\{K_{p+1}...K_M\}} E_{M-p-g+i-j}^{g-i}.$

Proof.
$PS1 = [T, T+1...T+(p-1), K_{p+1}...K_M], PT1 = [T, T+1...T+M-1],$
$PS2 = [K_{p+1}...K_M], PT2 = [T+p-1...T+M-1].$
$H_1(g, [T, PS], [T, PT]) = T \times H_1(g) + T \times H_1(g-1) \rightarrow$
$H_1(g, PS1, PT1) = [T+g-1]_g H_1(g, \sum K, PS1, PT1)$
$= [T+p-1]_p \sum_{i=0}^{p} \binom{p}{i} H_1(g-i, PS2, PT2).$
$= [T+p-1]_p \sum_{i=0}^{p} \binom{p}{i} [T+p-1+g-i]_{g-i} H_1(g-i, \sum K, PS2, PT2).$

□

$K_i = 0, g = M-p \rightarrow \sum_{j=0}^{p} F_j^{\{T...T+p-1\}} E_{p-j}^{M-p} = \sum_{i=0}^{p} \binom{p}{i} [T-1+M-i]_{p-i} E_i^{M-p-i}.$
$g = 1 \rightarrow \sum_{j=0}^{M-1} F_j^{\{T...T+p-1, K_{p+1}...K_M\}} = \sum_{i=0}^{p} \binom{p}{i} [T-1+p-i]_{p-i} \sum_{j=0}^{M-p-1+i} F_j^{\{K_{p+1}...K_M\}}.$
$p = M-1 \rightarrow \sum_{j=0}^{p} F_j^{\{T, T+1...T+p-1\}} = [T+p]_p.$

6 PT=PS AND ITS PROMOTION

$H_1(g, PT, PT) = \prod_{i=1}^{M} T_i \binom{M}{g}$, promoting it:

Theorem 6.1. $PS = [T_1, T_2...T_{M-p}, 0, -1... -(p-1)]$, *then*
$H_1(g) = \prod_{i=1}^{M} T_i \binom{M-p}{g-p}, H_2(g) = (-1)^{M-g} \prod_{i=1}^{M} T_i \binom{p}{M-g},$
$H_3(p) = \prod_{i=1}^{M} T_i, H_3(g \neq p) = 0, SUM(N) = \prod_{i=1}^{M} T_i \binom{N+T_M-p}{T_M+1}.$

Proof.
$H_1(g, M) = (K_M + g) H_1(g, M-1) + (T_M - M + g) H_1(g-1, M-1)$
Using induction and the recurrence relationship one can obtain $H_1(g)$.
$PT1 = [T_{M-p+1}...T_M]$, 2.2 $\rightarrow$
$H_2(g, [0... -(p-1)], PT1) = (-1)^{M-g} H_1(g, PT1, PT1) = (-1)^{M-g} \prod_{i=M-p+1}^{M} T_i \binom{p}{g}.$
$H_2(g) = \prod_{j=1}^{M-p} T_i \times H_2(g-(M-p), [0... -(p-1)], PT1) \rightarrow H_2(g).$
$H_3(g)$ can be obtained from the definition. $H_3(g) \rightarrow SUM(N)$.

□

Theorem 6.2. *$PS = [0, -1 ... -(M-p-1), T_{M-p+1} - (M-p) ... T_M - (M-p)]$, then $H_2(g) = (-1)^{M-g} \prod_{i=1}^{M} T_i \binom{M-p}{g-p}$, $H_1(g) = \prod_{i=1}^{M} T_i \binom{p}{M-g}$, $H_3(M-p) = \prod_{i=1}^{M} T_i, H_3(g \neq M-p) = 0, SUM(N) = \prod_{i=1}^{M} T_i \binom{N+T_M-M+p}{T_M+1}$.*

Proof.
2.2 & 6.1 $\rightarrow H_2(g)$.
$p = 0, H_1(M) = \prod_{i=1}^{M} T_i$. Recurrence relation & induction on p $\rightarrow H_1(g)$.
$H_3(g)$ can be obtained from the definition. $H_3(g) \rightarrow SUM(N)$.

□

Using similar methods:

Theorem 6.3. *$PS = [T_1, T_2 ... T_{M-p}, 0, -1 ... -(p-1)]$, $PT = [0, 1 ... (M-p-1), (M-p) - T_{M-p+1} ... (M-p) - T_M]$, then $H_3(g) = (-1)^g \prod_{i=1}^{M} T_i \binom{M-p}{g-p}$, $H_1(p) = (-1)^p \prod_{i=1}^{M} T_i, H_1(g \neq p) = 0$, $H_2(g) = (-1)^g \prod_{i=1}^{M} T_i \binom{M-p}{g}$, $SUM(N) = (-1)^p \prod_{i=1}^{M} T_i \binom{N-T_M-p}{1-T_M}, p > 0; SUM(N) = \prod_{i=1}^{M} T_i, p = 0$.*

7 TRANSFORMATION OF SUM(N)

$\prod_{i=1}^{M} (x + K_i) = \sum_{g=0}^{M} a_g x^{M-g} = \sum_{g=0}^{M} F_g^{\mathbb{K}} x^{M-g} = \nabla SUM(x+1, PS, [1, 2 ... M])$.
No need to know the value of K_i. Any polynomial can be converted to $\nabla SUM(N, PS, [1, 2 ...])$.

By choosing c and $K_1', K_2' ...$ appropriately, $SUM(N)$ can be converted to $c \times \nabla^q SUM(N, [K_1', K_2' ...], PT1)$.
However, it is generally necessary to solve higher-order equations to solve for $\mathbb{K}'$.
But specifies that every nested sum can be flattened, converting to $c \times \nabla^q SUM(N, PS, [1, 2 ...])$.

$(1) = \sum_{g=0}^{M} b_g \binom{N+Y}{g+X+1}$, $(2) = \sum_{g=0}^{M} c_g \binom{N+Y+g}{g+X+1}$, $(3) = \sum_{g=0}^{M} d_g \binom{N+Y+M+X-g}{M+1+X}$, can be converted to $c \times \nabla^{-X} SUM(N+Y-X, PS, [1, 2 ... M])$. $c = \frac{b_M}{M!} = \frac{c_M}{M!} = \frac{d_M}{\prod (1-K_i)}$. It is only necessary to solve the system of linear equations to find $F_j^{\mathbb{K}}$. so them can also be converted to $\sum_{g=0}^{M} a_g x^{M-g}$.
Special:
$b_0 = b_1 = ... = b_q = 0 \rightarrow K_1 = 0, K_2 = -1 ... K_q = -(q-1)$.
$c_0 = c_1 = ... = c_q = 0 \rightarrow K_1 = 1, K_2 = 2 ... K_q = q$.
$d_0 = d_1 = ... = d_q = 0 \rightarrow K_1 = 0, K_2 = -1 ... K_q = -(q-1)$.

$(1), (2), (3)$ can also be converted to $c \times \nabla SUM(N+Y-X, PS, [X+1, X+2 ... X+M])$.
$c = \frac{b_M}{(X+1)...(X+M)} = \frac{c_M}{(X+1)...(X+M)} = \frac{d_M}{\prod (X+1-K_i)}$.
Special: $c_0 = c_1 = ... = c_q = 0 \rightarrow K_1 = X+1, K_2 = X+2 ... K_q = X+q$.

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