FORMAL CALCULATION

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CONTENTS

ABSTRACT

Formal Calculation uses an auxiliary form to calculate various nested sums and provides results in three forms. In addition to computation, it is also a powerful tool for analysis, allowing one to study various numbers in a unified way. This article contains many results of two types of Stirling numbers, associated Stirling numbers, and Eulerian numbers, making a great generalization of Euler polynomials, Wilson's theorem, and Wolstenholme's theorem, showing that they are just special cases. Formal Calculation provides a novel method for obtaining combinatorial identities and analyzing q-binomial.This article has obtained many results in q-analogues, including inversion formulas for q-binomial coefficients. This article also introduces a theorem on symmetry.

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1 INTRODUCTION

Formal Calculation is introduced in [1] [2] [3], this article contains its summary and latest achievements.

Definition 1. The definition of ∇^p is recursive. $p \in \mathbb{Z}$,
$\nabla^0 f(n) = f(n)$, $\sum_{n=0}^{N-1} \nabla^1 f(n+1) = f(N)$, $\sum_{n=0}^{N-1} f(n+1) = \nabla^{-1} f(N)$, $\nabla^1 = \nabla$.

Definition 2. The definition of $SUM(N) = SUM(N, PS, PT)$ is recursive. $K_i, D_i \in$ Ring with identity elements.
$SUM(N, [K_1 : D_1], [T_1 = 1]) = \sum_{n=0}^{N-1} (K_1 + nD_1)$.
$SUM(N, [K_1 : D_1, K_2 : D_2], [T_1, T_2 = T_1 + 2 - p]) = \sum_{n=0}^{N-1} (K_2 + nD_2) \nabla^p SUM(n+1, [K_1 : D_1], [T_1])$.

If $f(N) = \sum A_i \binom{N_i}{M_i}$ and M_i is not changed with N, then $\nabla^p f(N) = \sum A_i \binom{N_i - p}{M_i - p}$.
$[K_1 : D, K_2 : D ... K_M : D]$ is abbreviated as $[K_1, K_2 ... K_M] : D$, $[K_1, K_2 ... K_M] : 1$ is abbreviated as $[K_1, K_2 ... K_M]$.
By default, this paper use:
PS=$[K_1 : D_1, K_2 : D_2 ... K_M : D_M]$,PT=$[T_1, T_2 ... T_M]$, PS1=[PS,$K_{M+1} : D_{M+1}$], PT1=[PT,$T_{M+1}$].
This is actually nested summation. For example:
$SUM(N, PS, [1, 2, 3 ... M]) = \sum_{n=0}^{N-1} \prod_{i=1}^{M} (K_i + nD_i)$.
$SUM(N, PS, [1, 3, 5 ... 2M-1]) = \sum_{n_M=0}^{N-1} (K_M + n_M D_M) ... \sum_{n_2=0}^{n_3} (K_2 + n_2 D_2) \sum_{n_1=0}^{n_2} (K_1 + n_1 D_1)$.
$SUM(N, PS, [1, 2, 4]) = \sum_{n_3=0}^{N-1} (K_3 + n_3 D_3) \sum_{n=0}^{n_3} (K_1 + nD_1)(K_2 + nD_2)$.
$SUM(N, PS, [1, 3, 4]) = \sum_{n_3=0}^{N-1} (K_3 + n_3 D_3)(K_2 + n_3 D_2) \sum_{n=0}^{n_3} (K_1 + nD_1)$.
The following use K to represent the set $[K_1, K_2 ... K_M]$, T to represent the set $[T_1, T_2 ... T_M]$.
Use the auxiliary form: $(K_1 + T_1)(K_2 + T_2) ... (K_M + T_M) = \sum \prod_{i=1}^{M} X_i,\ X_i = T_i$ or K_i.

Definition 3. X(T)=Number of $\{X_1, X_2 ... X_M\} \in T$.

Definition 4. X_{T-1}=Number of $\{X_1, X_2 ... X_{i-1}\} \in T$, X_{K-1}=Number of $\{X_1, X_2 ... X_{i-1}\} \in K$,
X_T=Number of $\{X_1, X_2 ... X_i\} \in T$, and also define X_K.

Obviously:$X_{T-1} + X_{K-1} = i - 1$.

1.1. *[1] SUM(N,PS,PT)=*

$$Form_1 \to \sum_{g=0}^{M} H_1(g) \binom{N + T_M - M}{N - 1 - g} = \sum_{g=0}^{M} H_1(g) \binom{N + T_M - M}{T_M - M + 1 + g},\ B_i = \begin{cases} (T_i - X_{K-1}) D_i, & X_i = T_i \\ K_i + X_{T-1} D_i, & X_i = K_i \end{cases};$$

$$Form_2 \to \sum_{g=0}^{M} H_2(g) \binom{N + T_M - M + g}{N - 1} = \sum_{g=0}^{M} H_2(g) \binom{N + T_M - M + g}{T_M - M + 1 + g},\ B_i = \begin{cases} (T_i - X_{K-1}) D_i, & X_i = T_i \\ K_i + (X_{K-1} - T_i) D_i, & X_i = K_i \end{cases};$$

$$Form_3 \to \sum_{g=0}^{M} H_3(g) \binom{N + T_M - g}{N - 1 - g} = \sum_{g=0}^{M} H_3(g) \binom{N + T_M - g}{T_M + 1},\ B_i = \begin{cases} -K_i + (T_i - X_{T-1}) D_i, & X_i = T_i \\ K_i + X_{T-1} D_i, & X_i = K_i \end{cases}.$$

The factors of $\prod X_i$ *cannot be exchanged.* $H_i(g)$*, short for* $H_i(g, PS, PT)$*, is also defined above as* $\sum_{X(T)=g} \prod_{i=1}^{M} B_i$.

The theorem is proved by induction. There are three forms because:

$$\sum_{n=0}^{N-1} n \binom{n+K}{M}$$
$$= (M+1) \binom{N+K}{M+2} + (M-K) \binom{N+K}{M+1}$$
$$= (M+1) \binom{N+K+1}{M+2} - (1-K) \binom{N+K}{M+1}$$
$$= (M-K) \binom{N+K+1}{M+2} + (1+K) \binom{N+K}{M+2}.$$

Definition 5. $F_M^{K=\{K_1, K_2 ... K_M\}} = \sum K_{I_1} K_{I_2} ... K_{I_M},\ a < b,\ I_a < I_b$, F_M^N is short for $F_M^{\{1,2...N\}}$, $F_0^K = 1$.

Definition 6. $E_M^{K=\{K_1, K_2 ... K_M\}} = \sum K_{I_1} K_{I_2} ... K_{I_M},\ a < b,\ I_a \leqslant I_b$, E_M^N is short for $E_M^{\{1,2...N\}}$, $E_0^K = 1$.

1.2. $\nabla SUM(n+1, PS, [1,2...M]) = \prod_{i=1}^{M}(K_i + nD_i)$.

1.3. *In SUM(N,[...PS...],[...T+1,T+2...T+M...]),* K_i *can exchange orders.*

1.4. $SUM(N, [L_1, L_2...L_q, PS], [L_1, L_2...L_q, PT]) = \prod_{i=1}^{q} L_i \times SUM(N, PS, PT)$. *So* T_1 *can great than 1,* $T_i \in \mathbb{N}$.

1.5. $SUM(N, [1, 1...1], [1, 2...M]) = SUM(N, [1, 1...1], [2, 3...M]) = 1^M + 2^M + ... + N^M$.

1.6. $SUM(N, [1, 1...1], [1, 3...2M-1]) = SUM(N, [1, 1...1], [3, 5...2M-1])$
$= \sum_{\lambda_1+...+\lambda_N=M, \lambda_i \geqslant 0} 1^{\lambda_1} 2^{\lambda_2} ... N^{\lambda_N} = E_M^N = S_2(N+M, N)$. S_2 *is Stirling numbers of the second kind.*

1.7. $SUM(N, [1, 2...M], [1, 3...2M-1]) = SUM(N, [2, 3...M], [3, 5...2M-1])$
$= \sum_{1 \leqslant i_1 < i_2 < ... < i_M \leqslant N+M-1} i_1 i_2 ... i_M = F_M^{N+M-1} = S_1(N+M, N)$. S_1 *is unsigned Stirling numbers of the first kind.*

Example 1.1:
$Form = (1+T_1)(2+T_2)(3+T_3)$, $\sum_{X(T)=1} \prod X_i = 1 \times 2 \times T_3 + 1 \times T_2 \times 3 + T_1 \times 2 \times 3$.
$H_1(1) = 1 \times 2 \times (T_3 - X_{K-1}) + 1 \times (T_2 - X_{K-1}) \times (3 + X_{T-1}) + T_1 \times (2 + X_{T-1}) \times (3 + X_{T-1})$
$= 1 \times 2 \times (5-2) + 1 \times (3-1) \times (3+1) + 1 \times (2+1) \times (3+1) = 26$.
$SUM(N, [1,2,3], [1,3,5]) = 1 \times 3 \times 5 \binom{N+2}{6} + 35 \binom{N+2}{5} + 26 \binom{N+2}{4} + 1 \times 2 \times 3 \binom{N+2}{3}$.

$SUM(N, [2,3], [3,5]) = 3 \times 5 \binom{N+3}{6} + (2 \times 4 + 3 \times 4) \binom{N+3}{5} + 2 \times 3 \binom{N+3}{4}$.
It also can be calculated in the Ring with identity elements. K_i, D_i can be a matrix.

1.8. $\prod(K_i + nD_i)$ *can be decomposed into three forms by 1.1 and* ∇.

2 PROPERTY

2.1 Relationships between $H(g)$

By definition:

1. $H_1(g, PS1, PT1) = H_1(g-1)(T_{M+1} - [M-(g-1)])D_{M+1} + H_1(g)(K_{M+1} + gD_{M+1})$.
2. $H_2(g, PS1, PT1) = H_2(g-1)(T_{M+1} - [M-(g-1)])D_{M+1} + H_2(g)(K_{M+1} + [M-g-T_{M+1}]D_{M+1})$.
3. $H_3(g, PS1, PT1) = H_3(g-1)(-K_{M+1} + (T_{M+1} - [g-1])D_{M+1}) + H_3(g)(K_{M+1} + gD_{M+1})$.

By utilizing these relationships and induction, it can be demonstrated that:

2.1. $H_1(g) = \sum_{k=g}^{M} H_2(k) \binom{k}{g} = \sum_{k=0}^{g} H_3(k) \binom{M-k}{M-g}$. *[2]*

Inversion can be used to obtain:

2.2. $H_2(g) = \sum_{k=g}^{M} (-1)^{k+g} H_1(k) \binom{k}{g}$, $H_3(g) = \sum_{k=0}^{g} (-1)^{k+g} H_1(k) \binom{M-k}{M-g}$.

Calculation with 2.1 can obtain:

2.3. $\sum_{g=0}^{M} H_1(g) = \sum_{g=0}^{M} H_2(g) 2^g = \sum_{g=0}^{M} H_3(g) 2^{M-g}$.

2.4. $\sum_{g=0}^{M} H_1(g) \binom{A}{B-g} = \sum_{g=0}^{M} H_2(g) \binom{A+g}{B} = \sum_{g=0}^{M} H_3(g) \binom{A+M-g}{B-g}$, $A, B \in \mathbb{N}$.

This indicates $Form_1 = Form_2 = Form_3 \rightarrow \sum_{g=0}^{M} H_1(g) \binom{A}{g} = \sum_{g=0}^{M} H_2(g) \binom{A+g}{g} = \sum_{g=0}^{M} H_3(g) \binom{A+M-g}{M}$.
Induction can be used to obtain: [2]

2.5. $\sum_{g=0}^{M} H_1(g) g \binom{A+1}{B-g} = \sum_{g=0}^{M} H_2(g) g \binom{A+g}{B-1} = \sum_{g=0}^{M} \{H_3(g) g \binom{A+M-g}{B-g} + M \times H_3(g) \binom{A+M-g}{B-1-g}\}$.

2.2 Property of H(g)

$\prod X_i = (\prod X_i \in T)(\prod X_i \in K)$. In some cases, H(g) is easy to calculate.

Definition 7. $H(g,T) = H(g,T,PS,PT) = \prod_{X_i \in T} B_i$, $H(g,\sum T) = \sum \prod_{X_i \in T} B_i$. Also define $H(g,K), H(g,\sum K)$

If $D_i = 1$ and $T_i + 1 = T_{i+1}$,then $H_1(g,T) = \prod_{i=1}^{g} T_i$, $H_1(g,T,PS,[1,2...M]) = g!$.
$H_1(g,\sum K,[1,1...1],PT) = E_{M-g}^{g+1}$.

2.6. *If* $D_i = 1$, $H_1(g,\sum K) = F_{M-g}^K E_0^g + F_{M-g-1}^K E_1^g + ... + F_0^K E_{M-g}^g$.

2.7. *If* $D_i = 1$, $H_1(g,\sum T) = F_g^T E_0^{M-g} - F_{g-1}^T E_1^{M-g} + ... + (-1)^{M-g} F_0^T E_g^{M-g}$.

2.8. *If* $D_i = 1$ *and* $K_{i+1} - K_i = T_{i+1} - T_i = 1$, $H_1(g)=\binom{M}{g} T_1...T_g \times K_{g+1}...K_M$.

2.9. *If PS=PT,* $H_1(g) = \prod_{i=1}^{M} T_i \binom{M}{N}$, $H_2(M) = H_3(0) = \prod_{i=1}^{M} T_i$, $H_2(g < M) = H_3(g > 0) = 0$.

2.10. $H_1(g,[AD:D,PS],[A,PT]) = AD(H_1(g-1) + H_1(g))$, $H_1(g,[1,PS],[1,PT]) = H_1(g-1) + H_1(g)$.

Definition 8. $E_p^q \odot ([T_1,T_2...T_M],C)$
$$= \sum_{\lambda_1+\lambda_2+...+\lambda_q = p, \lambda_i \geqslant 0} 1^{\lambda_1} 2^{\lambda_2} ... q^{\lambda_q} (T_1 + \lambda_1 C)(T_2 + \lambda_1 C + \lambda_2 C)...(T_{q-1} + \lambda_1 C + \lambda_2 C + ... + \lambda_{q-1} C).$$

[4] has proved: $\left\langle {M \atop g} \right\rangle = \left\langle {M \atop M-g-1} \right\rangle = E_{M-g-1}^{g+1} \odot ([1,1...1],1])$.

$\left\langle {M \atop g} \right\rangle$ is Eulerian numbers. It is known that there exists Worpitzky identity: $N^M = \sum_{g=0}^{M-1} \left\langle {M \atop g} \right\rangle \binom{N+g}{M}$.

For example: $\left\langle {5 \atop 2} \right\rangle = E_2^3 \odot ([1,1...],1) = \sum_{\lambda_1+\lambda_2+\lambda_3=2} 1^{\lambda_1} 2^{\lambda_2} 3^{\lambda_3} (1+\lambda_1)(1+\lambda_1+\lambda_2) = 66$.

By simple calculation:
$H_1(g,[1,1...1],PT = [T_i = T_1 + (C+1)(i-1)]) = E_{M-g}^{g+1} \odot (PT,C)$.
$H_3(g,[1,1...1],PT = [T_i = T_1 + (C+1)(i-1)]) = E_{M-g}^{g+1} \odot ([T_i = T_1 - 1 + C(i-1)],C+1)$.

2.3 Shape of numbers

In this section, if not specifically mentioned, $T_1 = 1, T_{i+1} - T_i = 1$ or 2.
To calculate $\sum_{1 \leqslant K_1 < K_2 < ... < K_M \leqslant N} K_1 K_2 ... K_M$ (*), products needs to be divided into 2^{M-1} categories.
There are **M-1** intervals between factors. If the interval=1,define it as **Continuity**. If the interval>1, define it as **Discontinuity**. Continuities, Discontinuities and their Positions are defined as **Shape**. So there have 2^{M-1} Shapes.
From the definition of nested sum:

$$\sum_{1 \leqslant K_1 < K_2 < K_3 \leqslant N} K_1 K_2 K_3$$
$$= SUM(N,[1,2,3],[1,2,3]) + SUM(N-1,[1,2,4],[1,2,4]) + SUM(N-1,[1,3,4],[1,3,4]) + SUM(N-2,[1,3,5],[1,3,5]).$$

Definition 9. PB(PT)=Number of $T_i + 1 < T_{i+1}$=Number of discontinuities.

$$(*) = \sum_{\text{All of the Shapes with factors}=M} SUM(N - PB(PT), PT, PT).$$
From 2.9 we can obtain a simple formula:

2.11. $SUM(N,PT,PT) = \prod_{i=1}^{M} T_i \binom{N+T_M}{T_M+1}$, *PT has no restrictions.*

This generalizes the famous formula $\sum_{n=0}^{N-1} \binom{n}{M} = \binom{N}{M+1}$.
It was discovered during the calculation of (*) which led to the birth of Formal Calculation.

2.12. *Number of products in SUM(N,PT,PT)* $= \binom{N+PB(PT)}{PB(PT)+1}$.

Definition 10. $MIN_g(M) = \sum_{PB(PT)=g} \prod_{i=1}^{M} T_i = 1 \times \sum_{PB(PT)=g} \prod_{i=2}^{M} T_i$.

This is the sum of the products of PTs with the same number of discontinuities.
By definition:

2.13. $MIN_g(M) = \sum \frac{(M+g)!}{i_1 i_2 ... i_g}, 2 \leqslant i_1 < i_2 < ... < i_g \leqslant M+g-1, i_{j+1} - i_j \geqslant 2$.

Based on the concept of Shape rather than 2.13, it is easier to understand.
For example: $MIN_2(5) = (12357) + (12457) + (13457) + (12467) + (13467) + (13567)$.
Here (...) is the product.

From the definition of nested sum, there exists general classification principles:

2.14. $SUM(N, [K_1 : D_1, K_2 : D_2 ... K_M : D_M], [T_1, T_2 ... T_M])$

$= SUM(N, PS, [T_1 ... T_i, T_{i+1} - 1 ... T_M - 1]) + SUM(N-1, [K_1 : D_1 ... K_i : D_i, K_{i+1} + D_{i+1} : D_{i+1} ... K_M + D_M : D_M], PT)$.

For example:
$SUM(N, [1,2,3], [1,3,5]) = SUM(N, [1,2,3], [1,2,4]) + SUM(N-1, [1,3,4], [1,3,5])$
$= SUM(N, [1,2,3], [1,2,3]) + SUM(N-1, [1,2,4], [1,2,4]) + SUM(N-1, [1,3,4], [1,3,4]) + SUM(N-2, [1,3,5], [1,3,5])$.
$(*) = SUM(N, [1, 2 ... M], [1, 3 ... 2M-1]) = \sum_{g=0}^{M-1} MIN_g(M) \binom{N}{g+1}$. It's exactly 1.7.

2.4 H(g) and Associated Stirling Numbers

Associated Stirling Numbers of the first kind $S_{1,r}(n,k)$ is defined as the number of permutations of a set of n elements having exactly k cycles, all length >= r.

1. $S_{1,r}(n,k) = \frac{n!}{k!} \sum_{i_1+i_2+...+i_k=n, i_j \geqslant r} \frac{1}{i_1 i_2 ... i_k}$.
2. $S_{1,r}(n+1,k) = nS_{1,r}(n,k) + (n)_{r-1} S_{1,r}(n-r+1, k-1), n \geqslant kr$.
3. $S_{1,r}(n,k) = \sum \frac{(n-1)!}{i_1 i_2 ... i_{k-1}}, r \leqslant i_1 < i_2 < ... < i_{k-1} \leqslant n-r, i_{j+1} - i_j \geqslant r$ [5] .

Derived from 2 and definition of H(g) or 3 and 2.13:

2.15. $MIN_g(M) = S_{1,2}(M+g+1, g+1)$.

Table 1: Table of $MIN_g(M) = S_{1,2}(M+g+1, g+1)$

	g=0	g=1	g=2	g=3	g=4	g=5	g=6
M=1	1						
M=2	2	3					
M=3	6	20	15				
M=4	24	130	210	105			
M=5	120	924	2380	2520	945		
M=6	720	7308	26432	44100	34650	10395	
M=7	5040	64224	303660	705320	866250	540540	135135

Associated Stirling Numbers of the second kind $S_{2,r}(n,k)$ is defined as the number of permutations of a set of n elements having exactly k blocks, all length >= r.

1. $S_{2,r}(n,k) = \frac{n!}{k!} \sum_{i_1+i_2+...+i_k=n, i_j \geqslant r} \frac{1}{i_1! i_2! ... i_k!}$.
2. $S_{2,r}(n+1,k) = kS_{2,r}(n,k) + \binom{n}{r-1} S_{2,r}(n-r+1, k-1), n \geqslant kr$.

Derived from 2:

2.16. $H_2(g, [1, 1 ... 1], [3, 5 ... 2M-1]) = S_{2,2}(M+g+1, g+1)$.

Table 2: Table of $H_2(g,[1,1...1],[3,5...2M-1]) = S_{2,2}(M+g+1,g+1)$

	g=0	g=1	g=2	g=3	g=4	g=5	g=6
M=1	1						
M=2	1	3					
M=3	1	10	15				
M=4	1	25	105	105			
M=5	1	56	490	1260	945		
M=6	1	119	1918	9450	17325	10395	
M=7	1	246	6825	56980	190575	270270	135135

$SUM(N,[2,3...M],[3,5...2M-1]) = S_{1,1}(N+M,N) = \frac{(N+M)!}{N!} \sum_{i_1+i_2+...+i_N=N+M, i_j \geqslant 1} \frac{1}{i_1 i_2 ... i_N}$

$= \sum_{g=0}^{M-1} S_{1,2}(M+g+1,g+1)\binom{N+M}{M+1+g} = \sum_{g=1}^{M} S_{1,2}(M+g,g)\binom{N+M}{M+g}$

$= \frac{(N+M)!}{N!} \sum_{g=1}^{M} [\sum_{i_1+...+i_N=N+M, i_j\geqslant 1, \text{Number of } i_j>1=g} \frac{1}{i_1 i_2 ... i_N}] = \sum_{g=1}^{M} f(*)$.

$f(*) = \frac{(N+M)!}{N!}\binom{N}{N-g} \sum_{i_1+...+i_g=g+M, i_j\geqslant 1} \frac{1}{i_1 i_2 ... i_g} = \binom{N+M}{M+g} S_{1,2}(M+g,g)$.

In the same way we can get:

2.17. $S_{1,r}(rN+M,N) = \sum_{g=1}^{M} \frac{1}{r^{N-g}} \frac{(rN-rg)!}{(N-g)!}\binom{rN+M}{rg+M} S_{1,r+1}(rg+M,g)$.

2.18. $S_{2,r}(rN+M,N) = \sum_{g=1}^{M} \frac{1}{(r!)^{N-g}} \frac{(rN-rg)!}{(N-g)!}\binom{rN+M}{rg+M} S_{2,r+1}(rg+M,g)$.

2.5 Table of H(g)

Table 3: Table of H(g)

PS	PT	$H_1(g)$	$H_2(g)$	$H_3(g)$
[1, 1...1]	[1, 2...M]	$g!E_{M-g}^{g+1} = g!S_2(M+1,g+1)$	$(-1)^{M-g} g!S_2(M,g)$	$\left\langle {M \atop g} \right\rangle$
[1, 1...1]	[2, 3...M]	$(g+1)!S_2(M,g+1)$	$(-1)^{M-1-g}(g+1)!S_2(M,g+1)$	$\left\langle {M \atop g} \right\rangle$
[1, 1...1]	[1, 3...2M − 1]	$E_{M-g}^{g+1} \odot (PT,1)$	$(-1)^{M-g} MIN_{g-1}(M)$	$E_{M-g}^{g+1} \odot ([0,1...],2)$
[1, 1...1]	[3, 5...2M − 1]	$S_{2,2}(M+1+g,g+1)$	$(-1)^{M-1-g} MIN_g(M)$	$E_{M-1-g}^{g+1} \odot ([2,3...],2)$
[1, 2...M]	[1, 3...2M − 1]	$MIN_{g-1}(M) + MIN_g(M)$	$1 \times (-1)^{M-g} E_{M-g}^{g} \odot ([3,5...],1)$	$1 \times E_g^{M-g} \odot ([2,3...],2)$
[2, 3...M]	[3, 5...2M − 1]	$MIN_g(M)$	$(-1)^{M-1-g} S_{2,2}(M+1+g,g+1)$	$E_g^{M-g} \odot ([2,3...],2)$

3 APPLICATION

3.1 Number analysis

3.1. *By using 2.1, 2.3 and 2.5, we can obtain:*

1. $\sum_{g=0}^{M} \left\langle {M \atop g} \right\rangle = M!,\ \sum_{g=1}^{M} (-1)^{M-g} g!S_2(M,g) = 1,\ \sum_{g=1}^{M} (-1)^{M-g} g \times g!S_2(M,g) = 2^M - 1.$
2. $g!S_2(M,g) = \sum_{k=g}^{M} (-1)^{M-k} k!S_2(M,k)\binom{K-1}{g-1} = \sum_{k=0}^{g-1} \left\langle {M \atop k} \right\rangle \binom{M-1-k}{M-g},\ 1 \leqslant g \leqslant M.$
3. $S_{1,2}(M+g,g) = \sum_{k=g}^{M} (-1)^{M-k} S_{2,2}(M+k,k)\binom{K-1}{g-1},\ S_{2,2}(M+g,g) = \sum_{k=g}^{M} (-1)^{M-k} S_{1,2}(M+k,k)\binom{K-1}{g-1}.$
4. $\sum_{g=1}^{M} g!S_2(M,g) = \sum_{g=1}^{M} (-1)^{M-g} g!S_2(M,g) 2^{g-1} = \sum_{g=1}^{M} \left\langle {M \atop M-g} \right\rangle 2^{M-g}.$

5. $\sum_{g=1}^{M} S_{2,2}(M+g,g) = \sum_{g=1}^{M} (-1)^{M-g} S_{1,2}(M+g,g) 2^{g-1}, \ \sum_{g=1}^{M} S_{1,2}(M+g,g) = \sum_{g=1}^{M} (-1)^{M-g} S_{2,2}(M+g,g) 2^{g-1}.$
6. $\sum_{g=1}^{M} g! S_2(M,g)(g-1)\binom{A+1}{g} = \sum_{g=1}^{M} (-1)^{M-g} g! S_2(M,g)(g-1)\binom{A+g-1}{g}$
7. $\sum_{g=0}^{M-1} (-1)^{M-1-g} MIN_g(M) = \sum_{g=1}^{M} (-1)^{M-g} S_{1,2}(M+g,g) = 1, \ \sum_{g=1}^{M} (-1)^{M-g} S_{2,2}(M+g,g) = M!.$

PS=PT=[1,2...M], $H_1(g) = H_1(M-g) = M!\binom{M}{g}$. Use 2.6 to obtain:

3.2. $M!\binom{M}{g} = g! \sum_{i=0}^{M-g} S_1(M+1,g+1+i) S_2(g+i,g) = (M-g)! \sum_{i=0}^{g} S_1(M+1,M+1-i) S_2(M-i,M-g).$

$H_1(g,[1,1...1],[1,2...M]) = g! S_2(M+1,g+1), \ F_i^{\{1,1...1\}} = \binom{M}{i}$. Use 2.6 to obtain:

3.3. $S_2(M+1,g+1) = \sum_{i=0}^{M-g} S_2(M-i,g)\binom{M}{i}.$

$H_1(g,[1,2...M],[1,2...M]) = H_1(g,[M,M-1...1],[1,2...M]) = M!\binom{M}{g}$. Use 2.7 to obtain obtain:

3.4. $g!\binom{M}{g} = \sum_{i=0}^{g} S_1(M+1,M+1-g+i) S_2(M-g+i,M-g)(-1)^i.$

$H_1(g,[K+i],[T+i])$, 2.8 $\to \binom{M}{g} T_1...T_g \times K_{g+1}...K_M$, 2.6$\to T_1...T_g(...)$, 2.7$\to K_{g+1}...K_M(...)$:

3.5. $\prod_{i=g+1}^{M} (K+i)\binom{M}{g} = F_{M-g}^{\{K+i\}} E_0^g + ... + F_0^{\{K+i\}} E_{M-g}^g, \ \prod_{i=1}^{g} (T+i)\binom{M}{g} = F_g^{\{T+i\}} E_0^{M-g} + ... + (-1)^g F_0^{\{T+i\}} E_g^{M-g}.$

3.2 Merge and Expand

3.6. *This SUM(N,PS,PT)=SUM(N,[1,1...1,PS],[1,1...1,PT]). It expands* $\sum_{g=0}^{M}(...)$ *to* $\sum_{g=0}^{M+(\text{number of 1 added})}(...)$.

Any $\sum_{g=0}^{M} a_g \binom{X}{Y+g}$ can be converted to $\frac{a_M}{M!}\nabla^q SUM(N+P,[K_1,K_2...K_M],[1,2...M])$,

2.2 provides the necessary and sufficient condition for $\sum_{g=0}^{M} H(g)\binom{X}{Y+g}$ to be merged into $\sum_{g=0}^{M-K}(...)\binom{X+K}{Y+K+g}$:

$H_2(g) = \sum_{x=g}^{M} (-1)^x H(x)\binom{x}{g} = 0, \ g < K$ or $H_3(g) = \sum_{x=0}^{g} (-1)^x H(x)\binom{M-x}{M-g} = 0, \ M-g < K.$

For example:

SUM(N,[1,2...M],[1,2...M]) and 2.11 can get $\sum_{x=g}^{M} (-1)^x \binom{M}{x}\binom{x}{g} = 0, \ 0 \leqslant g < M.$

$\sum_{n=0}^{N-1} \binom{M+dn}{M} = \frac{1}{M!} SUM(N,[1,2...M]:d,[1,2...M]) = \frac{1}{M!} \sum_{g=0}^{M} H_1(g)\binom{N}{1+g},$

If $M \geqslant kd$, then $B_i(X_i = K_i) = i + (X_{K-1} - i)d \to H_2(g<k) = 0 \to \sum_{n=0}^{N-1}\binom{M+dn}{M} = \sum_{g=0}^{M-k}(...)\binom{N+k}{1+g+k}.$

After a simple calculation, it can be written as:

3.7. *Necessary and sufficient conditions for merging,* $0 \leqslant g < K \leqslant M$*:*

1. $\sum_{n=0}^{M} H(n)\binom{X}{Y+n} \to \sum_{n=0}^{M-K}(...)\binom{X+K}{Y+K+n} : \sum_{x=0}^{M} (-1)^x H(x)\binom{P \pm x}{g} = 0.$
2. $\sum_{n=0}^{M} H(n)\binom{X+n}{Y+n} \to \sum_{n=0}^{M-K}(...)\binom{X+K+n}{Y+K+n} : \sum_{x=0}^{M} H(x)\binom{P \pm x}{g} = 0.$
3. $\sum_{n=0}^{M} H(n)\binom{X-n}{Y} \to \sum_{n=0}^{M-K}(...)\binom{X-K-n}{Y-K} : \sum_{x=0}^{M} H(x)\binom{P \pm x}{g} = 0.$

3.8. $\sum_{g=0}^{M}\binom{M}{g}\binom{A+T+g}{A}\binom{X}{Y+g} = \sum_{g=0}^{A}\binom{A+T}{g+T}\binom{M+T+g}{M+T}\binom{X+M-A}{Y+M-A+g}, \ A \geqslant 0.$

Proof. It can be proved by induction, but it is cumbersome.
SUM(N,[T+1,T+2...T+M],[T+A+1,T+A+2...T+A+M])

$= \sum_{g=0}^{M} \binom{M}{g} [T+A+g]_g [T+M]_{M-g} \binom{N+T+A}{T+A+1+g}$

$= \sum_{g=0}^{M} \binom{M}{g} \frac{(T+A+g)!}{(T+A)!} \frac{(T+M)!}{(T+g)!} \binom{N+T+A}{T+A+1+g} = \frac{A!(T+M)!}{(T+A)!} \sum_{g=0}^{M} \binom{M}{g} \binom{A+T+g}{A} \binom{N+T+A}{T+A+1+g}$

$= \frac{(T+M)!}{(T+A)!} SUM(N, [T+1, T+2...T+A], [T+M+1, T+M+2...T+M+A])$

$= \frac{(T+M)!}{(T+A)!} \sum_{g=0}^{A} \binom{A}{g} \frac{(T+M+g)!}{(T+M)!} \frac{(T+A)!}{(T+g)!} \binom{N+T+M}{T+M+1+g}.$

$\sum_{g=0}^{M} \binom{M}{g} \binom{A+T+g}{A} \binom{N+T+A}{T+A+1+g} = \sum_{g=0}^{A} \frac{(T+M+g)!}{(T+M)!} \frac{(T+A)!}{(T+g)!} \frac{A!}{(A-g)!g!} \frac{1}{A!} \binom{N+T+M}{T+M+1+g}.$ □

$g := M-g \rightarrow \sum_{g=0}^{M} \binom{M}{g} \binom{A+T+M-g}{A} \binom{X}{Y+g} = \sum_{g=0}^{A} \binom{A+T}{g} \binom{A+M+T-g}{M+T} \binom{X+M-A}{Y+M-A+g}, \ A \geqslant 0.$

When A>M, it is an expansion; When A<M, it is a merge. Combining with 3.7:

3.9. $\sum_{g=0}^{M} (-1)^g \binom{M}{g} \binom{X_1 \pm g}{A} \binom{X_2 \pm g}{B} = 0, \ M > A+B, \ A, \ B \geqslant 0.$

Using induction we can get:

3.10. $\sum_{g=0}^{M} (-1)^g \binom{M}{g} \binom{T+g}{M+K} = \begin{cases} 0, K<0 \\ (-1)^M \binom{T}{K}, K \geqslant 0 \end{cases}; \sum_{g=0}^{M} (-1)^g \binom{M}{g} \binom{T-g}{M+K} = \begin{cases} 0, K<0 \\ \binom{T}{K}, K \geqslant 0 \end{cases}, \ T \pm g \in \mathbb{Z}$

This helps to understand the differential sequence.

3.3 Congruence

P is prime. K_i, D is any integer, D ≠0.

3.11. $(P, D) = 1, \ SUM(P, [K_1, K_2...K_M] : D, [1, 2...M]) \equiv \begin{cases} 0 \pmod{P}, M<P-1 \\ -1 \pmod{P}, M=P-1 \end{cases}.$

Proof. If M=P-1, $SUM(P) \equiv H_1(P-1) \binom{P}{P} \equiv H_1(P-1) \equiv (P-1)!D^{P-1} \equiv -1 \pmod{P}$. □

If a product has a factor that is divisible by P then ignore it and change the factor to its minimum positive residue, then we can obtain many congruence. Wilson's Theorem is a special case.
For Example: $A, B, C \in \mathbb{N}$,

$1^A 2^B + 2^A 3^B + ... + (P-2)^A (P-1)^B \equiv 1^A 3^B + ... + (P-3)^A (P-1)^B + (P-1)^A 1^B \equiv \begin{cases} 0 \pmod{P}, A+B<P-1 \\ -1 \pmod{P}, A+B=P-1 \end{cases}.$

$1^A 2^B 3^C + 2^A 3^B 4^C + ... + (P-3)^A (P-2)^B (P-1)^C \equiv \begin{cases} 0 \pmod{P}, A+B+C<P-1 \\ -1 \pmod{P}, A+B+C=P-1 \end{cases}.$

3.12. $\sum_{0<K_i,K_j<P,K_i \neq K_j} K_1^{\lambda_1} K_2^{\lambda_2} ... K_q^{\lambda_q} \equiv \begin{cases} -1 \pmod{P}, \lambda_1+\lambda_2+...+\lambda_q=P-1 \\ 0 \pmod{P}, \lambda_1+\lambda_2+...+\lambda_q<P-1 \end{cases}, \ \lambda_i \in \mathbb{N}.$

Wolstenholme's Theorem is also a special case. P>3.

1. Wolstenholme's Theorem: $(P-1)! \sum_{n=1}^{P-1} \frac{1}{n} = \sum_{0<K_i,K_j<P,K_i \neq K_j} K_1 K_2 ... K_{P-2} \equiv 0 \pmod{P^2}$.
2. $\sum_{n=1}^{P-1} n^{P-2} \equiv 0 \pmod{P^2}$.

They are two extremes. In fact, there have:

3.13. $\sum_{0<K_i,K_j<P,K_i \neq K_j} K_1^{C_1} K_2^{C_2} ... K_q^{C_q} \equiv 0 \pmod{P^2}, \ C_1 + C_2 + ... + C_q = P-2, \ C_i > 0.$

Proof.
If $X \pmod{P^2} \equiv 0$ and $X+Y \pmod{P^2} \equiv 0$ then $Y \pmod{P^2} \equiv 0$. The Sum has symmetry.
For $\sum AB^{P-3}, A \neq B$:
If $P-A \neq B$ then add AB^{P-3} with $(P-A)B^{P-3}$ to PB^{P-3}.
If $P-A = B$ then add AB^{P-3} with BB^{P-3} to PB^{P-3}.

So $\sum AB^{P-3} + \sum B^{P-2} = xP\sum B^{P-3},\ 0 < x < P \to \sum AB^{P-3} \equiv 0 \pmod{P^2}$.

Similarly:
For $\sum ABC^{P-4},\ A \neq B \neq C$, treat P-A=B, P-A=C, $P - A \neq B,\ C$ separately.
$\sum ABC^{P-4} + x\sum B^2C^{P-4} + y\sum BC^{P-3} \equiv 0 \bmod P^2,\ 0 < x,\ y < P$
$\to \sum ABC^{P-4} + x\sum B^2C^{P-4} \equiv 0 \pmod{P^2},\ 0 < x < P.$
$\sum ABC^{P-4} + \sum B^2C^{P-4} = \sum(\binom{P}{2} - B)BC^{P-4} + \sum B^2C^{P-4} = \binom{P}{2}\sum BC^{P-4} \equiv 0 \pmod{P^2}$
$\to \sum B^2C^{P-4} \equiv 0 \bmod P^2 \to \sum ABC^{P-4} \equiv 0 \bmod P^2.$
Prove the conclusion in a similar way... □

3.14. $E_{P-2}^{P-1} = S_2(2P-3, P-1) \equiv 0 \pmod{P^2};\ E_{P-2}^{P} = S_2(2P-2, P) \equiv 0 \pmod{P^2}$.

For example:
$S_2(7,4) = 350,\ S_2(8,5) = 1050 \equiv 0 \pmod{25};\ S_2(11,6) = 179487,\ S_2(12,7) = 627396 \equiv 0 \pmod{49}$.

4 COMBINATORIAL IDENTITIES

Definition 11. R-FOLD SUM: $\sum_{(r)}^{N} f(k) = \sum_{k_r=1}^{N} \dots \sum_{k_2=1}^{k_3}\sum_{k_1=1}^{k_2} f(k_1) = \sum_{k_r=0}^{N-1} \dots \sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2} f(k_1+1)$.

By the definition of nested sum:

4.1. $\sum_{(r)}^{N} \nabla^p SUM(k, PS, PT) = \nabla^{p-r} SUM(N, PS, PT)$.

4.2. $\sum_{k_r=1}^{N} \dots \sum_{k_2=1}^{k_3}\sum_{k_1=1}^{k_2} \nabla SUM(k_r, PS, [1, 2 \dots M]) = SUM(N, PS, PT = [T_i = i + r - 1])$.

Proof.
$PS1 = [1:0, 1:0 \dots 1:0, PS],\ PT1 = [1, 3 \dots 2(r-1) - 1, 1 + 2(r-1), 2 + 2(r-1) \dots M + 2(r-1)]$.
$B_i = \begin{cases} (T_i - X_{T-1})D_i = 0,\ X_i \in T \\ K_i + X_{K-1}D_i = 1,\ X_i \in K \end{cases} \to H_1(g > M, PS1, PT1) = 0,\ H_1(g \leqslant M, PS1, PT1) = H_1(g, PS, PT)$.
$SUM(N, PS1, PT1) = \sum_{g=0}^{M} H_1(g, PS, PT)\binom{N+r-1}{r+g} = \sum_{k_r=1}^{N} \nabla SUM(k_r, PS, [1, 2 \dots M]) \dots \sum_{k_2=1}^{k_3}\sum_{k_1=1}^{k_2} 1.$ □

4.3. $\sum_{k_x=1}^{N} \dots \sum_{k_2=1}^{k_3}\sum_{k_1=1}^{k_2} \nabla SUM(k_r, PS, [1, 2 \dots M]) = \nabla^{r-x} SUM(N, PS, [T_i = i + r - 1])$.

For example:
$(*) \sum_{k_r=1}^{N} \dots \sum_{k_2=1}^{k_3}\sum_{k_1=1}^{k_2} \binom{k_i}{j} = \frac{1}{j!}\nabla^{i-r} SUM(N+1, [0, -1, -2 \dots -(j-1)], [T_x = x + (i-1)])$.
$H_1(g < j) = 0,\ H_1(j) = \frac{(j+i-1)!}{(i-1)!},\ T_j - j = i - 1 \to$
$(*) = \frac{1}{j!}\nabla^{i-r}\frac{(j+i-1)!}{(i-1)!}\binom{(N+1)+(i-1)}{j+1+(i-1)} = \binom{j+i-1}{i-1}\binom{N+i-(i-r)}{j+i-(i-r)} = \binom{j+i-1}{i-1}\binom{N+r}{j+r}$. [6]

Using induction to prove:

4.4. $\sum_{0 \leqslant n_1 \leqslant \dots \leqslant n_M \leqslant N-1} (K + n_1D_1 + \dots + n_MD_M) = (D_1 + 2D_2 + \dots + MD_M)\binom{N+M-1}{M+1} + K\binom{N+M-1}{M}$.

For example:
$\sum_{0 \leqslant n_1 \leqslant \dots \leqslant n_p \leqslant n} (n_1 + \dots + n_p) = \sum_{n_p=0}^{n}\sum_{n_{p-1}=0}^{n_p} \dots \sum_{n_1=0}^{n_2} (n_1 + \dots + n_p) = \binom{p+1}{2}\binom{n+p}{p+1} = \frac{pn}{2}\binom{n+p}{p}$. [6]
1.2, 1.3 and 1.4 can be used to derive combinatorial identities.

4.5. $\binom{n+A}{A}\binom{n+M+B}{M} =$

1. $\sum_{g=0}^{M}\binom{A+g}{g}\binom{M+B}{M-g}\binom{n+A}{A+g}$.
2. $\sum_{g=0}^{M}(-1)^{M-g}\binom{A+g}{g}\binom{A-B}{M-g}\binom{n+A+g}{A+g}$.

3. $\sum_{g=0}^{M} \binom{A-B}{g}\binom{M+B}{M-g}\binom{n+A+M-g}{A+M}$.

Proof.
$= \frac{1}{A!M!}\nabla SUM(N,[1,2...A,B+1...B+M],[1,2...A+M]) = \frac{1}{M!}\nabla SUM(N,[B+1...B+M],[A+1...A+M])$.
$H_1(g) = \binom{M}{g}(A+1)...(A+g)(B+g+1)...(B+M) = \binom{M}{g} g!\binom{A+g}{g}(M-g)!\binom{B+M}{M-g}$.
Using a similar method to obtain $H_2(g),\ H_3(g)$. ☐

4.6. $\binom{n+X}{A}\binom{n+Y}{M} = \sum_{x=0}^{A}\binom{M+x}{x}\binom{M+X-Y}{A-x}\binom{n+Y}{M+x},\ 0 \leqslant Y \leqslant M.$

Proof.
$= \frac{1}{A!M!}\nabla SUM(N,[X,X-1...X-A+1,Y,Y-1...Y-M+1],[1,2...A+M])$
$= \frac{1}{A!M!}\nabla SUM(N,[1,2...Y,0,-1,-2...-(M-Y)+1,X,X-1...X-A+1],[1,2...A+M])$
$= \frac{Y!}{A!M!}\nabla SUM(N,[0,-1,-2...-(M-Y)+1,X,X-1...X-A+1],[Y+1,Y+2...A+M])$.
If $H_1(g) \neq 0$ then $X_1, X_2...X_{M-Y} \in T \to H_1(g < M-Y) = 0$, Let $C = A+M-Y$.
If $H_1(g \geqslant M-Y) \neq 0$, Number of $X \in K = \binom{C-M+Y}{C-g} \to H_1(g) = \binom{C-M+Y}{C-g}[X+M-Y]_{C-g}[Y+1]^g$.
x:=-(M-Y-g)→ $H_1(M-Y+x) = \frac{A!}{(A-x)!x!}\frac{(M+X-Y)!}{(M+X-Y-A+x)!}\frac{(M+x)!}{Y!}$. ☐

4.7. $\prod_{i=1}^{M}(A+2i+n) = \binom{n+A}{A}^{-1}\sum_{g=0}^{M}(2(M-g)-1)!!\binom{2M-g}{g}[A+1]^g\binom{n+A+g}{A+g},\ A \geqslant 0.$

Proof.
$PS = [A+2,A+4...A+2M],\ PT = [A+1,A+2...A+M],\ PT1 = [1,3...2(M-g)-1]$.
$H_2(g,\sum K) = SUM(g+1,PT1,PT1)) = (2(M-g)-1)!!\binom{2M-g}{g},\ H_2(g,T) = [A+1]^g$.
$\binom{n+A}{A}\prod_{i=1}^{M}(A+2i+n) = \frac{1}{A!}\nabla SUM(N,[1,2...A,PS],[1,2...A,PT]) = \nabla SUM(N,PS,PT)$. ☐

4.8. *SUM(N,[A+1,A+3...A+2M-1],[1,3...2M-1])=* $\sum_{g=0}^{M}[A]^{M-g}(2g-1)!!\binom{M+g}{2g}\binom{N+M-1+g}{M+g}$.

Proof. $H_2(g,\sum T) = SUM(M-g+1,[1,3...2g-1],[1,3...2g-1])) = (2g-1)!!\binom{M+g}{2g},\ H_2(g,K) = [A]^{M-g}$. ☐

4.9. *SUM(N,[A,A+1...A+M-1]:2,[1,3...2M-1])=*$\binom{M+N-1}{M}[A+M+N-2]_M$.

Proof.
$SUM(g+1,[A,A+1...A+M-1-g]:2,[1,3...2(M-g)-1])$
$= H_1(g,\sum K,[A,A+1...A+M-1],[1,2...M]) = \binom{M}{g}[A+M-1]_{M-g}$. ☐

$SUM(N,[1,2...M]:2,[1,3...2M-1]) = M!\binom{N+M-1}{M}^2,\ 1+3+...+(2N-1) = N^2$.

Using 4.9 and 4.5 ,we can obtain:
$\frac{1}{M!}H(g,[A+1,A+2...A+M]:2,[1,3...2M-1]) = H(g,[A+1,A+2...A+M],[M+1,M+2...2M])$.

4.10. $SUM(N,[A+2,A+4...A+2M]:3,[1,3...2M-1]) = \sum_{g=0}^{M}\binom{A+N-1+g}{g}\binom{N+M-1-g}{M-g}\frac{[M+N-g]^M}{2^{M-g}}$.

Proof.
$PS1 = [A+2,A+4...A+2M],\ PT1 = [A+1,A+2...A+M]$.
$H_1(g,PS1,PT1) = [A+1]^g SUM(g+1,[A+2,A+4...A+2(M-g)]:3,[1,3...2(M-g)-1]))$
$= \sum_{k=g}^{M} H_2(g,PS1,PT1)\binom{k}{g}$, 4.7 →$= \sum_{k=g}^{M}(2(M-k)-1)!!\binom{2M-k}{k}[A+1]^k\binom{k}{g}$. ☐

5 MATRIX OF SUM(N)

Consider H(g) as variables, list SUM(N),SUM(N+1)...SUM(N+M), we can obtain a (M+1)×(M+1) matrix.
Let $P = N + T_M - M,\ Q = N - 1$, corresponding to the three forms, define $A_{1,2,3}(P, Q, M) =$

$$\begin{pmatrix} \binom{P}{Q} & \cdots & \binom{P}{Q-M} \\ \vdots & \ddots & \vdots \\ \binom{P+M}{Q+M} & \cdots & \binom{P+M}{Q} \end{pmatrix}, \begin{pmatrix} \binom{P}{Q} & \cdots & \binom{P+M}{Q} \\ \vdots & \ddots & \vdots \\ \binom{P+M}{Q+M} & \cdots & \binom{P+2M}{Q+M} \end{pmatrix}, \begin{pmatrix} \binom{P+M}{Q} & \cdots & \binom{P}{Q-M} \\ \vdots & \ddots & \vdots \\ \binom{P+2M}{Q+M} & \cdots & \binom{P+M}{Q} \end{pmatrix}$$

.

5.1. $\| A_1(P, Q, M) \| = \| A_2(P, Q, M) \| = \| A_3(P, Q, M) \|,\ \| A(P, Q, M) \| = \| A(P, P-Q, M) \|$. [2]

5.2. $\| A(P, 0, M) \| = 1,\ \| A(P, 1, M) \| = \binom{P+M}{1+M},\ \| A(P, Q > 1, M) \| = \prod_{g=0}^{Q-1} \frac{\binom{P+M-g}{1+M}}{\binom{1+M-g}{1+M}}$. [2]

If SUM(N) or ∇ SUM(N) is easy to obtained, then H(g) can be calculated with the Cramer's law. Below, $T_M \geqslant M$.

5.3. $H_1(g) = \sum_{k=1}^{g+1} (-1)^{g+1+k} \binom{T_M-M+1+g}{T_M-M+k} SUM(k) = \sum_{k=1}^{g+1} (-1)^{g+1+k} \binom{T_M-M+g}{T_M-M+k-1} \nabla SUM(k)$.

(1) $\nabla SUM(N, [1, 1...1], [2, 3...M]) = N^M \rightarrow S_2(M, g) = \frac{1}{g!} \sum_{k=0}^{g} (-1)^{g+k} \binom{g}{k} k^M = \frac{1}{g!} \sum_{k=0}^{g} (-1)^k \binom{g}{k} (g-k)^M$.

5.4. $z(k) = \sum_{i=1}^{k} (-1)^{i+k} \binom{k}{i} \nabla SUM(k),\ H_2(g) = \sum_{k=g+1}^{M+1} (-1)^{g+k-1} \binom{k-1}{g} z(k)$.

$\nabla SUM(N, [1, 1...1], [2, 3...M]) = N^M \rightarrow z(k) = \sum_{i=1}^{k} (-1)^{i+k} \binom{k}{i} i^M = k! S_2(M, k) \rightarrow$ 3.1.2.

5.5. $H_3(g) = \sum_{k=1}^{g+1} (-1)^{g+1+k} \binom{2+T_M}{g+1-k} SUM(k) = \sum_{k=1}^{g+1} (-1)^{g+1+k} \binom{1+T_M}{g+1-k} \nabla SUM(k)$.

(2) $\nabla SUM(N, [1, 1...1], [2, 3...M]) = N^M \rightarrow \left\langle {M \atop g} \right\rangle = \sum_{k=1}^{g+1} (-1)^{1+g+k} \binom{M+1}{g+1-k} k^M = \sum_{k=0}^{g} (-1)^k \binom{M+1}{k} (g+1-k)^M$.
(1) and (2) are already known formula.

6 EULERIAN POLYNOMIALS AND BEYOND

In this section, $q \neq 0,\ q \neq 1$. By induction it can be shown that:

6.1. $\sum_{n=0}^{N-1} q^n \binom{n+K}{M} = q^N \sum_{g=0}^{M} (-1)^g \frac{\binom{N+K-1-g}{M-g}}{(q-1)^{g+1}} + \frac{q^{M-K}}{(1-q)^{M+1}}$.

Definition 12. $A_q^M = \sum_{k=0}^{M} (1-q)^{M-k} q^k S_2(M, k) k!,\ A_q^0 = 1,\ A_q^1 = q$.

Table 4: Table of A_q^M

	M=0	M=1	M=2	M=3	M=4	M=5	M=6	OEIS
A_2^M	1	2	6	26	150	1082	9366	A000629
A_3^M	1	3	12	66	480	4368	47712	A123227
A_4^M	1	4	20	132	1140	12324	160020	A201355

$n^M = \nabla SUM(n, [1, 1...1], [2, 3...M]) =$
$\sum_{g=0}^{M} S_2(M, g) g! \binom{n}{g} = \sum_{g=0}^{M} (-1)^{M-g} S_2(M, g) g! \binom{n+g}{g} = \sum_{g=0}^{M} \left\langle {M \atop g} \right\rangle \binom{n+g}{M}$.
$\sum_{n=0}^{N-1} q^n n^M = \sum_{g=0}^{M} S_2(M, g) g! \sum_{n=0}^{N-1} q^n \binom{n}{g} = \sum_{g=0}^{M} S_2(M, g) g! \{ q^N \sum_{k=0}^{g} (-1)^k \frac{\binom{N-1-k}{g-k}}{(q-1)^{k+1}} + \frac{q^g}{(1-q)^{g+1}} \}$

$= \frac{q^N}{(q-1)^{M+1}} \sum_{g=0}^{M} S_2(M,g)g! \sum_{k=0}^{M} (-1)^k \binom{N-1-k}{g-k} (q-1)^{M-k} + \frac{\sum_{g=0}^{M} S_2(M,g)g!(1-q)^{M-g}q^g}{(1-q)^{M+1}}$

$= \frac{q^N}{(q-1)^{M+1}} \sum_{k=0}^{M} (-1)^k(q-1)^{M-k} \sum_{g=0}^{M} S_2(M,g)g! \binom{N-1-k}{g-k} + \frac{A_q^M}{(1-q)^{M+1}}$

$= \frac{q^N}{(q-1)^{M+1}} \sum_{k=0}^{M} (-1)^k(q-1)^{M-k}\nabla^k(N-1)^M + \frac{A_q^M}{(1-q)^{M+1}}(*).$

Use the $From_2$ and $From_3$ of n^M, the first part of (*) keep same, we can obtain:

6.2. $A_q^M = q \sum_{k=0}^{M} (q-1)^{M-k}S_2(M,k)k! = \sum_{g=0}^{M} \langle {M \atop g} \rangle q^{M-g} = \sum_{g=0}^{M} \langle {M \atop g} \rangle q^{1+g}.$

$n^M = \nabla SUM(n,[1,1...1],[1,2...M]) = \sum_{g=0}^{M} S_2(M+1,g+1)g! \binom{n-1}{g}$, we can obtain:

6.3. $A_q^M = \sum_{k=0}^{M} (1-q)^{M-k}q^{k+1}S_2(M+1,k+1)k!,\ M > 0.$

6.4. $A_q^M = \sum_{k=0}^{M} (q-1)^{M-k}S_2(M+1,k+1)k!.$

Proof.

$S_2(M+1,k+1) = \frac{1}{(k+1)!} \sum_{j=0}^{k+1} (-1)^{j+k+1} \binom{k+1}{j} j^{M+1} = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{j+k} \binom{k}{j} (j+1)^M.$

By definition of difference:

$\nabla^k(N-1)^M = \sum_{j=0}^{k} (-1)^j \binom{k}{j}(N-1-j)^M = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \sum_{g=0}^{M} \binom{M}{g}N^g(j+1)^{M-g}(-1)^{M-g}$

$= \sum_{g=0}^{M} (-1)^{M-g-k} \binom{M}{g} N^g \sum_{j=0}^{k} (-1)^{j+k} \binom{k}{j}(j+1)^{M-g} = \sum_{g=0}^{M} (-1)^{M-g-k} \binom{M}{g} N^g S_2(M-g+1,k+1)k! \rightarrow$

$(*) = \frac{q^N}{(q-1)^{M+1}} \sum_{k=0}^{M} (-1)^k(q-1)^{M-k}\{\sum_{g=0}^{M} (-1)^{M-g-k} \binom{M}{g} N^g S_2(M-g+1,k+1)k!\} + \frac{A_q^M}{(1-q)^{M+1}}$

$= \frac{q^N}{(q-1)^{M+1}} \sum_{g=0}^{M} (-1)^{M-g}(q-1)^g \binom{M}{g} N^g \sum_{k=0}^{M} (q-1)^{M-k-g}S_2(M-g+1,k+1)k! + \frac{A_q^M}{(1-q)^{M+1}}(**)$

If N = o then (*) = o$\rightarrow \frac{q^N}{(q-1)^{M+1}}(...) + \frac{A_q^M}{(1-q)^{M+1}} = 0.$ □

$A_q^{M-g} = \sum_{k=0}^{M} (q-1)^{M-k-g}S_2(M-g+1,k+1)k!$, (**) can obtain:

6.5. $\sum_{n=0}^{N-1} q^n n^M = \frac{q^N}{(q-1)^{M+1}} \sum_{g=0}^{M} (-1)^{M-g}(q-1)^g \binom{M}{g} A_q^{M-g}N^g + \frac{A_q^M}{(1-q)^{M+1}}.$

The Eulerian polynomials $A_M(t) : \sum_{i=0}^{\infty} t^i i^M = \frac{tA_M(t)}{(1-t)^{M+1}},\ A_M(t) = \sum_{g=0}^{M-1} \langle {M \atop g} \rangle t^g.$

$|q| < 1,\ \lim_{N\to\infty} \sum_{n=0}^{N-1} q^n n^M = \frac{A_q^M}{(1-t)^{M+1}} \rightarrow A_t^M = tA_M(t)$. There are five expressions for $A_M(t)$.

Eulerian numbers and polynomials is just a special case, we can handler:

$X = T_M - M - p,\ \sum_{n=0}^{N-1} q^n \nabla^p SUM(n+Y,PS,PT)$

$$= \begin{cases} \sum_{g=0}^{M} H_1(g) \sum_{n=0}^{N-1} q^n \binom{n+Y+X}{X+1+g} = \sum_{g=0}^{M} H_1(g)\{q^N \sum_{k=0}^{X+1+g} (-1)^k \frac{\binom{N+Y+X-1-k}{X+1+g-k}}{(q-1)^{k+1}} + \frac{q^{1+g-Y}}{(1-q)^{X+2+g}}\} \\ \sum_{g=0}^{M} H_2(g) \sum_{n=0}^{N-1} q^n \binom{n+Y+X+g}{X+1+g} = \sum_{g=0}^{M} H_2(g)\{q^N \sum_{k=0}^{X+1+g} (-1)^k \frac{\binom{N+Y+X+g-1-k}{X+1+g-k}}{(q-1)^{k+1}} + \frac{q^{1-Y}}{(1-q)^{X+2+g}}\} \\ \sum_{g=0}^{M} H_3(g) \sum_{n=0}^{N-1} q^n \binom{n+Y+X+M-g}{X+1+M} = \sum_{g=0}^{M} H_3(g)\{q^N \sum_{k=0}^{X+1+M} (-1)^k \frac{\binom{N+Y+X+M-g-1-k}{X+1+M-k}}{(q-1)^{k+1}} + \frac{q^{1+g-Y}}{(1-q)^{X+2+M}}\} \end{cases}.$$

6.6. $\sum_{g=0}^{M} H_1(g)(1-q)^{M-g}q^{g+1} = q \sum_{g=0}^{M} H_2(g)(1-q)^{M-g} = \sum_{g=0}^{M} H_3(g)q^{g+1}.$

Here q can take any value, which is magical. $q = 0.5 \to$ 2.3.

Definition 13. $A_q(PS, PT) =$ 6.6.

6.7. $X = T_M - p,\ \sum_{n=0}^{N-1} q^n \nabla^p SUM(n+Y) = \frac{q^N}{(q-1)^{X+2}} \sum_{k=0}^{M} (q-1)^{X-k}(-1)^k \nabla^{p+K-1} SUM(n+Y-2) + \frac{A_q(PS,PT)q^{-Y}}{(1-q)^{X+2}}.$

6.8. $|q| < 1,\ \sum_{n=0}^{\infty} q^n \nabla^p SUM(n+Y, PS, PT) = \frac{A_q(PS,PT)q^{-Y}}{(1-q)^{T_M+2-p}}.$

We can handler 6.8 of SUM(N,[a,a...a]:d,[1,2...M]), SUM(N,[1,1...1,2,2...2...k,k...k],[1,2...kM]). Many results of [7], [8] can be obtained by this.

7 FORMAL CALCULATION OF Q-BINOMIAL

7.1 Concept

q-Binomial: $\begin{bmatrix} N \\ M \end{bmatrix}_q = \frac{(q^N-1)(q^{N-1}-1)...(q^{N-M+1}-1)}{(q^M-1)(q^{M-1}-1)...(q^1-1)},\ q \neq 0,\ 1$, abbreviated as G_M^N.

1. $G_0^N = 1, G_{M<0,M>N}^N = 0, G_M^N = G_{N-M}^N$.
2. $G_M^N = q^M G_M^{N-1} + G_{M-1}^{N-1} = G_M^{N-1} + q^{N-M} G_{M-1}^{N-1}$.
3. $\sum_{n=0}^{N-1} q^n G_M^{n+K} = q^{M-K} G_{M+1}^{N+K}$.
4. $G_K^M = \sum_{w \in \Omega(0^{M-K}, 1^K)} q^{inv(w)}$ [9]. $w_1...w_M$ with M-K(zeros) and K(ones), inv(.) denotes the inversion statistic.

The Formal Calculation use $q^n(K_i + G_1^n D_i)$ instead of $K_i + q^n D_i$.

Definition 14. The definition of ∇_q^p is recursive.

$p \in \mathbb{N},\ \nabla_q^0 f(n) = f(n),\ \sum_{n=0}^{N-1} q^n \nabla_q^1 f(n+1) = f(N),\ \sum_{n=0}^{N-1} q^n f(n+1) = \nabla_q^{-1} f(N).$

Definition 15. The definition of $SUM_q(N) = SUM_q(N, PS, PT)$ is recursive.

$SUM_q(N, [K_1 : D_1], [T_1 = 1]) = \sum_{n=0}^{N-1} q^n (K_1 + G_1^n D_1).$

$SUM_q(N, [K_1 : D_1, K_2 : D_2], [T_1, T_2 = T_1 + 2 - p]) = \sum_{n=0}^{N-1} q^n (K_2 + G_1^n D_2) \nabla^p SUM_q(n+1, [K_1 : D_1], [T_1]).$

7.1. $\sum_{n=0}^{N-1} q^n G_1^n G_M^{n+K},\ M > 0,\ M \geqslant K$

$= q^{2(M-K)+1} G_1^{M+1} G_{M+2}^{N+K} + q^{M-K} G_1^{M-K} G_{M+1}^{N+K}$

$= q^{M-2K-1} G_1^{M+1} G_{M+2}^{N+K+1} + q^{M-K}(G_1^{M-K} - q^{-K-1} G_1^{M+1}) G_{M+1}^{N+K}$

$= (q^{2(M-K)+1} G_1^{M+1} - q^{2M-K+2} G_1^{M-K}) G_{M+2}^{N+K} + q^{M-K} G_1^{M-K} G_{M+2}^{N+K+1}.$

Use this to prove:

7.2. *[2]* $SUM_q(N, PS, PT) =$

$$Form_1 \to \sum_{g=0}^{M} H_1^q(g) G_{N-1-g}^{N+T_M-M} = \sum_{g=0}^{M} H_1^q(g) G_{T_M-M+1+g}^{N+T_M-M},\ B_i = \begin{cases} q^{1+(T_i-T_{i-1})X_{T-1}} G_1^{T_i-X_{K-1}} D_i,\ X_i = T_i \\ q^{(T_i-T_{i-1}-1)X_{T-1}} (K_i + G_1^{X_{T-1}} D_i),\ X_i = K_i \end{cases}$$

$$Form_2 \to \sum_{g=0}^{M} H_2^q(g) G_{N-1}^{N+T_M-M+g} = \sum_{g=0}^{M} H_2^q(g) G_{T_M-M+1+g}^{N+T_M-M+g},\ B_i = \begin{cases} q^{-(T_i-X_{K-1})} G_1^{T_i-X_{K-1}} D_i,\ X_i = T_i \\ K_i - q^{-(T_i-X_{K-1})} G_1^{T_i-X_{K-1}} D_i,\ X_i = K_i \end{cases}$$

$$Form_3 \to \sum_{g=0}^{M} H_3^q(g) G_{N-1-g}^{N+T_M-g} = \sum_{g=0}^{M} H_3^q(g) G_{T_M+1}^{N+T_M-g},\ B_i = \begin{cases} q^{1+(T_i-T_{i-1}-1)X_{T-1}} \{(q^{X_{T-1}} G_1^{T_i} - q^{T_i} G_1^{X_{T-1}}) D_i - K_i q^{T_i}\},\ X_i = T_i \\ q^{(T_i-T_{i-1}-1)X_{T-1}} (K_i + G_1^{X_{T-1}} D_i),\ X_i = K_i \end{cases}.$$

$$H^q(g) = \sum_{X(T)=g} \prod_{i=1}^{M} B_i,\ \lim_{q \to 1} SUM_q(N) = SUM(q),\ \lim_{q \to 1} H^q(g) = H(g).$$

7.2 Property

7.3. $\nabla_q^1 SUM_q(N, PS, [1, 2...M]) = \prod_{i=1}^{M}(K_i + D_i G_1^n)$.

7.4. *In* $SUM_q(N, [...PS...], [...T+1, T+2...T+M...])$, K_i *can exchange orders.*

$Form_2$ is simplest, $X = T_M - M - p$. By using 3, we can obtain:

7.5. $\nabla_q^p SUM_q(N) = \sum_{g=0}^{M} H_1^q(g) G_{X+1+g}^{N+X} q^{-gp} = \sum_{g=0}^{M} H_2^q(g) G_{X+1+g}^{N+X+g} = \sum_{g=0}^{M} H_3^q(g) G_{X+M+1}^{N+X+M-g} q^{-gp}$.

Definition 16. $n_{q-} = q^{-n} G_1^n$, $n_{q+} = q^n G_1^n$, $n_{q-}! = n_{q-}...2_{q-}1_{q-}$, $0_{q-}! = 0$, $n_{q+}!$ is similar.

By using $Form_2$, we can obtain:

7.6. $SUM_q(N, [L1_{q-}, L2_{q-}...LQ_{q-}, PS], [L_1, L_2...L_Q, PT]) = \prod_{i=1}^{Q} L_{i_{q-}} SUM_q(N, PS, PT)$. *So* T_1 *can great than 1,* $T_i \in \mathbb{N}$.

7.7. $SUM_q(N, [T1_{q-}, T2_{q-}...TM_{q-}], [T_1, T_2...T_M]) = \prod_{i=1}^{M} T_{i_{q-}} G_{T_M+1}^{N+T_M}$.

$SUM_q(N, [1_{q-}, 2_{q-}...M_{q-}], [1, 2...M]) \to \sum_{n=0}^{N-1} q^n G_1^{n+1} G_1^{n+2}...G_1^{n+M} = G_1^1 G_1^2...G_1^M G_{M+1}^{N+M}$.
By definition and 4:

7.8. In $H^q(g)$, $\sum_{X_i \in K} \prod q^{X_T} = G_{M-g}^M = G_g^M$.

7.3 Application

7.9. $\sum_{0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant ... \leqslant \lambda_M \leqslant N} q^{\lambda_1+\lambda_2+...+\lambda_M} = G_M^{N+M} = G_N^{N+M}$. *[6]*

Proof. $SUM_q(N+1, [1, 1...1] : 0, [1, 3...2M-1]) \to H_2^q(g > 0) = 0$, $H_2^q(0) = 1 \to \sum_{\lambda_M=0}^{N} q^{\lambda_M} ... \sum_{\lambda_1=0}^{\lambda_2} q^{\lambda_1} = G_M^{N+M}$. □

7.10. $\sum_{1 \leqslant \lambda_1 < \lambda_2 < ... < \lambda_M \leqslant N+M} q^{\lambda_1+\lambda_2+...+\lambda_M} = q^{\binom{M+1}{2}} G_M^{N+M}$.

Proof. $\sum_{\lambda_M=0}^{N} q^{M+\lambda_M} ... \sum_{\lambda_2=0}^{\lambda_3} q^{2+\lambda_2} \sum_{\lambda_1=0}^{\lambda_2} q^{1+\lambda_1} = q^{1+2+...+M} \sum_{\lambda_M=0}^{N} q^{\lambda_M} ... \sum_{\lambda_2=0}^{\lambda_3} q^{\lambda_2} \sum_{\lambda_1=0}^{\lambda_2} q^{\lambda_1}$. □

7.11. $\sum_{A \leqslant \lambda_1 < \lambda_2 < ... < \lambda_M \leqslant B} q^{\lambda_1+\lambda_2+...+\lambda_M} = q^{\binom{M+1}{2}+(A-1)M} G_M^{B-A+1}$, $A, B \in \mathbb{Z}$.

By simply following the definition of product, we can obtain:

7.12. $\prod_{i=1}^{M}(1 + q^{A+i} z) = \sum_{g=0}^{M} q^{\binom{g+1}{2}+Ag} G_g^M z^g$.

A=-1 or 1,it's **q-Binomial Theorem**: $\prod_{i=1}^{M}(1 + q^{i-1} z) = \sum_{g=0}^{M} q^{\binom{g}{2}} G_g^M z^g$, $\prod_{i=1}^{M}(1 + q^i z) = \sum_{g=0}^{M} q^{\binom{g+1}{2}} G_g^M z^g$

.

If $T_i + 1 = T_{i+1}$, $D_i = 1$, $K_i = K_{i_{q-}}$:
B_i of $H_{1,2,3}^q(g) = \begin{cases} X_i = T_i \\ X_i = K_{i_{q-}} \end{cases} = \begin{cases} q^{X_T} G^{T_i - X_{K-1}} \\ q^{-K_i} G^{K_i + X_{T-1}} \end{cases}, = \begin{cases} q^{-(T_i - X_{K-1})} G^{T_i - X_{K-1}} \\ q^{-K_i} G^{K_i - T_i + X_{K-1}} \end{cases}, = \begin{cases} q^{1+X_T} G^{T_i - K_i - X_{T-1}} \\ q^{-K_i} G^{K_i + X_{T-1}} \end{cases}$.
It's similar to 1.1. Replace each B_i to $G_1^{B_i}$ in $H(g, [K_1, K_2...K_M], PT)$ and multiply by $q^?$ to obtain $H^q(g)$.
If there is another $K_i + 1 = K_{i+1}$, 7.11 can be used to obtain general formulas.
Expansion of 2:

7.13. $G_{M+1}^{N+M} = \sum_{g=0}^{M} q^{(g+1)g} G_g^M G_{1+g}^N$, $G_{M+1+P}^{N+M+P} = \sum_{g=0}^{M} q^{(g+1+P)g} G_g^M G_{P+1+g}^{N+P}$.

Proof.

$SUM_q(N,[1_{q-},2_{q-}...M_{q-}],[1,2...M]) = M_{q-}!G_{M+1}^{N+M} = \sum_{g=0}^{M} H_1^q(g)G_{1+g}^N = SUM_q(N,[M_{q-},...,1_{q-}],PT).$

$H_1^q(g,T) = g_{q+}!,\ H_1^q(g,\sum K) = G_1^M G_1^{M-1}...G_1^{g+1} q^{-(M+1)(M-g)} \sum_{1\leqslant\lambda_1<...<\lambda_{M-g}\leqslant M} q^{\lambda_1+...+\lambda_{M-g}}.$

$$\frac{H_1^q(g)}{M_{q-}!} = \frac{q^{\binom{g+1}{2}} q^{-(M+1)(M-g)} \sum_{1\leqslant\lambda_1<...<\lambda_{M-g}\leqslant M} q^{\lambda_1+...+\lambda_{M-g}}}{q^{-(1+2+...+M)}} = \frac{q^{\binom{g+1}{2}} q^{-(M+1)(M-g)} q^{\binom{M-g+1}{2}} G_{M-g}^M}{q^{-\binom{M+1}{2}}} = q^{(g+1)g} G_g^M.$$ □

7.14. $G_M^N = \sum_{g=0}^{M} (-1)^g q^{\frac{g^2}{2}} G_g^M \begin{bmatrix} N+M-g \\ 2M \end{bmatrix}_{q^{\frac{1}{2}}}.$

Proof.

$SUM_q(N+1,[q^1:(q-1)q^1, q^2:(q-1)q^2...q^M:(q-1)q^M],[1,3...2M-1])$

$= \sum_{\lambda_M=0}^{N} q^{2\lambda_M+M} ... \sum_{\lambda_2=0}^{\lambda_3} q^{2\lambda_2+2} \sum_{\lambda_1=0}^{\lambda_2} q^{2\lambda_1+1} = q^{\binom{M+1}{2}} \sum_{0\leqslant\lambda_1\leqslant...\leqslant\lambda_M\leqslant N} q^{2(\lambda_1+...+\lambda_M)} = q^{\binom{M+1}{2}} \begin{bmatrix} N+M \\ M \end{bmatrix}_{q^2}.$

B_i of $H_3^q(g) = \begin{cases} q^{1+X_{T-1}}\{(q^{X_{T-1}} G_1^{T_i} - q^{T_i} G_1^{X_{T-1}})D_i - K_i q^{T_i}\} = -q^{i+1+2X_{T-1}} = -q^{i-1+2X_T},\ X_i = T_i \\ q^{X_{T-1}}(K_i + G_1^{X_{T-1}} D_i) = q^{X_{T-1}}(q^i + G_1^{X_{T-1}} q^i(q-1)) = q^{2X_{T-1}+i} = q^{i+2X_T},\ X_i = K_i \end{cases}.$

Factor q^i is present in all of them here, so $q^{(1+2+...+M)}$ can be extracted.

$H_3^q(g) = (-1)^g q^{(1+2+...+M)-g+2(1+2+...+g)} \sum_{X_i\in K} \prod q^{2X_T} = (-1)^g q^{\binom{M+1}{2}+g^2} \begin{bmatrix} M \\ g \end{bmatrix}_{q^2}.$

$q^{\binom{M+1}{2}} \begin{bmatrix} N+M \\ M \end{bmatrix}_{q^2} = \sum_{g=0}^{M} H_3^q(g) G_{2M}^{N+2M-g} = q^{\binom{M+1}{2}} \sum_{g=0}^{M} (-1)^g q^{g^2} \begin{bmatrix} M \\ g \end{bmatrix}_{q^2} G_{2M}^{N+2M-g}.$

□

Definition 17. ${}_qE_M^N = \sum_{1\leqslant\lambda_1\leqslant\lambda_2\leqslant..\leqslant\lambda_M\leqslant N} G_1^{\lambda_1} G_1^{\lambda_2} ... G_1^{\lambda_M} = S_2^q(N+M,N).$

7.15. $(G_1^N)^M = (1+q+...+q^{N-1})^M = \sum_{g=1}^{M} g_{q+}! S_2^q(M,g) G_g^N q^{-g}.$

Proof.

$\nabla_q^1 SUM(N,[1_{q-},1_{q-}...1_{q-}],[1,2...M]) = \prod(\frac{1}{q} + \frac{q^n-1}{q-1}) = q^{-M}(G_1^{n+1})^M = q^{-M}(G_1^N)^M$

$= q^{-1}\nabla_q^1 SUM(N,[1_{q-},1_{q-}...1_{q-}],[2,3...M]) = q^{-1}\sum_{g=0}^{M-1} H_1^q(g) G_{1+g}^N q^{-g}.$

$B_i = \begin{cases} q^{1+X_{T-1}} G_1^{(i+1)-X_{K-1}} = q^{X_T} G_1^{1+X_T} = q^{-1} q^{1+X_T} G_1^{1+X_T},\ X_i\in T \\ q^{-1} + G_1^{X_{T-1}} = q^{-1} G_1^{1+X_{T-1}} = q^{-1} G_1^{1+X_T},\ X_i\in K \end{cases}.$

$H_1^q(g,T) = q^{-(g+1)}(g+1)_{q+}!, H_1^q(g,\sum K) = q^{-(M-1-g)}\, {}_qE_{M-1-g}^{g+1}.$

$(G_1^N)^M = q^{M-1} \sum_{g=0}^{M-1} q^{-(g+1)}(g+1)_{q+} q^{-(M-1-g)} S_2^q(M,g+1)! G_{1+g}^N q^{-g}.$ □

By using $PS = [K_i+1 : D_i(q-1)]$ and PT=[1,2...M], we can obtain:

7.16. $\sum_{n=0}^{N-1} q^n \prod_{i=1}^{M} (K_i + D_i q^n) = \sum_{g=0}^{M} H_1^q(g) G_{1+g}^N,\ B_i = \begin{cases} q^{X_T}(q^{X_T}-1)D_i,\ X_i\in T \\ K_i + q^{X_T} D_i,\ X_i\in K \end{cases}.$

$K_i = 1,\ D_i = q^{-i}z \to \nabla SUM(M+1) = \prod_{i=1}^{M} (1+q^{M-i}z) = \sum_{g=0}^{M} H_1^q(g) G_g^M q^{-g} (*).$

$B_i = \begin{cases} q^{X_T}(q^{X_T}-1)q^{-i}z,\ X_i = T_i \\ 1+q^{X_T}q^{-i}z = 1+q^{-1-X_{K-1}}z = 1+q^{-X_K}z,\ X_i = K_i \end{cases}.$

$H_1^q(g) = q^{1+2+...+g} z^g (q^1-1)...(q^g-1)[\sum_{-M\leqslant\lambda_1<...<\lambda_g\leqslant-1} q^{\lambda_1+...+\lambda_g}](1+q^{-1}z)...(1+q^{-(M-g)}z)$

$(*) = \sum_{g=0}^{M} q^{\binom{g+1}{2}} z^g (q^1-1)...(q^g-1) q^{\binom{g+1}{2}-g(M+1)} G_g^M (1+q^{-1}z)...(1+q^{-(M-g)}z) G_g^M q^{-g}.$

$= \sum_{g=0}^{M} q^{g^2-g(M+1)-\binom{M-g+1}{2}} z^g (q^M-1)...(q^{M-g+1}-1)(q^1+z)...(q^{(M-g)}+z) G_g^M.$

Combining q-Binomial Theorem, we can obtain:

7.17. $\sum_{g=0}^{M} q^{\binom{g}{2}} z^g G_g^M (q^M - 1)...(q^{M-g+1} - 1)(q^{M-g} + z)...(q^1 + z) = q^{\binom{M+1}{2}} \sum_{g=0}^{M} q^{\binom{g}{2}} z^g G_g^M.$

Using 7.16 and induction, we can obtain:

7.18. $\sum_{n=0}^{N-1} \prod_{i=1}^{M} (K_i + D_i q^n) = \sum_{g=1}^{M} f(g) G_g^N + N \prod K_i, f(g) = \sum \prod B_i, \ B_i = \begin{cases} 1, X_i \in T, \ X_{T-1} = 0 \\ q^{X_{T-1}}(q^{X_{T-1}} - 1)D_i, \ X_i \in T, X_{T-1} > 0 \\ K_i, X_i \in K, \ X_{T-1} = 0 \\ K_i + q^{X_{T-1}-1} D_i, X_i \in K, \ X_{T-1} > 0 \end{cases}.$

7.19. $q^{Mn} = \sum_{g=0}^{M} G_g^M \prod_{i=0}^{g-1} (q^n - q^i) = q^{-M} \sum_{g=0}^{M} q^g \begin{bmatrix} M \\ g \end{bmatrix}_{q^{-1}} \prod_{i=1}^{g} (q^n - q^{-i}) = \sum_{g=0}^{M} G_g^M (-1)^g q^{\binom{g}{2}} G_M^{n+M-g}.$

Proof.
$q^{Mn} = \nabla_q^1 SUM(N, [1, 1...1] : q - 1, [1, 2...M])$
$= \sum_{g=0}^{M} H_1^q(g) G_g^n q^{-g} = \sum_{g=0}^{M} H_2^q(g) G_g^{n+g} = \sum_{g=0}^{M} H_3^q(g) G_M^{n+M-g} q^{-g}.$
$B_i = \begin{cases} q^{1+X_{T-1}} G_1^{i-X_{K-1}} (q-1) = q^{X_T}(q^{X_T} - 1), \ X_i \in T \\ 1 + G_1^{X_{T-1}} (q-1) = q^{X_{T-1}} = q^{X_T}, \ X_i \in K \end{cases}, \ H_1^q(g, T) = (q-1)^g g_{q+}!, \ H_1^q(g, \sum K) = G_g^M.$
$B_i = \begin{cases} q^{-(1+X_{T-1})}(q^{1+X_{T-1}} - 1) = q^{-X_T}(q^{X_T} - 1), \ X \in T \\ q^{-(1+X_{T-1})} = q^{-1} q^{-X_T}, \ X \in K \end{cases}, \ H_2^q(g, T) = (q-1)^g g_{q-}!, \ H_2^q(g, \sum K) = q^{-(M-g)} \begin{bmatrix} M \\ g \end{bmatrix}_{q^{-1}}.$
$B_i = \begin{cases} -q^{1+X_{T-1}} = -q^{X_T}, \ X \in T \\ q^{X_{T-1}} = q^{X_T}, \ X \in K \end{cases}, \ H_3^q(g, T) = (-1)^g q^{\binom{g+1}{2}}, \ H_3^q(g, \sum K) = G_g^M.$

□

7.20. $\sum_{k_r=1}^{N} ... \sum_{k_2=1}^{k_3} \sum_{k_1=1}^{k_2} q^{k_1+k_2+...+k_r} \nabla_q^p SUM_q(k_1, PS, PT) = \nabla_q^{p-r} SUM_q(N, PS, PT).$

7.21. $\sum_{k_x=1}^{N} ... \sum_{k_2=1}^{k_3} \sum_{k_1=1}^{k_2} q^{k_1+k_2+...+k_x} \nabla_q^1 SUM_q(k_r, PS, [1, 2...M]) = \nabla_q^{r-x} SUM_q(N, PS, [T_i = i + (r-1)]).$

7.22. $\sum_{0 \leqslant n_1 \leqslant ... \leqslant n_M \leqslant N-1} q^{n_1+n_2+...+n_M} (K + G_1^{n_1} D_1 + ... + G_1^{n_M} D_M)$
$= \{q^1 D_M + q^2 (D_M + D_{M+1}) + ... + q^M (D_M + D_{M+1} + ... + D_1)\} G_{M+1}^{N+M-1} + K G_M^{N+M-1}.$

7.23. $(K + qD)^M = (q-1) D \sum_{g=0}^{M-1} (K + D)^g (K + qD)^{M-1-g} + (K + D)^M.$

Proof. $PS = [K + D, K + D...K + D] : (q-1)D, PT = [1, 2...M].$
$B_i = \begin{cases} q^{X_T}(q^{X_T} - 1)D, \ X \in T \\ K + D q^{X_T}, \ X \in K \end{cases}, H_1^q(1) = q^1(q^1 - 1) D \sum_{a+b=M-1, a,b \geqslant 0} (K + D)^a (K + qD)^b.$
$H_1^q(0) = (K + D)^M.$
$SUM_q(2) - SUM_q(1) = H_1^q(1) + H_1^q(0) G_1^2 - H_1^q(0) = H_1^q(1) + q H_1^q(0) = q(K + qD)^M.$ □

7.4 Relationships between $H^q(g)$

7.24. $PT = [1, 2...M], \ H_1^q(g) = q^{g(g+1)} \sum_{k=g}^{M} H_2^q(k) G_g^k.$ [3]

7.25. $PT = [1, 2...M], H_1^q(g) = \sum_{k=0}^{g} H_3^q(k) G_{M-g}^{M-k} q^{(g+1)(g-k)}.$

Proof. Direct verification when M=1, assuming M holds.
$(*) H_1^q(PS1, PT1, g) = q^g G_1^g H_1^q(g-1) + (K_{M+1} + G_1^g D_{M+1}) H_1^q(g)$
$= q^g G_1^g H_1^q(g-1) \sum_{x=0}^{M} H_3^q(x) G_{M-g+1}^{M-x} q^{g(g-1-x)} + (K_{M+1} + G_1^g D_{M+1}) \sum_{x=0}^{M} H_3^q(x) G_{M-g}^{M-x} q^{(g+1)(g-x)}.$
$(**) H_1^q(PS1, PT1, g) = \sum_{x=0}^{M+1} H_3^q(PS1, PT1, x) G_{M+1-g}^{M+1-x} q^{(g+1)(g-x)}.$
$= \sum_{x=0}^{M} H_3^q(x) \{\frac{q^{M+2} - q^{X+1}}{q-1} D_{M+1} - K_{M+1} q^{M+2}\} G_{M+1-g}^{M-x} q^{(g+1)(g-x-1)}$

$+\sum_{x=0}^{M} H_3^g(x)(K_{M+1}+G_1^x D_{M+1})G_{M+1-g}^{M+1-x}q^{(g+1)(g-x)}$.
Items containing K_{M+1}:
$K_{M+1}G_{M+1-g}^{M+1-x}q^{(g+1)(g-x)} - q^{M+2}K_{M+1}G_{M+1-g}^{M-x}q^{(g+1)(g-x-1)} = K_{M+1}G_{M-g}^{M-x}q^{(g+1)(g-x)}$.

Items does not contain K_{M+1} :
In $(*) = q^g G_1^g D_{M+1} G_{M-g+1}^{M-x} q^{g(g-1-x)} + G_1^g D_{M+1} G_{M-g}^{M-x} q^{(g+1)(g-x)}$.
Divide by $D_{M+1}q^g q^{g(g-1-x)}(q-1)^{-1} = (q^g-1)G_{M+1-g}^{M-x} + (q^g-1)G_{M-g}^{M-x}q^{g-x} = (q^g-1)G_{M+1-g}^{M+1-x}$.
In $(**) = \frac{q^{M+2}-q^{x+1}}{q-1} D_{M+1} G_{M+1-g}^{M-x} q^{(g+1)(g-x-1)} + G_1^x D_{M+1} G_{M+1-g}^{M+1-x} q^{(g+1)(g-x)}$.
Divide by $D_{M+1}q^g q^{g(g-1-x)}(q-1)^{-1}$
$= (q^{M+1-x}-1)G_{M+1-g}^{M-x} + (q^g - q^{g-x})G_{M+1-g}^{M+1-x} = (q^g-1)G_{M+1-g}^{M+1-x}$. □

7.26. $(q^M-1)...(q^{M-K+1}-1) = \sum_{g=0}^{K}(-1)^g q^{\binom{K-g+1}{2}} G_g^M G_{M-K}^{M-g} = \sum_{g=0}^{K}(-1)^g q^{\binom{K-g+1}{2}+(K-g)(M-K)} G_g^K$.

Proof. 7.25 and 7.19 can obtain the first equation, the second equation is derived from 7.12. □

Similarly, using induction to prove:

7.27. $PT = [1,2...M],\ H_2^q(g) = \sum_{k=g}^{M}(-1)^{k+g}G_g^k q^{-k(k+1)+\binom{k-g}{2}}H_1^q(k)$.

7.28. $PT = [1,2...M], H_3^q(g) = \sum_{k=0}^{g}(-1)^{k+g}G_{M-g}^{M-k}q^{(g+1)(g-k)-\binom{g-k}{2}}H_1^q(k)$.

7.29. *PT = [1,2...M] and* $D_i = 1$, $H_1^q(g,\sum K) = F_{M-g}^K \times {}_qE_0^g + F_{M-g-1}^K \times {}_qE_1^g + ... + F_0^K \times {}_qE_{M-g}^g$.

This indicates any $\sum_{g=0}^{M} a_g G_{Y+g}^X$ can be converted to $\frac{a_M}{M_{q+}!}\nabla^A SUM_q(N+B,[K_1,K_2...K_M],[1,2...M])$.
Using 7.27 , 7.28 and 7.5, we can obtain the inversion formulas.

7.30. *If* $\sum_{g=0}^{M} a_g G_{Y+g}^X = \sum_{g=0}^{M} b_g G_{Y+g}^{X+g} = \sum_{g=0}^{M} c_g G_{Y+M}^{X+M-g}$:

1. $a_g q^{(1-Y)g} = q^{g(g+1)}\sum_{k=g}^{M} b_k G_g^k = \sum_{k=0}^{g} c_k G_{M-g}^{M-k} q^{(g+1)(g-k)} q^{(1-Y)k}$.
2. $b_g = \sum_{k=g}^{M}(-1)^{k+g}G_g^k q^{-k(k+1)+\binom{k-g}{2}} a_k q^{(1-Y)k}$.
3. $c_g q^{(1-Y)g} = \sum_{k=0}^{g}(-1)^{k+g}G_{M-g}^{M-k}q^{(g+1)(g-k)-\binom{g-k}{2}} a_k q^{(1-Y)k}$.

7.5 Merge and Expand

7.31. *Necessary and sufficient conditions for merging,* $0 < K \leqslant M$:

1. $\sum_{n=0}^{M} H(n)G_{Y+n}^X \to \sum_{n=0}^{M-K}(...)G_{Y+K+n}^{X+K} : \sum_{x=g}^{M}(-1)^x G_g^x q^{-x(x+1)+\binom{x-g}{2}}H(x)q^{(1-Y)x} = 0,\ 0 \leqslant g < K$.
2. $\sum_{n=0}^{M} H(n)G_{Y+n}^X \to \sum_{n=0}^{M-K}(...)G_{Y+K+n}^{X+K} : \sum_{x=0}^{g}(-1)^x G_{M-g}^{M-x} q^{(g+1)(g-x)-\binom{g-x}{2}}H(x)q^{(1-Y)x} = 0,\ 0 \leqslant M-g < K$.
3. $\sum_{n=0}^{M} H(n)G_{Y+n}^{X+n} \to \sum_{n=0}^{M-K}(...)G_{Y+K+n}^{X+K+n} : \sum_{x=g}^{M} H(x)G_g^x = 0,\ 0 \leqslant g < K$.
4. $\sum_{n=0}^{M} H(n)G_{M+Y}^{X-n} \to \sum_{n=0}^{M-K}(...)G_{M+Y-K}^{X-K-n} : \sum_{x=0}^{g} H(x)G_{M-g}^{M-x}q^{(g+1)(g-x)}q^{(1-Y)x} = 0, 0 \leqslant M-g < K$

Using Y=1 and 7.13, we can obtain:

7.32. $\sum_{x=0}^{M}(-1)^x G_g^x G_x^M q^{\binom{x-g}{2}} = 0,\ 0 \leqslant g < M,\ \sum_{x=0}^{M}(-1)^x G_x^M q^{\binom{x}{2}} = 0$.

7.33. *P=A+T+1-Y, M > A ⩾ 0,*
$$\sum_{g=0}^{M} G_g^M G_A^{A+T+g} G_{Y+g}^{X} q^{g(g+1+T-P)} = \sum_{x=0}^{A} G_{x+T}^{A+T} G_{M+T}^{M+T+x} G_{Y+M-A+x}^{X+M-A} q^{x(x+1+T-P)}.$$

Proof.

$SUM_q(N,[(T+1)_{q-},(T+2)_{q-}...(T+M)_{q-}],[T+A+1...T+A+M])$, B_i of $H_1^q(g) = \begin{cases} q^{X_T} G_1^{T+A+i-X_T}, & X_i=T_i \\ q^{-(T+i)} G_1^{T+i+X_T}, & X_i=K_i \end{cases}$

$= \sum_{g=0}^{M} G_{A+T+1+g}^{X} q^{\frac{g(1+g)}{2}} G_1^{T+A+1}...G_1^{T+A+g} \times G_1^{T+M}...G_1^{T+g+1} q^{-(M-g)(T+M+1)} \sum_{1\leqslant\lambda_1<...<\lambda_{M-g}\leqslant M} q^{\sum\lambda_i}$

$= \frac{(q^A-1)...(q-1)}{(q-1)^M} \sum_{g=0}^{M} G_{A+T+1+g}^{X} q^{\frac{g(1+g)}{2}-(M-g)(T+M+1)+\frac{(M-g+1)(M-g)}{2}} \times (q^{T+M}-1)...(q^{T+A+1}-1) G_A^{A+T+g} G_g^M$

$= q^{-(T+A+1)} G_1^{T+A+1}...q^{-(T+M)} G_1^{T+M} SUM_q(N,[(T+1)_{q-}....(T+A)_{q-}],[T+M+1...T+M+A])$

$= q^{\frac{-(M-A)(T+A+1+T+M)}{2}} G_1^{T+A+1}...G_1^{T+M} \sum_{x=0}^{A} G_{M+T+1+x}^{X+M-A} q^{\frac{x(1+x)}{2}-(A-x)(T+A+1)+\frac{(A-x+1)(A-x)}{2}}$

$\times G_1^{T+M+1}..G_1^{T+M+x} \times G_1^{T+A}...G_1^{T+x+1} G_x^A.$

$\to \sum_{g=0}^{M} G_{A+T+1+g}^{X} q^{\frac{g(1+g)-(M-g)(g+M+1+2T)}{2}} G_A^{A+T+g} G_g^M$

$= q^{\frac{-(M-A)(T+A+1+T+M)}{2}} \sum_{x=0}^{A} G_{M+T+1+x}^{X+M-A} q^{\frac{x(1+x)-(A-x)(x+A+1+2T)}{2}} G_{M+T}^{M+T+x} G_{x+T}^{A+T}$

$\to \sum_{g=0}^{M} G_g^M G_A^{A+T+g} G_{A+T+1+g}^{X} q^{g(g+1+T)} = \sum_{x=0}^{A} G_{x+T}^{A+T} G_{M+T}^{M+T+x} G_{M+T+1+x}^{X+M-A} q^{x(x+1+T)}.$ □

The difference between this and 3.8 is that M>A is required.

$A=0 \to \sum_{g=0}^{M} G_g^M G_{T+1+g}^{N} q^{g(g+1+T)} = \sum_{g=0}^{M} G_g^M G_{T+1+M-g}^{N} q^{(M-g)(M-g+1+T)} = G_{T+1+M}^{N+M}$

$K=T+1+M \to \sum_{g=0}^{K} G_g^M G_{K-g}^{N} q^{(M-g)(K-g)} = G_K^{N+M}$. It's the **q-Vandermorde Theorem**.

$0\leqslant B+A<M$, 7.31 $\to \sum_{g=0}^{M} G_g^M G_A^{A+T+g} G_B^g q^{-gA+\binom{g-B}{2}}(-1)^g = 0 \to \sum_{g=0}^{M} G_g^M G_A^{X_1+g} G_B^g q^{\binom{g}{2}-g(A+B)}(-1)^g = 0.$

A and B have symmetry, 2→.

7.34. $\sum_{g=0}^{M} G_g^M G_A^{X_1+g} G_B^{X_2+g} q^{\binom{g}{2}-g(A+B)}(-1)^g = 0,\ 0\leqslant B+A<M.$

7.35. $\sum_{g=0}^{M} (-1)^g G_g^M G_{M+K}^{X+g} q^{\binom{M+1-g}{2}+(M-g)K} = (-1)^M G_K^X,\ X+g\geqslant 0,\ K\geqslant 0.$

Proof.

$SUM_q(N,[(T+1)_{q-},(T+2)_{q-}...(T+M)_{q-}],[T+K+M+1,T+K+M+2...T+K+2M]).$

$H_1^q(g) = q^{\frac{g(1+g)}{2}-(M-g)(T+M+1)+\binom{M-g+1}{2}} \times G_1^{T+K+M+1}...G_1^{T+K+M+g} \times G_1^{T+M}...G_1^{T+1+g} \times G_1^{T+M+1} G_g^M.$

$H_2^q(0) = \sum_{g=0}^{M} (-1)^g H_1^q(g) q^{-g(g+1)+\frac{g(g-1)}{2}-(T+K+M)g} = (-1)^M q^{-M(T+1+T+M)} G_1^{K+M} G_1^{K+M-1}...G_1^{K+1}.$

$\to (-1)^M G_1^{K+M}...G_1^{K+1} = \sum_{g=0}^{M} (-1)^g q^{\binom{M-g+1}{2}+(M-g)K} \times G_1^{T+K+M+g}...G_1^{T+K+M+1} \times G_1^{T+M}...G_1^{T+1+g} \times G_g^M.$

$\to (-1)^M G_K^{T+K+M} = \sum_{g=0}^{M} (-1)^g q^{\binom{M-g+1}{2}+(M-g)K} \times G_{M+K}^{T+K+M+g} G_g^M.$ □

7.6 Matrix of $SUM_q(N)$

Let $P=N+T_M-M, Q=N-1$, corresponding to the three forms, define $A_{1,2,3}^q(P,Q,M) =$

$$\begin{pmatrix} G_Q^P & \cdots & G_{Q-M}^P \\ \vdots & \ddots & \vdots \\ G_{Q+M}^{P+M} & \cdots & G_Q^{P+M} \end{pmatrix}, \begin{pmatrix} G_Q^P & \cdots & G_Q^{P+M} \\ \vdots & \ddots & \vdots \\ G_{Q+M}^{P+M} & \cdots & G_{Q+M}^{P+2M} \end{pmatrix}, \begin{pmatrix} G_Q^{P+M} & \cdots & G_{Q-M}^P \\ \vdots & \ddots & \vdots \\ G_{Q+M}^{P+2M} & \cdots & G_Q^{P+M} \end{pmatrix}$$

.

7.36. $\| A_2^q(P,Q,M) \| = \| A_2^q(P,P-Q,M) \| = \frac{G_Q^P}{G_0^P}\frac{G_{Q+1}^{P+2}}{G_1^{P+2}}\ldots\frac{G_{Q+M}^{P+2M}}{G_M^{P+2M}} q^{(P+1)+2(P+2)+\ldots+M(P+M)}$.

$\| A_2^q(P,0,M) \| = q^{(P+1)+2(P+2)+\ldots+M(P+M)}$, $\| A_2^q(P,1,M) \| = G_{M+1}^{P+M} q^{(P+1)+2(P+2)+\ldots+M(P+M)}$.

7.37. $\| A_{1,3}^q(P,Q,M) \| = \frac{G_Q^P}{G_0^P}\frac{G_{Q+1}^{P+2}}{G_1^{P+2}}\ldots\frac{G_{Q+M}^{P+2M}}{G_M^{P+2M}} q^{Q\binom{M+1}{2}}$.

8 MULTI-PARAMETER FORMAL CALCULATION

2-parameters Formal Calculation calculate nested sum of $K_i + \binom{n}{1} D_{1,i} + \binom{n}{2} D_{2,i}$.
$SUM(N,[K_1],[T_{1,1}=1:D_{1,1}],[T_{2,1}=1:D_{2,1}]) = \sum_{n=0}^{N-1}(K_1 + D_{1,1}n + D_{2,1}\binom{n}{2})$.
$SUM(N,[K_1,K_2],[T_{1,1}:D_{1,1},T_{1,2}=T_{1,1}+2-p:D_{1,2}],[T_{2,1}=T_{1,1}:D_{2,1},T_{2,2}=T_{1,2}:D_{2,2}])$
$= \sum_{n=0}^{N-1}(K_2 + D_{1,2}n + D_{2,2}\binom{n}{2})\nabla^p SUM(N,[K_1],[T_{1,1}:D_{1,1}],[T_{2,1}:D_{2,1}])$.
The definition of $SUM(N,PS,PT_1,PT_2)$ is recursive. There is always $T_i = T_{1,i} = T_{2,i}$.
Use the Form: $(K_1 + T_{1,1} + T_{2,1})(K_2 + T_{1,2} + T_{2,2})\ldots(K_M + T_{1,M} + T_{2,M})$.
Use T_1 to represent the set $\{T_{1,1}, T_{1,2}\ldots T_{1,M}\}$, T_2 to represent the set $\{T_{2,1}, T_{2,2}\ldots T_{2,M}\}$.

Definition 18. X(PT_1)=Number of $\{X_1\ldots X_M\} \in T_1$, X($PT_2$)=Number of $\{X_1\ldots X_M\} \in T_2$, X(PT)=X($PT_1$)+2X($PT_2$).

Definition 19. X_{PT1}=Number of $\{X_1\ldots X_i\} \in T_1$, X_{PT2}=Number of $\{X_1\ldots X_i\} \in T_2$, $X_{PT} = X_{PT1} + 2X_{PT2}$.

8.1. $SUM(N,PS,PT_1,PT_2)$

$$= \sum_{g=0}^{2M} H_1(g)\binom{N+T_M-M}{T_M-M+1+g},\ B_i = \begin{cases} K_i + X_{PT}D_{1,i} + \binom{X_{PT}}{2}D_{2,i},\ X_i = K_i \\ (T_i - i + X_{PT})D_{1,i} + (T_i - i + X_{PT})(X_{PT}-1)D_{2,i},\ X_i = T_{1,i} \\ \binom{T_i - i + X_{PT}}{2}D_{2,i},\ X_i = T_{2,i} \end{cases}$$

$$= \sum_{g=0}^{2M} H_2(g)\binom{N+T_M-M+g}{T_M-M+1+g},\ B_i = \begin{cases} K_i - (T_i - i + X_{PT}+1)D_{1,i} + \binom{T_i - i + X_{PT}+2}{2}D_{2,i},\ X_i = K_i \\ (T_i - i + X_{PT})D_{1,i} - (T_i - i + X_{PT})(T_i - i + X_{PT}+1)D_{2,i},\ X_i = T_{1,i} \\ \binom{T_i - i + X_{PT}}{2}D_{2,i}, X_i = T_{2,i} \end{cases}$$

$$= \sum_{g=0}^{2M} H_3(g)\binom{N+T_M+M-g}{T_M+M+1},\ B_i = \begin{cases} K_i + X_{PT-1}D_{1,i} + \binom{X_{PT-1}}{2}D_{2,i},\ X_i = K_i \\ -2K_i + (T_i + i - 1 - 2X_{PT-1})D_{1,i} + (T_i + i - X_{PT-1})D_{2,i},\ X_i = T_{1,i} \\ K_i - (T_i + i - 1 - X_{PT-1})D_{1,i} + \binom{T_i + i - X_{PT-1}}{2}D_{2,i},\ X_i = T_{2,i} \end{cases}.$$

According to this way, it can be extended to multi-parameter $SUM(N)$ and $SUM_q(N)$. This formula is complex and has not been studied in terms of analysis yet.

9 A THEOREM OF SYMMETRY

In this section, $T_i \geqslant i$.
By 2.9, we have $H_1(g,PT,PT) = \prod_{i=1}^{M} T_i\binom{M}{g} = \prod_{i=1}^{M} T_i\binom{M}{g,M-g}$.
Promoted it: Set T come from p Source: $S_1, S_2\ldots S_p$.

Definition 20. $Diff(S_x,S_x) = 0, Diff(S_x,S_y) = -Diff(S_y,S_x) = 1,\ x > y$.

Definition 21. $Diff(T_i,T_j) = Diff(S_x,S_y),\ T_i \in S_x, T_j \in S_y$.

Definition 22. $W(g_1,g_2\ldots g_p,[T_1,T_2\ldots T_M]) = \sum_{g_1+g_2+\ldots+g_p=M, g_i=|S_i|}\prod_{i=1}^{M}(T_i + \sum_{j<i} Diff(T_j,T_i))$
In set T, g_1 come from S_1, g_2 comes from $S_2\ldots$ g_M comes from S_M.

9.1. $W(g_1,g_2\ldots g_p,[T_1,T_2\ldots T_M]) = \prod_{i=1}^{M} T_i\binom{M}{g_1,g_2,\ldots,g_M}$.

Definition 23. $W_q(g_1,g_2\ldots g_p,[T_1,T_2\ldots T_M]) = \sum_{g_i+g_2+\ldots+g_p=M, g_i=|S_i|}\prod_{i=1}^{M} G_1^{T_i+\sum_{j<i} Diff(T_j,T_i)} q^{\sum_{j<i, Diff(T_j,T_i)=-1} 1}$.

Definition 24. $G^{M}_{g_1,g_2 \dots g_p} = \frac{(q^M-1)(q^{M-1}-1)\dots(q^1-1)}{\prod_{i=0}^{p}(q^{g_i}-1)(q^{g_i-1}-1)\dots(q^i-1)}$, $g_1+g_2+\dots+g_p = M$.

9.2. $W_q(g_1, g_2 \dots g_p, [T_1, T_2 \dots T_M]) = (\prod_{i=1}^{M} G_1^{T_i}) G^{M}_{g_1,g_2 \dots g_p}$, $T_i \geqslant i$.

Proof.
$W(1,1,[T_1,T_2]) = G_1^{T_1}G_1^{T_2+1} + G_1^{T_1}G_1^{T_2-1}q = G_1^{T_1}G_1^{T_2}G_1^2$. Holds
Suppose $W_q(g_1, g_2, PT)$ holds,
$W_q(g_1, g_2+1, [PT, T_{M+1}]) = T_{M+1} \in Source_1 + T_{M+1} \in Source_2$
$= W_q(g_1,g_2,PT)G_1^{T_{M+1}+g_1} + W_q(g_1-1,g_2+1,PT)G_1^{T_{M+1}-(g_2+1)}q^{g_2+1}$
$= (\prod_{i=1}^{M} G_1^{T_i})G^{M}_{g_1,g_2}G_1^{T_{M+1}+g_1} + (\prod_{i=1}^{M} G_1^{T_i})G^{M}_{g_1-1,g_2+1}G_1^{T_{M+1}-(g_2+1)}q^{g_2+1}$.
Just need to prove:
$G^M_{g_1}G_1^{T_{M+1}+g_1} + G^M_{g_1-1}G_1^{T_{M+1}-(M-g_1+1)}q^{M-g_1+1} = G_1^{T_{M+1}}G^{M+1}_{g_1}$.
$(\text{Right side}) \times \frac{(q^{M-g_1+1}-1)}{G^M_{g_1}} = (q^{T_{M+1}-1}+\dots+q+1)(q^{M+1}-1) = (1)$.
$(\text{Left side}) \times \frac{(q^{M-g_1+1}-1)}{G^M_{g_1}} = (q^{M-g_1+1}-1)G_1^{T_{M+1}+g_1} + (q^{g_1}-1)G_1^{T_{M+1}-(M-g_1+1)}q^{M-g_1+1}$
$= (q^{M-g_1+1}-1)(q^{T_{M+1}+g_1-1}+\dots+q+1) + (q^{g_1}-1)(q^{T_{M+1}-1}+\dots+q^{M-g_1+2}+q^{M-g_1+1}) = (2)$.
(1)-(2)=0→It's holds when p=2.

$W_q(g_1, g_2+g_3, [T_1,T_2 \dots T_M]) = (\prod_{i=1}^{M} G_1^{T_i})G^{g_1+g_2+g_3}_{g_1,g_2+g_3}$.
Every product has g_2+g_3 factors come from $Source_2$, divide them to $g_2 \times Source_2 + g_3 \times Source_3$.
g_1-factors are invariant, (g_2+g_3)-factors are variant.
$\sum\prod(\text{variant factors}) = W_q(g_2, g_3, [X_1, X_2 \dots X_{g_2+g_3}]) = \prod_{i=1}^{g_2+g_3} G_1^{X_i}G^{g_2+g_3}_{g_2,g_3}$.
$W_q(g_1,g_2,g_3,[T_1,T_2 \dots T_M]) = (\prod_{i=1}^{M} G_1^{T_i})G^{g_1+g_2+g_3}_{g_1,g_2+g_3}G^{g_2+g_3}_{g_2,g_3} = (\prod_{i=1}^{M} G_1^{T_i})G^{g_1+g_2+g_3}_{g_1,g_2,g_3}$. □

For example:
$W_q(1,2,[1,2,3]) = S_1S_2S_2 + S_2S_1S_2 + S_2S_2S_1 = G_1^1G_1^3G_1^4 + G_1^1G_1^1qG_1^4 + G_1^1G_1^2G_1^1q^2 = G_1^1G_1^2G_1^3G^3_{1,2}$.
When $W_q(1,2,[1,2,3])$ changes to $W_q(1,1,1,[1,2,3])$
$= S_1S_2S_3 + S_1S_3S_2 + S_2S_1S_3 + S_3S_1S_2 + S_2S_3S_1 + S_3S_2S_1$
$= G_1^1\{G_1^3G_1^5 + G_1^3G_1^3q\} + G_1^1q\{G_1^1G_1^5 + G_1^1G_1^3q\} + G_1^1q^2\{G_1^1G_1^3 + G_1^1G_1^1q\}$
$= G_1^1\{G_1^3G_1^4G^2_{1,1}\} + G_1^1q\{G_1^1G_1^4G^2_{1,1}\} + G_1^1q^2\{G_1^1G_1^2G^2_{1,1}\} = G_1^1G_1^2G_1^3G^3_{1,2} \times G^2_{1,1} = G_1^1G_1^2G_1^3G^3_{1,1,1}$.

10 DECLARATIONS

10.1 Competing interests

The authors have no competing interests to declare that are relevant to the content of this article.

10.2 Data availability statement

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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