On the origin of *abc*-triples

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Abstract: The rules describing emergence of *abc*-triples formed by the set of roots for Pell's equations $x^2 - Dy^2 = N$ with $N = \pm 1$ and $N = \pm 2$ are revealed.
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1 Introduction

The *abc*-conjecture, concerning some deep connections between additive and multiplicative properties of integers, is a challenging issue in Diophantine analysis. It's proof [1], extremely long and hard to comprehend, is not accepted by professionals in this particular field of mathematics, therefore it's confirmation remains controversial [2]. We offer a completely different approach to this problem, the first step of which is successfully accomplished.

2 Background information

2.1 Radicals and *abc*-conjecture

The radical of a positive integer n is the product of all distinct prime factors of n. It is the largest square-free divisor of n and therefore is sometimes defined as the square-free kernel of n. Here we use the notification $R(n)$ for radical function, but frequently $rad(n)$ can be found. Thus $R(n) \leq n$. For $n = 2^2 \cdot 5^3 \cdot 17 = 8500$ we have $R(8500) = 2 \cdot 5 \cdot 17 = 170$.

Radical $R(n)$ is a multiplicative function – for two relatively prime integers $a \perp b$ we have $R(ab) = R(a) \cdot R(b)$. Comparison of n and $R(n)$ shows how far n is from being squarefree, therefore $R(n)$ is mostly used for analysis of multiplicative and additive properties of integers, as in *abc*-conjecture.

Assume that a, b and c are pairwise relatively prime positive integers, $a < b < c$ and $a + b = c$ (they are additively related). Then generally $R(abc) > c$, but rarely we have $R(abc) < c$; there are exactly 418 such cases for $c < 100000$, see [3]. These special cases are called *abc*-triples (sometimes *abc*-hits) and *abc*-conjecture postulates that $R(abc)$ is usually not much smaller than c. More precise formulation of *abc*-conjecture is the following.

Definition 2.1. For every $\epsilon > 0$ there exist only finitely many triples (a, b, c) of coprime positive integers satisfying $a + b = c$, for which

$$R(abc)^{1+\epsilon} < c. \tag{1}$$

Some other formulations of *abc*-conjecture see [4, 5], but here we restrict ourselves by finding new sequences of *abc*-triples and relations revealed by them.

There are infinitely many *abc*-triples, because it is possible to construct infinite sequences of them.

Example 2.2.

$$\begin{aligned} 2^6 - 1 &= (2^3 - 1)(2^3 + 1) = 7 \cdot 9 = 63, \\ 2^{12} - 1 &= (2^6 - 1)(2^6 + 1) = 63 \cdot (2^6 + 1), \text{etc.}, \end{aligned}$$

and $(2^6 - 1)|(2^{6k} - 1), k = 1, 2, 3...$

So $a = 1$, $b = 2^{6k} - 1$ and $c = 2^{6k}$. Now $b = 63N$, where N is natural and

$$R(abc) = R(1) \cdot R(63N) \cdot R(2^{6k}) = 1 \cdot R(63N) \cdot 2 \leq 2 \cdot R(63) \cdot R(N).$$

As maximal $R(N)$ value is $R(N) = N$, we have $R(abc) \leq 42N$; $c = b + 1 = 63N + 1$ and $R(abc) < c$ for all $k = 1, 2, 3, ...$ Therefore $(1, 2^{6k} - 1, 2^{6k})$ for $k = 1, 2, 3, ...$ represents an infinite sequence of *abc*-triples.

Excellent tables of known *abc*-triples can be found at Bart de Smit's site [3, 6]. Their analysis led to the following

Theorem 2.3. *If a, b and c are different pairwise relatively prime positive integers forming an abc-triple $a + b = c$ with $R(abc) < c$, then components b and c are not square-free.*

Proof. We arrange them traditionally $a < b < c$.

1. Assume that b is square-free, it can be composite (product of first powers of primes). Then $R(b) = b$. We need $R(a) \cdot b \cdot R(c) < c$, so $R(a)$ and $R(c)$ must be small. Minimal $R(a) = 1$ and minimal $R(c) = 2$, so minimal $R(abc) = 2b > c$, because $a < b$ and $c = a + b$.

2. Assume that c is square-free, it can be composite. Then $R(c) = c$. Again minimal $R(a) = 1$ and minimal $R(b) = 2$, so minimal $R(abc) = 2c > c$.

3. Assume that components b and c are square-free, they can be composite. Then $R(b) = b$ and $R(c) = c$. Minimal $R(a) = 1$ gives minimal $R(abc) = b \cdot c$, but minimal $R(b) = 2$, so minimal $R(abc) = 2c > c$. Equally, minimal $R(c) = 2$ gives $R(abc) = 2b > c$, because $a < b$ and $c = a + b$.

In all three cases we obtain contradictions, which proves the theorem. □

Theorem 2.3 allows classification of *abc*-equations according to their components composition. Cases with b or c as perfect squares are clearly generalized Pell's equations $x^2 - Dy^2 = N$ with positive or negative N; if none of b or c are perfect squares we have equation $ax^2 - by^2 = N$, whose solution also is Pell's type (see [7]). This induced further investigation of Pell's equations as simple generators for *abc*-triples.

2.2 Continuants, continued fractions and Pell's equations

In our exposition we will intensively use the concept of continuants, invented by L. Euler and rarely used today. Therefore Chapter 13 of [8] is highly recommended. Basic theory about continued fractions and Pell's equations can be found in many number theory textbooks, as [9, 10]. Here we only accent without proofs some well-known facts about these items.

1. For the sequence of natural numbers $a_0, a_1, ..., a_n$ the continuant (in Muir's notation – simple continuant) $K(a_0, a_1, ..., a_n)$ is defined recursively:

- $K(a_0) = a_0$, but for empty set $K() = 1$;
- for all $n \geq 1$ we have $K(a_0, a_1, ..., a_n) = a_n \cdot K(a_0, a_1, ..., a_{n-1}) + K(a_0, a_1, ..., a_{n-2})$.

We allow $a_0 = 0$ for concordance with continued fractions.

2. Continuant inversion: $K(a_0, a_1, ..., a_n) = K(a_n, ..., a_1, a_0)$.

3. Two forms of continuant splitting.

- Between two elements a_l and a_{l+1}:
  $K(a_0, a_1, ..., a_n) = K(a_0, a_1, ..., a_l) \cdot K(a_{l+1}, ..., a_n) + K(a_0, a_1, ..., a_{l-1}) \cdot K(a_{l+2}, ..., a_n)$.
- If $a_l = a_{l1} + a_{l2}$, then we can split at position a_l:
  $K(a_0, a_1, ..., a_n) = K(a_0, a_1, ..., a_{l1}) \cdot K(a_{l+1}, ..., a_n) + K(a_0, a_1, ..., a_{l-1}) \cdot K(a_{l2}, ..., a_n)$.

4. From Laplace's theorem (or from properties of continued fractions) the following can be derived.

$$K(a_0, a_1, ..., a_n) \cdot K(a_1, a_2, ..., a_{n-1}) - K(a_0, a_1, ..., a_{n-1}) \cdot K(a_1, a_2, ..., a_n) = (-1)^{n-1}. \quad (2)$$

5. If $[a_0; a_1, ..., a_n]$ is a finite simple continued fraction ($a_0, a_1, ..., a_n$ are positive integers, $a_0 = 0$ is allowed), then

$$[a_0; a_1, ..., a_n] = \frac{K(a_0, a_1, ..., a_n)}{K(a_1, a_2, ..., a_n)}$$

and $K(a_0, a_1, ..., a_n) \perp K(a_1, a_2, ..., a_n)$.

6. For a non-square natural number D the representation of irrational $\sqrt{D}$ in the form of a simple continued fraction is periodic: $\sqrt{D} = [a_0; \overline{a_1, a_2, ..., a_s}]$. Here a_0 is an integer part of $\sqrt{D}$, $a_s = 2a_0$, but the sequence $a_1, a_2, ..., a_{s-1}$ is palindromic. The bar above $a_1, a_2, ..., a_s$ indicate the period. In the following we denote this palindromic component by π, so $\sqrt{D} = [a_0; \overline{\pi, 2a_0}]$.

7. We have a non-square natural number D. If the length of the palindromic sequence π for $\sqrt{D}$ expression is odd (and the length of the period $\overline{\pi 2a_0}$ is even), then:

- there are not solutions for the negative Pell's equation $x^2 - Dy^2 = -1$ for this particular D value;
- all solutions for the corresponding positive Pell's equation $x^2 - Dy^2 = 1$ are given by

$$\frac{x_0}{y_0} = \frac{K(a_0, \pi)}{K(\pi)}; \quad \frac{x_1}{y_1} = \frac{K(a_0, \pi, 2a_0, \pi)}{K(\pi, 2a_0, \pi)}; \quad \text{etc.}$$

8. We have a non-square natural number D. If the length of the palindromic sequence π for $\sqrt{D}$ expression is even (so the length of the period $\overline{\pi 2a_0}$ is odd), then:

- all solutions for the negative Pell's equation $x^2 - Dy^2 = -1$ are given by

$$\frac{x_0}{y_0} = \frac{K(a_0,\pi)}{K(\pi)}; \quad \frac{x_1}{y_1} = \frac{K(a_0,\pi,2a_0,\pi,2a_0,\pi)}{K(\pi,2a_0,\pi,2a_0,\pi)}; \quad \text{etc.};$$

- all solutions for the corresponding positive Pell's equation $x^2 - Dy^2 = 1$ are given by

$$\frac{x_0}{y_0} = \frac{K(a_0,\pi,2a_0,\pi)}{K(\pi,2a_0,\pi)}; \quad \frac{x_1}{y_1} = \frac{K(a_0,\pi,2a_0,\pi,2a_0,\pi,2a_0,\pi)}{K(\pi,2a_0,\pi,2a_0,\pi,2a_0,\pi)}; \quad \text{etc.}$$

2.3 Pell's equations and *abc*-triples

A lot of items in Bart de Smit's table of 418 smallest *abc*-triples [3] are formed by solutions for some Pell's equations – fundamental or any higher roots. The very first *abc*-triple $(1, 2^3, 3^2)$ represents fundamental solution $x_0 = 3$ and $y_0 = 2$ for positive Pell's equation $x^2 - Dy^2 = 1$ with discriminant $D = 2$. The 3rd *abc*-triple $(1, 2^4 \cdot 3, 7^2)$ gives roots $x_1 = 7$ and $y_1 = 4$ for positive Pell's equation with $D = 3$. Few items in [3] correspond to roots of negative Pell's equation, for example, *abc*-triple $\#32$ or $(1, 2^{10}, 5^2 \cdot 41)$ represents fundamental roots $x_0 = 32$ and $y_0 = 5$ for equation $x^2 - 41y^2 = -1$. Such examples also appear among items in files of big *abc*-triples [6]. As palindrome-containing continuants constitute different roots for positive (and, if they exist, also for negative) Pell's equation with given non-square D, divisibility interrelations of these continuants were investigated.

3 Divisibility interrelations of positive/negative Pell's equation roots

Theorem 3.1. *Continuant* $K(\pi, \underbrace{2a_0, \pi}_{k\text{-times}})$ *is a divisor of all continuants* $K(\pi, \underbrace{2a_0, \pi}_{l\text{-times}})$*. Here* $l = k + n(k+1)$*,* $k = 0, 1, 2, \ldots$ *and* $n = 0, 1, 2, \ldots$*.*

Proof. **1.** $k = 0, n = 0$ and $K(\pi)|K(\pi)$ – trivially.

$k = 0, n = 1$ and we must compare $K(\pi)$ with $K(\pi, 2a_0, \pi)$.
$K(\pi, 2a_0, \pi) = K(\pi, a_0 + a_0, \pi) = K(\pi, a_0) \cdot K(\pi) + K(\pi) \cdot K(a_0, \pi) = 2K(\pi) \cdot K(a_0, \pi)$, therefore $K(\pi)|K(\pi, 2a_0, \pi)$.

$k = 0, n = 2$, we must compare $K(\pi)$ and $K(\pi, 2a_0, \pi, 2a_0, \pi)$.
$K(\pi, 2a_0, \pi, 2a_0, \pi) = K(\pi, 2a_0, \pi, a_0) \cdot K(\pi) + K(\pi, 2a_0, \pi) \cdot K(a_0, \pi)$. In view of result for $k = 0, n = 1$ we obtain $K(\pi)|K(\pi, 2a_0, \pi, 2a_0, \pi)$. For $n = 3, 4, \ldots$ we can act analogously.

2. $k = 1, n = 0$ and $K(\pi, 2a_0, \pi)|K(\pi, 2a_0, \pi)$ – trivially.

$k = 1, n = 1$ and we must compare $K(\pi, 2a_0, \pi)$ with $K(\pi, 2a_0, \pi, 2a_0, \pi, 2a_0, \pi)$.
$K(\pi, 2a_0, \pi, 2a_0, \pi, 2a_0, \pi) = K(\pi, 2a_0, \pi, a_0) \cdot K(\pi, 2a_0, \pi) + K(\pi, 2a_0, \pi) \cdot K(a_0, \pi, 2a_0, \pi)$
$= 2K(a_0, \pi, 2a_0, \pi) \cdot K(\pi, 2a_0, \pi)$, therefore $K(\pi, 2a_0, \pi)|K(\pi, 2a_0, \pi, 2a_0, \pi, 2a_0, \pi)$.

$k = 1, n = 2$ and the dividend
$K(\pi, 2a_0, \pi, 2a_0, \pi, 2a_0, \pi, 2a_0, \pi, 2a_0, \pi)$
$= K(\pi, 2a_0, \pi, 2a_0, \pi, 2a_0, \pi, a_0) \cdot K(\pi, 2a_0, \pi) + K(\pi, 2a_0, \pi, 2a_0, \pi, 2a_0, \pi) \cdot K(a_0, \pi, 2a_0, \pi)$.
As we just obtained $K(\pi, 2a_0, \pi) | K(\pi, 2a_0, \pi, 2a_0, \pi, 2a_0, \pi)$, the necessary
$K(\pi, 2a_0, \pi) | K(\pi, 2a_0, \pi, 2a_0, \pi, 2a_0, \pi, 2a_0, \pi, 2a_0, \pi)$ follows.

In each case we cut off from the dividend the corresponding divisor fragment, this inductively proves the theorem. □

Theorem 3.1 suggests that each so defined continuant with the number of palindromes specified in the left section of the Table 1 is a divisor of all continuants of the same type, specified in the right section of the Table 1.

Table 1.

Divisor	Dividends (trivial cases, as $1\pi \vert 1\pi$, are omitted)				
1π	2π	3π	4π	5π	...
2π	4π	6π	8π	10π	...
3π	6π	9π	12π	15π	...
4π	8π	12π	16π	20π	...
5π	10π	15π	20π	25π	...
...	...	...	...	...	...

Roots of the negative Pell's equation $x^2 - Dy^2 = -1$ have $1, 3, 5, ...$ palindromic units, so their divisibility table is the following.

Table 2.

Divisor	Dividends (trivial cases, as $y_0 \vert y_0$, are omitted)				
y_0	y_1	y_2	y_3	y_4	...
y_1	y_4	y_7	y_{10}	y_{13}	...
y_2	y_7	y_{12}	y_{17}	y_{22}	...
y_3	y_{10}	y_{17}	y_{24}	y_{31}	...
...	...	...	...	...	...

Roots of the positive Pell's equation $x^2 - Dy^2 = 1$ have $1, 2, 3, 4, ...$ palindromic units, which are π or $\pi, 2a_0, \pi$ type. This gives the following divisibility.

Table 3.

Divisor	Dividends (trivial cases, as $y_0 \vert y_0$, are omitted)				
y_0	y_1	y_2	y_3	y_4	...
y_1	y_3	y_5	y_7	y_9	...
y_2	y_5	y_8	y_{11}	y_{14}	...
y_3	y_7	y_{11}	y_{15}	y_{19}	...
...	...	...	...	...	...

Theorem 3.2. *Continuant* $K(a_0, \pi, \underbrace{2a_0, \pi}_{k\text{-times}})$ *is a divisor of all continuants* $K(a_0, \pi, \underbrace{2a_0, \pi}_{l\text{-times}})$*. Here* $l = k + 2n(k+1)$*,* $k = 0, 1, 2, ...$ *and* $n = 0, 1, 2, ...$

Proof. **1.** $k = 0, n = 0$ and $K(a_0, \pi)|K(a_0, \pi)$ – trivially.

$k = 0, n = 1$ and we must compare $K(a_0, \pi)$ with $K(a_0, \pi, 2a_0, \pi, 2a_0, \pi)$.
$K(a_0, \pi, 2a_0, \pi, 2a_0, \pi) = K(a_0, \pi, 2a_0, \pi, a_0) \cdot K(\pi) + K(a_0, \pi, 2a_0, \pi) \cdot K(a_0, \pi)$
$= [K(a_0, \pi, a_0) \cdot K(\pi, a_0) + K(a_0, \pi) \cdot K(a_0, \pi, a_0)] \cdot K(\pi) + K(a_0, \pi, 2a_0, \pi) \cdot K(a_0, \pi)$
$= K(a_0, \pi) \cdot [2K(\pi) \cdot K(a_0, \pi, a_0) + K(a_0, \pi, 2a_0, \pi)]$, therefore $K(a_0, \pi)|K(a_0, \pi, 2a_0, \pi, 2a_0, \pi)$.

$k = 0, n = 2$ and the dividend $K(a_0, \pi, 2a_0, \pi, 2a_0, \pi, 2a_0, \pi, 2a_0, \pi)$
$= K(a_0, \pi, 2a_0, \pi, 2a_0, \pi, 2a_0, \pi, a_0) \cdot K(\pi) + K(a_0, \pi, 2a_0, \pi, 2a_0, \pi, 2a_0, \pi) \cdot K(a_0, \pi)$
$= K(a_0, \pi, 2a_0, \pi, 2a_0, \pi, 2a_0, \pi) \cdot K(a_0, \pi) + K(\pi) \cdot [K(a_0, \pi, 2a_0, \pi, 2a_0, \pi) \cdot K(a_0, \pi, a_0)$
$+K((a_0, \pi, 2a_0, \pi, 2a_0, \pi, a_0) \cdot K(a_0, \pi)]$. For $k = 0, n = 1$ we just obtained $K(a_0, \pi)|K(a_0, \pi, 2a_0, \pi, 2a_0, \pi)$, therefore the first summand in square brackets divides by $K(a_0, \pi)$ and we obtain necessary $K(a_0, \pi)|K(a_0, \pi, 2a_0, \pi, 2a_0, \pi, 2a_0, \pi, 2a_0, \pi)$.

2. $k = 1, n = 0$ and $K(a_0, \pi, 2a_0, \pi)|K(a_0, \pi, 2a_0, \pi)$ – trivially.

$k = 1, n = 1$ and the dividend $K(a_0, \pi, 2a_0, \pi, 2a_0, \pi, 2a_0, \pi, 2a_0, \pi, 2a_0, \pi)$
$= K(a_0, \pi, 2a_0, \pi, 2a_0, \pi, 2a_0, \pi, a_0) \cdot K(\pi, 2a_0, \pi)$
$+K(a_0, \pi, 2a_0, \pi, 2a_0, \pi, 2a_0, \pi) \cdot K(a_0, \pi, 2a_0, \pi)$. The first summand gives $K(\pi, 2a_0, \pi) \cdot [K(a_0, \pi, 2a_0, \pi) \cdot K(a_0, \pi, 2a_0, \pi, a_0) + K(a_0, \pi, 2a_0, \pi, a_0) \cdot K(a_0, \pi, 2a_0, \pi)]$, so it is divisible by $K(a_0, \pi, 2a_0, \pi)$. Therefore we get
$K(a_0, \pi, 2a_0, \pi)|K(a_0, \pi, 2a_0, \pi, 2a_0, \pi, 2a_0, \pi, 2a_0, \pi, 2a_0, \pi)$.

Remark. It seems useful in continuant expressions to label longer repeating $\pi, 2a_0, ..., \pi$ sequences accordingly to their number of π units. Thus sequence $a_0, \pi, 2a_0, \pi$ can be labelled as $a_0, 2\pi$, sequence with four palindromes $\pi, 2a_0, \pi, 2a_0, \pi, 2a_0, \pi$ can be labelled as 4π. For instance, $K(a_0, \pi, 2a_0, \pi, 2a_0, \pi, 2a_0, \pi, 2a_0, \pi, 2a_0, \pi)$ can be shortened to $K(a_0, 6\pi)$.

$k = 1, n = 2$ and the dividend
$K(a_0, 10\pi) = K(a_0, 8\pi, a_0) \cdot K(\pi, 2a_0, \pi) + K(a_0, 8\pi) \cdot K(a_0, \pi, 2a_0, \pi)$. The first summand gives $K(\pi, 2a_0, \pi) \cdot [K(a_0, \pi, 2a_0, \pi) \cdot K(a_0, 6\pi, a_0) + K(a_0, \pi, 2a_0, \pi, a_0) \cdot K(a_0, 6\pi)]$ and we already know that $K(a_0, \pi, 2a_0, \pi)|K(a_0, 6\pi)$. So both summands are divisble by $K(a_0, \pi, 2a_0, \pi)$ and we get the necessary $K(a_0, \pi, 2a_0, \pi)|K(a_0, 10\pi)$.

Uniform splitting inductively proves the theorem. □

Theorem 3.2 establishes the following divisibility rule for the specified continuants.

Table 4.

Divisor	Dividends (trivial cases, as $1\pi|1\pi$, are omitted)				
1π	3π	5π	7π	9π	...
2π	6π	10π	14π	18π	...
3π	9π	15π	21π	27π	...
4π	12π	20π	28π	36π	...
5π	15π	25π	35π	45π	...
...	...	...	...	...	...

For roots of the negative Pell's equation $x^2 - Dy^2 = -1$ this gives the following divisibility.

Table 5.

Divisor	Dividends (trivial cases, as $x_0 \vert x_0$, are omitted)				
x_0	x_1	x_2	x_3	x_4	...
x_1	x_4	x_7	x_{10}	x_{13}	...
x_2	x_7	x_{12}	x_{17}	x_{22}	...
x_3	x_{10}	x_{17}	x_{24}	x_{31}	...
...	...	...	...	...	...

For roots of the positive Pell's equation $x^2 - Dy^2 = 1$ divisibility table is the following.

Table 6.

Divisor	Dividends (trivial cases, as $x_0 \vert x_0$, are omitted)				
x_0	x_2	x_4	x_6	x_8	...
x_1	x_5	x_9	x_{13}	x_{17}	...
x_2	x_8	x_{14}	x_{20}	x_{26}	...
x_3	x_{11}	x_{19}	x_{27}	x_{35}	...
...	...	...	...	...	...

Theorem 3.3. *Continuant* $K(a_0, \pi, \underbrace{2a_0, \pi}_{k\text{-times}})$ *is a divisor of all continuants* $K(\pi, \underbrace{2a_0, \pi}_{l\text{-times}})$. *Here* $l = 2k + 1 + 2n(k + 1)$, $k = 0, 1, 2, ...$ *and* $n = 0, 1, 2, ...$

Proof. **1.** $k = 0, n = 0$ and we must compare $K(a_0, \pi)$ with $K(\pi, 2a_0, \pi)$.
$K(\pi, 2a_0, \pi) = K(\pi, a_0) \cdot K(\pi) + K(\pi) \cdot K(a_0, \pi) = 2K(\pi) \cdot K(a_0, \pi)$, therefore $K(a_0, \pi)|K(\pi, 2a_0, \pi)$ – this was in Theorem 3.1.

$k = 0, n = 1$ and we must compare $K(a_0, \pi)$ with $K(4\pi)$.
$K(4\pi) = K(3\pi) \cdot K(a_0, \pi) + K(a_0, 3\pi) \cdot K(\pi)$. As $K(a_0, \pi)|K(a_0, 3\pi)$, which was shown for $k = 0, n = 1$ in Theorem 3.2, this confirms divisibility $K(a_0, \pi)|K(4\pi)$.

$k = 0, n = 2$ and the dividend $K(6\pi) = K(5\pi) \cdot K(a_0, \pi) + K(a_0, 5\pi) \cdot K(\pi)$.
As $K(a_0, \pi)|K(a_0, 5\pi)$, which was shown for $k = 0, n = 2$ in Theorem 3.2, this confirms divisibility $K(a_0, \pi)|K(6\pi)$.

2. $k = 1, n = 0$ and we must compare $K(a_0, \pi, 2a_0, \pi)$ with $K(4\pi)$.
As $K(4\pi) = 2K(\pi, 2a_0, \pi) \cdot K(a_0, \pi, 2a_0, \pi)$, this confirms $K(a_0, \pi, 2a_0, \pi)|K(4\pi)$.

$k = 1, n = 1$. Now the dividend $K(8\pi) = K(a_0, 6\pi) \cdot K(\pi, 2a_0, \pi) + K(6\pi) \cdot K(a_0, \pi, 2a_0, \pi)$.
The divisibility $K(a_0, \pi, 2a_0, \pi)|K(a_0, 6\pi)$ was shown at $k = 1, n = 1$ in Theorem 3.2, therefore $K(a_0, \pi, 2a_0, \pi)|K(8\pi)$.

The long expressions for $k = 1, n = 2$ are left for reader.

In total, this inductively confirms the theorem. □

Theorem 3.3 establishes the following divisibility rule for the specified continuants.

Table 7.

Divisor	Dividends				
1π	2π	4π	6π	8π	...
2π	4π	8π	12π	16π	...
3π	6π	12π	18π	24π	...
4π	8π	16π	24π	32π	...
5π	10π	20π	30π	40π	...
...	...	...	...	...	...

Roots of the negative Pell's equation $x^2 - Dy^2 = -1$ have $1, 3, 5, ...$ palindromic units, so divisibility of the type $x_i|y_j$ do not exist for them.

For roots of the positive Pell's equation $x^2 - Dy^2 = 1$ the following divisibility table exists.

Table 8.

Divisor	Dividends				
x_0	y_1	y_3	y_5	y_7	...
x_1	y_3	y_7	y_{11}	y_{15}	...
x_2	y_5	y_{11}	y_{17}	y_{23}	...
x_3	y_7	y_{15}	y_{23}	y_{31}	...
...	...	...	...	...	...

From $\pi|\pi$ divisibility tables (Tables 1, 4 and 7) also more complex situations can be analyzed, where any roots of positive Pell's equation become divisors of any higher roots of the corresponding negative Pell's equation (with the same D value) and vice versa, but they are not the subject of given article.

4 More about continuants and Pell's equations

1. For Equation (2) with $a_n = a_0$ and palindromic $a_1, a_2, ..., a_{n-1} = \pi$ we have

$$K(a_0, \pi, a_0) \cdot K(\pi) - K^2(a_0, \pi) = (-1)^{n-1}. \tag{3}$$

Here $n - 1$ is the length of π, so for odd length palindrome we obtain positive Pell's equation

$$K^2(a_0, \pi) - \frac{K(a_0, \pi, a_0)}{K(\pi)} \cdot K^2(\pi) = 1. \tag{4}$$

Even length palindrome π gives negative Pell's equation

$$K^2(a_0, \pi) - \frac{K(a_0, \pi, a_0)}{K(\pi)} \cdot K^2(\pi) = -1. \tag{5}$$

Thus this main watershed is established by determinant properties (and continuants are special form of determinants, see [8]).

If we substitute odd length π by longer chains $(\pi, 2a_0, \pi)$, $(\pi, 2a_0, \pi, 2a_0, \pi)$, etc., these items also have odd length and they produce higher roots of positive Pell's equation. For even length

π we obtain alternation between roots of negative Pell's equation $y_1 = K(\pi, 2a_0, \pi, 2a_0, \pi)$, $y_2 = K(\pi, 2a_0, \pi, 2a_0, \pi, 2a_0, \pi, 2a_0, \pi)$, etc. (all these items have even length), and roots of positive Pell's equation $y_1 = K(\pi, 2a_0, \pi)$, $y_2 = K(\pi, 2a_0, \pi, 2a_0, \pi, 2a_0, \pi)$, etc. (and these items have odd length).

Relation

$$D = \frac{K(a_0, \pi, a_0)}{K(\pi)} = \frac{K(a_0, \pi, 2a_0, \pi, a_0)}{K(\pi, 2a_0, \pi)} = \frac{K(a_0, \pi, 2a_0, \pi, 2a_0, \pi, a_0)}{K(\pi, 2a_0, \pi, 2a_0, \pi)} = ... = const \qquad (6)$$

for given particular π. If this π has odd length, we can write it from Equation (4) in the following form.

$$D = \frac{K^2(a_0, \pi) - 1}{K^2(\pi)} = \frac{K^2(a_0, 2\pi) - 1}{K^2(2\pi)} = \frac{K^2(a_0, 3\pi) - 1}{K^2(3\pi)} = \frac{K^2(a_0, 4\pi) - 1}{K^2(4\pi)} = ... = const. \qquad (7)$$

For even length π we must alternatively use Equations (4) and (5), so obtaining expression with alternating signs.

$$D = \frac{K^2(a_0, \pi) + 1}{K^2(\pi)} = \frac{K^2(a_0, 2\pi) - 1}{K^2(2\pi)} = \frac{K^2(a_0, 3\pi) + 1}{K^2(3\pi)} = \frac{K^2(a_0, 4\pi) - 1}{K^2(4\pi)} = ... = const. \qquad (8)$$

2. Now we consider relation

$$D = \frac{K^2(a_0, n\pi) - 1}{K^2(n\pi)} = \frac{[K(a_0, n\pi) - 1] \cdot [K(a_0, n\pi) + 1]}{K^2(n\pi)}.$$

It's denominator is square, but the numerator is formed from the first and the third of three consecutive natural numbers, therefore these numbers:

- either are coprime (and $K(a_0, n\pi)$ is even), but then their product is not natural square;
- or their single common divisor is 2. But, as they differ by 2, their product $K^2(a_0, n\pi) - 1$ cannot have prime divisor 2 in even power.

Thus this relation cannot be rational square (or – integral square).

3. For an odd length palindrome π the following binomial expression takes place.

$$K(a_0, n\pi) + \sqrt{D} \cdot K(n\pi) = [K(a_0, \pi) + \sqrt{D} \cdot K(\pi)]^n = \sum_{i=0}^{n} \binom{n}{i} K^{n-i}(a_0, \pi) \cdot [\sqrt{D} \cdot K(\pi)]^i.$$

$$\text{Here } D = \frac{K^2(a_0, n\pi) - 1}{K^2(n\pi)} = const, n = 1, 2, 3...$$

We express separately

$$K(a_0, n\pi) = K^n(a_0, \pi) + \sum_{i=1} \binom{n}{2i} K^{n-2i}(a_0, \pi) \cdot D^i \cdot K^{2i}(\pi);$$

$$K(n\pi) = \sum_{i=1} \binom{n}{2i-1} K^{n-2i+1}(a_0, \pi) \cdot D^{i-1} \cdot K^{2i-1}(\pi).$$

For $n = 3, 5$ and 7 we have the following values.

$$K(a_0, 3\pi) = K(a_0, \pi) \cdot [K^2(a_0, \pi) + 3D \cdot K^2(\pi)];$$

$$K(3\pi) = K(\pi) \cdot [3K^2(a_0, \pi) + D \cdot K^2(\pi)].$$

$$K(a_0, 5\pi) = K(a_0, \pi) \cdot [K^4(a_0, \pi) + 10K^2(a_0, \pi) \cdot D \cdot K^2(\pi) + 5D^2 \cdot K^4(\pi)];$$

$$K(5\pi) = K\pi) \cdot [5K^4(a_0, \pi) + 10K^2(a_0, \pi) \cdot D \cdot K^2(\pi) + D^2 \cdot K^4(\pi)].$$

$$K(a_0, 7\pi) = K(a_0, \pi) \cdot [K^6(a_0, \pi) + 21K^4(a_0, \pi) \cdot D \cdot K^2(\pi) + 35K^2(a_0, \pi) \cdot D^2 \cdot K^4(\pi) + 7D^3 \cdot K^6(\pi)];$$

$$K7\pi) = K(\pi) \cdot [7K^6(a_0, \pi) + 35K^4(a_0, \pi) \cdot D \cdot K^2(\pi) + 21K^2(a_0, \pi) \cdot D^2 \cdot K^4(\pi) + D^3 \cdot K^6(\pi)].$$

A lot of corollaries can be made here, we show only few:

- if n is an odd prime p, $p|D$, but $p \nmid K(\pi)$, then $p|K(p\pi)$;
- if n is an odd prime p, $p\|D$ and $p\|K(\pi)$, then $p^2\|K(p\pi)$.

5 Two basic theorems

Experimental calculations revealed that *abc*-triples $(1, Dy_i^2, x_i^2)$ and $(1, x_j^2, Dy_j^2)$ arise according to some rules between root pairs of positive/negative Pell's equations.

Theorem 5.1. *Now π is an odd length palindromic unit. If roots of positive Pell's equation (x_i, y_i), obtained from simple continued fraction $[a_0; \pi, \underbrace{2a_0, \pi}_{k\text{-times}}]$ constitute an abc-triple $(1, Dy_i^2, x_i^2)$, then abc-triples $(1, Dy_j^2, x_j^2)$ are formed by all roots (x_j, y_j), obtained from simple continued fractions $[a_0; \pi, \underbrace{2a_0, \pi}_{l\text{-times}}]$. Here $l = k + n(k+1)$, $k = 0, 1, 2, ...$ and $n = 0, 1, 2, ...$*

Proof. **1.** $k = 0, n = 0$. Now $[a_0; \pi] = \dfrac{K(a_0, \pi)}{K(\pi)}$ and *abc*-equation $a + b = c$ is positive Pell's equation

$$1 + [K^2(a_0, \pi) - 1] = K^2(a_0, \pi).$$

If fundamental roots of this equation give an *abc*-triple, then

$$R[(K^2(a_0, \pi) - 1) \cdot K^2(a_0, \pi)] < K^2(a_0, \pi).$$

Numbers in square brackets are coprime, therefore

$$R[K^2(a_0, \pi) - 1] \cdot R[K^2(a_0, \pi)] < K^2(a_0, \pi) \tag{9}$$

– this is our point of departure.

Remark. While from radical properties $R[K^2(a_0, \pi)] \leq K(a_0, \pi)$, these initial conditions do not mean $R[K^2(a_0, \pi) - 1] < K(a_0, \pi)$, significant is the product of two radicals in the left side of (9). Thus, for the positive Pell's equation with $D = 5$ we have $x_0 = 9$ and $y_0 = 4$, which gives

$$R[K^2(a_0, \pi) - 1] = 10 > K(a_0, \pi) = 9.$$

Here $R(abc) = 30 < 81 = x_0^2$, so $(1, 80, 81)$ is an *abc*-triple.

2. Now we pick $n = 1$ and must justify an inequality

$$R[K^2(a_0, \pi, 2a_0, \pi) - 1] \cdot R[K^2(a_0, \pi, 2a_0, \pi)] < K^2(a_0, \pi, 2a_0, \pi). \tag{10}$$

From radical properties

$$R[K^2(a_0, \pi, 2a_0, \pi)] \le K(a_0, \pi, 2a_0, \pi). \tag{11}$$

In view of Equation (3) for π odd length $K(a_0, \pi, 2a_0, \pi) = 2K^2(a_0, \pi) - 1$, so

$$K^2(a_0, \pi, 2a_0, \pi) - 1 = 4K^4(a_0, \pi) - 4K^2(a_0, \pi) + 1 - 1 = 4K^2(a_0, \pi) \cdot [K^2(a_0, \pi) - 1].$$

As numbers $K^2(a_0, \pi)$ and $K^2(a_0, \pi) - 1$ are consecutive, we can ignore 4 as factor in radical calculations and

$$\begin{aligned} R[K^2(a_0, \pi, 2a_0, \pi) - 1] &= R[K^2(a_0, \pi)] \cdot R[K^2(a_0, \pi) - 1] \\ &< K^2(a_0, \pi) \le 2K^2(a_0, \pi) - 1 = K(a_0, \pi, 2a_0, \pi). \end{aligned}$$

By multiplying this inequality with (11) we get the necessary justification of Inequality (10).

The proof is analogous for all continuants with even number of palindromic units, because it depends on properties of halved continuants $K(a_0, n\pi, 2a_0, n\pi) = 2K^2(a_0, n\pi) - 1$.

3. So we must justify Theorem 5.1 for continuants with an odd number of palindromic units and the simplest case is $k = 0$ and $n = 2$ – three palindromic units or 3π. From Eq. (4) and (6):

$$\frac{K^2(a_0, 3\pi) - 1}{K^2(a_0, \pi) - 1} = \frac{K^2(3\pi)}{K^2(\pi)} = A^2$$

$$\text{and} \qquad \frac{K^2(a_0, 3\pi)}{K^2(a_0, \pi)} = B^2.$$

A and B are natural numbers (see Theorems 3.1 and 3.2).

$$R(B^2) \le B \qquad \text{or} \qquad R[\frac{K^2(a_0, 3\pi)}{K^2(a_0, \pi)}] \le \frac{K(a_0, 3\pi)}{K(a_0, \pi)}; \tag{12}$$

$$R(A^2) \le A \qquad \text{or} \qquad R[\frac{K^2(a_0, 3\pi) - 1}{K^2(a_0, \pi) - 1}] \le \frac{K(3\pi)}{K(\pi)}; \tag{13}$$

and our point of departure is Equation (9) or

$$R[K^2(a_0, \pi) - 1] \cdot R[K^2(a_0, \pi)] < K(a_0, \pi) \cdot K(a_0, \pi). \tag{14}$$

In the left side of (14) there is a product of two coprime radicals, so the right side of (14) is also splitted in two factors. As $R[K^2(a_0, \pi)] \le K(a_0, \pi)$, the maximal value for one of these factors in the right side of (14) is $K(a_0, \pi)$. But, for inequality (14) to be satisfied, the maximal value of the other factor, derived from $R[K^2(a_0, \pi) - 1]$, cannot exceed $K(a_0, \pi) - 1$, because this is the greatest natural number coprime to $K(a_0, \pi)$ and not exceeding it. Therefore:

$$R[K^2(a_0, \pi) - 1] \cdot R[K^2(a_0, \pi)] \le K(a_0, \pi) \cdot [K(a_0, \pi) - 1]. \tag{15}$$

We multiply (12), (13) and (15):

$$R[K^2(a_0,3\pi)-1]\cdot R[K^2(a_0,3\pi)] \le K(a_0,3\pi)\cdot[K(a_0,\pi)-1]\cdot\frac{K(3\pi)}{K(\pi)}. \qquad (16)$$

Now we compare $[K(a_0,\pi)-1]\cdot\dfrac{K(3\pi)}{K(\pi)}$ and $K(a_0,3\pi)$. Both are natural numbers, so square them. We have:

$$[K(a_0,\pi)-1]^2\cdot\frac{K^2(a_0,3\pi)-1}{K^2(a_0,\pi)-1} \quad \text{and} \quad K^2(a_0,3\pi);$$

$$\underbrace{\frac{K(a_0,\pi)-1}{K(a_0,\pi)+1}}_{<1}\cdot\underbrace{[K^2(a_0,3\pi)-1]}_{<K^2(a_0,3\pi)} \quad \text{and} \quad K^2(a_0,3\pi).$$

So sign $<$ can be used and from (16) we get the necessary:

$$R[K^2(a_0,3\pi)-1]\cdot R[K^2(a_0,3\pi)] < K(a_0,3\pi)\cdot K(a_0,3\pi) = K^2(a_0,3\pi). \qquad (17)$$

All cases with an odd number of palindromic units can be treated analogously. So the theorem is confirmed for $k=0$ and all n values – with even and odd numbers of palindromic units.

For higher k values we denote unit $\pi, 2a_0, \pi$ as π'. Since π' is an odd length palindrome, the proof reduces to discussed cases. This completes the proof of Theorem 5.1. $\square$

Theorem 5.2. *Now π is an even length palindromic unit. If roots of positive or negative Pell's equation (x_i, y_i), obtained from simple continued fraction $[a_0;\pi,\underbrace{2a_0,\pi}_{k\text{-times}}]$ produce an abc-triple, then abc-triples are produced by all roots (x_j, y_j), obtained from simple continued fractions $[a_0;\pi,\underbrace{2a_0,\pi}_{l\text{-times}}]$. Here $l = k + n(k+1)$, $k = 0,1,2,...$ and $n = 0,1,2,...$*

Proof. **1.** This time π is an even length palindromic unit, which gives the following set of *abc*-equations (observe alternation).

$$\begin{aligned}
1+K^2(a_0,\pi) &= [K^2(a_0,\pi)+1].\\
1+[K^2(a_0,2\pi)-1] &= K^2(a_0,2\pi).\\
1+K^2(a_0,3\pi) &= [K^2(a_0,3\pi)+1].\\
1+[K^2(a_0,4\pi)-1] &= K^2(a_0,4\pi).
\end{aligned}$$

In Theorem 5.2 we deliberately do not indicate composition of *abc*-triples, because it depends on the number of involved palindromes. If solution with one π unit is an *abc*-triple, then

$$R[K^2(a_0,\pi)]\cdot R[K^2(a_0,\pi)+1] < K^2(a_0,\pi)+1.$$

2. We elongate sequence to 2π, so objects for comparison are

$$R[K^2(a_0,2\pi)]\cdot R[K^2(a_0,2\pi)-1] \quad \text{and} \quad K^2(a_0,2\pi). \qquad (18)$$

From radical properties:

$$R[K^2(a_0,2\pi)] \le K(a_0,2\pi). \qquad (19)$$

$$K(a_0, 2\pi) = \underbrace{K(a_0, \pi, a_0) \cdot K(\pi)}_{=K^2(a_0,\pi)+1} + K^2(a_0, \pi) = 2K^2(a_0, \pi) + 1.$$

Then $K^2(a_0, 2\pi) - 1 = 4K^2(a_0, \pi) \cdot [K^2(a_0, \pi) + 1]$ and factor 4 can be ignored in radical calculations. We have:

$$R[K^2(a_0, 2\pi)-1] = R[K^2(a_0, \pi)] \cdot R[K^2(a_0, \pi)+1] < K^2(a_0, \pi)+1 < 2K^2(a_0, \pi)+1 = K(a_0, 2\pi).$$

By multiplying this with (19) we obtain necessary:

$$R[K^2(a_0, 2\pi)] \cdot R[K^2(a_0, 2\pi) - 1] < K^2(a_0, 2\pi). \tag{20}$$

Again – there will be analogous proof for all continuants with an even number of palindromic units due to splitting (for π even length)

$$K(a_0, n\pi, 2a_0, n\pi) = 2K^2(a_0, n\pi) + 1.$$

3. We elongate sequence to 3π, so objects for comparison are:

$$R[K^2(a_0, 3\pi)] \cdot R[K^2(a_0, 3\pi) + 1] \quad \text{and} \quad K^2(a_0, 3\pi) + 1.$$

$$\frac{K^2(a_0, 3\pi) + 1}{K^2(a_0, \pi) + 1} = \frac{K^2(3\pi)}{K^2(\pi)} = A^2$$

$$\text{and} \quad \frac{K^2(a_0, 3\pi)}{K^2(a_0, \pi)} = B^2.$$

A and B are natural numbers (see Theorems 3.1 and 3.2).

$$R(B^2) \leq B \quad \text{or} \quad R[\frac{K^2(a_0, 3\pi)}{K^2(a_0, \pi)}] \leq \frac{K(a_0, 3\pi)}{K(a_0, \pi)}. \tag{21}$$

$$R(A^2) \leq A \quad \text{or} \quad R[\frac{K^2(a_0, 3\pi) + 1}{K^2(a_0, \pi) + 1}] \leq \frac{K(3\pi)}{K(\pi)}. \tag{22}$$

$$R[K^2(a_0, \pi) + 1] \cdot R[K^2(a_0, \pi)] < K^2(a_0, \pi) + 1, \tag{23}$$

$$\text{or} \qquad R[K^2(a_0, \pi) + 1] \cdot R[K^2(a_0, \pi)] \leq K^2(a_0, \pi). \tag{24}$$

The product of two coprime radicals in the left side of (24) cannot be a square, therefore sign $\leq$ is changed again:

$$R[K^2(a_0, \pi) + 1] \cdot R[K^2(a_0, \pi)] < K^2(a_0, \pi) = K(a_0, \pi) \cdot K(a_0, \pi). \tag{25}$$

Reasoning, analogous to Theorem 5.1, gives:

$$R[K^2(a_0, \pi) + 1] \cdot R[K^2(a_0, \pi)] \leq K(a_0, \pi) \cdot [K(a_0, \pi) - 1]. \tag{26}$$

We multiply (21), (22) and (26):

$$R[K^2(a_0, 3\pi) + 1] \cdot R[K^2(a_0, 3\pi)] \leq K(a_0, 3\pi) \cdot [K(a_0, \pi) - 1] \cdot \frac{K(3\pi)}{K(\pi)}. \tag{27}$$

Now we compare

$$[K(a_0,\pi)-1]\cdot\frac{K(3\pi)}{K(\pi)} \quad \text{and} \quad K(a_0,3\pi).$$

Both are natural numbers, so square them. We have:

$$[K^2(a_0,\pi)+1-2K(a_0,\pi)]\cdot\frac{K^2(a_0,3\pi)+1}{K^2(a_0,\pi)+1} \quad \text{and} \quad K^2(a_0,3\pi);$$

$$\underbrace{[1-\frac{2K(a_0,\pi)}{K^2(a_0,\pi)+1}]}_{<1}\cdot[K^2(a_0,3\pi)+1] \quad \text{and} \quad K^2(a_0,3\pi).$$

Clearly $\leq$ sign must be used:

$$[K(a_0,\pi)-1]\cdot\frac{K(3\pi)}{K(\pi)} \leq K(a_0,3\pi).$$

Now we obtain from (27) the necessary:

$$R[K^2(a_0,3\pi)+1]\cdot R[K^2(a_0,3\pi)] \leq K^2(a_0,3\pi) < [K^2(a_0,3\pi)+1]. \tag{28}$$

Similar reasoning confirms the same rule in all other cases with odd number of π units. In terms of Theorem 5.2 this covers situation $k=0$ and all n values, but with even length palindrome π. For greater k values substitution of longer $\pi, 2a_0, ..., \pi$ chains by π' gives odd or even length palindromes π', whose behaviour is previously analysed. This completes the proof of theorem. □

Of course, both Theorems 5.1 and 5.2 can be combined in one, omitting the accent on the π length parity, but then the proof will be less clear.

Theorems 5.1 and 5.2 suggest that each primary *abc*-triple with the number of palindromes specified in the left column of the Table 9 induces an infinite sequence of secondary *abc*-triples, specified in the right columns of the Table 9.

Table 9.

Primary	Secondary				
1π	2π	3π	4π	5π	...
2π	4π	6π	8π	10π	...
3π	6π	9π	12π	15π	...
4π	8π	12π	16π	20π	...
5π	10π	15π	20π	25π	...
...	...	...	...	...	...

If fundamental roots of positive or negative Pell's equation for given specified D value produce an *abc*-triple, then all higher roots for this D also produce *abc*-triples. The sequence of such D values for positive Pell's equation is the following:

$$D = 2, 5, 7, 8, 12, 13, 14, 18, 20, 21, 27, 28, 29, 31, 32, 39, ...$$

From them, the following are of type $K(a_0, \pi), K(\pi)$:

$$D = 7, 8, 12, 14, 18, 20, 21, 27, 28, 31, 32, 39, 45, 46, 47, 48, ...,$$

but the following are of type $K(a_0, \pi, 2a_0, \pi), K(\pi, 2a_0, \pi)$:

$$D = 2, 5, 13, 29, 41, 50, 53, 73, 74, 85, 89, 109, 113, 122, ...$$

The corresponding sequence of such D values for negative Pell's equation is the following:

$$D = 41, 73, 89, 109, 125, 250, 338, 457, 610, 634, 761, 778, ...$$

Here all fundamental roots are of $K(a_0, \pi), K(\pi)$ type. Of course, all roots of positive Pell's equation for these D values produce *abc*-triples.

The following experimental Table 10 illustrates emerging of *abc*-triples from higher roots of positive/negative Pell's equations with specified number of palindromes (T means "True" – we get an *abc*-triple; F means "False").

Table 10.

	Number of π units																			
D	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
2	F	T	F	T	F	T	T	T	F	T	F	T	F	T	T	T	F	T	F	T
3	F	T	T	T	F	T	F	T	T	T	F	T	F	T	T	T	F	T	F	T
5	F	T	F	T	T	T	F	T	F	T	F	T	F	T	T	T	F	T	F	T
6	F	T	T	T	T	T	F	T	T	T	F	T	F	T	T	T	F	T	F	T
10	F	F	F	T	T	T	F	T	T	T	F	T	F	F	T	T	F	T	T	T
11	F	F	F	T	T	T	F	T	T	T	T	T	F	F	T	T	F	T	F	T
13	F	T	T	T	T	T	F	T	T	T	F	T	T	T	T	T	F	T	F	T
15	F	T	T	T	T	T	F	T	T	T	F	T	F	T	T	T	F	T	F	T
17	F	F	F	T	F	T	F	T	F	F	F	T	F	F	T	T	T	T	F	T
19	F	F	F	F	T	T	F	T	T	T	F	T	T	T	T	T	T	T	T	T
22	F	F	T	T	F	T	T	T	T	F	T	T	F	T	T	T	F	T	F	T
23	F	T	T	T	T	T	F	T	T	T	F	T	F	T	T	T	F	T	F	T

Characteristic and easily understandable are columns of "False" at 13π, 17π and 19π (here only primary *abc*-triples can emerge), as well as columns of "True" at 15π, 16π, 18π, 20π. For $D = 17$ here is a longer experimental sequence (Table 11), limited by my laptop's performance.

Table 11.

	Number of π units															
D	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
17	F	F	F	T	F	T	F	T	F	F	F	T	F	F	T	T

	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
	T	T	F	T	F	T	F	T	F	F	F	T	F	T	F	T

	33	34	35	36	37	38	39	40	41	42	43	44	45	46
	F	T	F	T	F	F	T	T	F	T	F	T	T	F

In Table 11 for $D = 17$ we see the first appearance of *abc*-triple at 4π. This primary *abc*-triple gives further secondary *abc*-triples at 8π, 12π, 16π, 20π, 24π, 28π, 32π, 36π, 40π and 44π. The next primary at 6π gives secondary triples at 12π, 18π, 24π, 30π, 36π, 42π; secondary triples from different primary sources can overlap. The next primary is at 15π, it gives secondary triples at 30π and 45π. Then primary at 17π gives secondary at 34π; primary at 22π gives secondary at 44π, remains one primary at 39π.

6 Generalized Pell's equation

Classical theory of the Diophantine equation

$$m^2 - Dn^2 = N \tag{29}$$

with $N \neq \pm 1$ tells us the following.

- Every solution $m + n \cdot \sqrt{D}$ of the Equation (29) can generate associated solutions of it, which are of the form

$$(m + n \cdot \sqrt{D}) \cdot (x + y \cdot \sqrt{D}) = \underbrace{m \cdot x + n \cdot y \cdot D}_{\text{other } m \text{ value}} + \underbrace{(m \cdot y + n \cdot x)}_{\text{other } n \text{ value}} \cdot \sqrt{D}. \tag{30}$$

  Here $x + y \cdot \sqrt{D}$ is any solution of the corresponding positive Pell's equation $x^2 - Dy^2 = 1$ or Pell's resolvent. There exists a solution associativity criterion and the set of all such associated solutions is called a class. As there are infinitely many solutions of Pell's resolvent, every such class contains an infinity of solutions to Equation (29), among them the fundamental one $m_0 + n_0 \cdot \sqrt{D}$ is defined. For one particular pair of D and N values there can be zero, one or more solution classes, characterized by their fundamental solutions.

- If one given class C consists of the solutions $m_i + n_i \cdot \sqrt{D}, i = 1, 2, 3, ...$ to Equation (29), the solutions $m_i - n_i \cdot \sqrt{D}, i = 1, 2, 3, ...$ also satisfy Equation (29) and constitute a class C', which is conjugate of C.

- Generally speaking, conjugate classes are distinct. If classes C and C' coincide, they are called ambiguous and they have a common ambiguous fundamental solution.

Useful online tools for finding fundamental solutions of Equation (29) are [11].

Example 6.1. For $D = 7$ and $N = 2, ..., 30$ we have the following fundamental solutions to Equation (29).

Table 12. Equation $m^2 - 7n^2 = N$.

N	Fundamental solutions m_0/n_0			N	Fundamental solutions m_0/n_0		
2	3/1 ambiguous			18	9/3 ambiguous;	5/1;	-5/1
4	2/0 ambiguous			21		7/2;	-7/2
8	6/2 ambiguous			25	5/0 ambiguous		
9	3/0 ambiguous;	4/1;	-4/1	29		6/1;	-6/1
16	4/0 ambiguous			...			

Remark. As our final objective is *abc*-equation with pairwise coprime components, we deliberately discard non-corresponding solutions of Equation (29). Remaining ones from Table 12 are shown in Table 13.

Table 13. Equation $m^2 - 7n^2 = N$.

N	Fundamental solutions m_0/n_0
2	3/1 ambiguous
9	4/1; -4/1
18	5/1; -5/1
29	6/1; -6/1

If we can find fundamental values $m_0 = K(\rho, \omega)$ and $n_0 = K(\omega)$, which satisfy an equation

$$m_0^2 - Dn_0^2 = N \quad \text{or} \quad K^2(\rho, \omega) - \frac{K^2(a_0, \pi) - 1}{K^2(\pi)} \cdot K^2(\omega) = N \tag{31}$$

with integral $N \neq \pm 1$, then we can find infinitely many positive solutions of Equation (29), each times elongating continuant sequences from the left side in the following way for π odd length:

$$\frac{m_0}{n_0} = \frac{K(\rho, \omega)}{K(\omega)}; \quad \frac{m_1}{n_1} = \frac{K(a_0, \pi, a_0 + \rho, \omega)}{K(\pi, a_0 + \rho, \omega)}; \quad \frac{m_2}{n_2} = \frac{K(a_0, \pi, 2a_0, \pi, a_0 + \rho, \omega)}{K(\pi, 2a_0, \pi, a_0 + \rho, \omega)}; \quad \text{etc.}$$

For π even length an increment contains two π units:

$$\frac{m_0}{n_0} = \frac{K(\rho, \omega)}{K(\omega)}; \quad \frac{m_1}{n_1} = \frac{K(a_0, \pi, 2a_0, \pi, a_0 + \rho, \omega)}{K(\pi, 2a_0, \pi, a_0 + \rho, \omega)}; \quad \text{etc.}$$

Here ρ is zero or positive integer, but ω is an empty set or a sequence of one or more positive integers.

We split m_1 and n_1 for π odd length:

$$\begin{aligned} m_1 &= K(a_0, \pi, a_0 + \rho, \omega) = K(a_0, \pi) \cdot K(\rho, \omega) + K(a_0, \pi, a_0) \cdot K(\omega) \\ &= K(a_0, \pi) \cdot K(\rho, \omega) + K(\pi) \cdot K(\omega) \cdot D; \end{aligned} \tag{32}$$

$$n_1 = K(\pi, a_0 + \rho, \omega) = K(\pi) \cdot K(\rho, \omega) + K(a_0, \pi) \cdot K(\omega). \tag{33}$$

The result agrees with an Equation (30).

Example 6.2. $D = 21, N = 4. K(\rho, \omega) = K(4, 1) = 5, K(\omega) = K(1) = 1.$

$$\frac{m_1}{n_1} = \frac{K(4, \boxed{1,1,2,1,1}, 4+4, 1)}{K(\boxed{1,1,2,1,1}, 4+4, 1)}; \quad \frac{m_2}{n_2} = \frac{K(4, \boxed{1,1,2,1,1}, 8, \boxed{1,1,2,1,1}, 4+4, 1)}{K(\boxed{1,1,2,1,1}, 8, \boxed{1,1,2,1,1}, 4+4, 1)}.$$

An odd length π unit $1, 1, 2, 1, 1$ is highlighted in the box. As $K() = 1 = K(1)$, the value of ω in such cases is ambiguous and in calculations $m_1 = K(1, 1, 2, 1, 1, 8, 1) = K(1, 1, 2, 1, 1, 9)$ – similarly to simple continued fractions. All higher roots also will end by $2, 1, 1, 9$.

Example 6.3. $D = 13, N = 4. K(\rho, \omega) = K(3, 1, 2) = 11, K(\omega) = K(1, 2) = 3.$

$$\frac{m_1}{n_1} = \frac{K(3, \boxed{1,1,1,1}, 6, \boxed{1,1,1,1}, 3+3, 1, 2)}{K(\boxed{1,1,1,1}, 6, \boxed{1,1,1,1}, 3+3, 1, 2)};$$

$$\frac{m_2}{n_2} = \frac{K(3, \boxed{1,1,1,1}, 6, \boxed{1,1,1,1}, 6, \boxed{1,1,1,1}, 6, \boxed{1,1,1,1}, 3+3, 1, 2)}{K(\boxed{1,1,1,1}, 6, \boxed{1,1,1,1}, 6, \boxed{1,1,1,1}, 6, \boxed{1,1,1,1}, 3+3, 1, 2)}.$$

An even length π unit $1, 1, 1, 1$ is highlighted in the box.

Example 6.4. $D = 5, N = 4. K(\rho, \omega) = K(2, 1) = 3, K(\omega) = K(1) = 1.$

$$\frac{m_1}{n_1} = \frac{K(2, \square, 4, \square, 2+2, 1)}{K(\square, 4, \square, 2+2, 1)}; \quad \frac{m_2}{n_2} = \frac{K(2, \square, 4, \square, 4, \square, 4, \square, 2+2, 1)}{K(\square, 4, \square, 4, \square, 4, \square, 2+2, 1)}.$$

Empty set $\square$ is an even length palindrome π. Again ω value ambiguity is present.
In calculations these empty sets do not show up and $m_1 = K(2, 4, 4, 1) = K(2, 4, 5) = 47$; $n_1 = K(4, 4, 1) = K(4, 5) = 21$; $47^2 - 5 \cdot 21^2 = 4$.

Example 6.5. $D = 2, N = -343. K(\rho, \omega) = K(0, 1, 4, 3) = 13, K(\omega) = K(1, 4, 3) = 16.$

$$\frac{m_1}{n_1} = \frac{K(1, \square, 2, \square, 1+0, 1, 4, 3)}{K(\square, 2, \square, 1+0, 1, 4, 3)}; \quad \frac{m_2}{n_2} = \frac{K(1, \square, 2, \square, 2, \square, 2, \square, 1+0, 1, 4, 3)}{K(\square, 2, \square, 2, \square, 2, \square, 1+0, 1, 4, 3)}.$$

Empty set $\square$ is an even length palindrome π. If $\rho = 0, K(\rho, \omega) < K(\omega)$. All higher roots for this example in continuant form will have a long sequence of two's, ending by 1, 1, 4, 3.

Higher solutions of the negative branch or conjugate class C' are products of $\pm(m_0 - n_0 \cdot \sqrt{D})$ with higher roots of Pell's resolvent $[K(a_0, \pi) + \sqrt{D} \cdot K(\pi)]^n$ (to avoid long and clumsy continuant expressions we restrict our exposition with odd length π units):

$$m'_1 = \pm m_0 \cdot K(a_0, \pi) \mp n_0 \cdot D \cdot K(\pi); \tag{34}$$

$$n'_1 = \pm m_0 \cdot K(\pi) \mp n_0 \cdot K(a_0, \pi); \tag{35}$$

$$m'_2 = \pm m_0 \cdot K(a_0, \pi, 2a_0, \pi) \mp n_0 \cdot D \cdot K(\pi, 2a_0, \pi); \tag{36}$$

$$n'_2 = \pm m_0 \cdot K(\pi, 2a_0, \pi) \mp n_0 \cdot K(a_0, \pi, 2a_0, \pi). \tag{37}$$

From Equation (36):

$$\begin{aligned} m'_2 &= \pm m_0 \cdot [K(a_0, \pi) \cdot K(a_0, \pi) + \underbrace{K(a_0, \pi, a_0) \cdot K(\pi)}_{=D \cdot K^2(\pi)}] \mp n_0 \cdot D \cdot 2K(\pi) \cdot K(a_0, \pi) \\ &= K(a_0, \pi) \cdot [\pm m_0 \cdot K(a_0, \pi) \mp n_0 \cdot D \cdot K(\pi)] + D \cdot K(\pi) \cdot [\pm m_0 \cdot K(\pi) \mp n_0 \cdot K(a_0, \pi)] \\ &= K(a_0, \pi) \cdot m'_1 + D \cdot K(\pi) \cdot n'_1. \end{aligned} \tag{38}$$

From Equation (37):

$$\begin{aligned} n'_2 &= \pm m_0 \cdot 2K(\pi) \cdot K(a_0, \pi) \mp n_0 \cdot [K^2(a_0, \pi) + D \cdot K^2(\pi)] \\ &= K(\pi) \cdot [\pm m_0 \cdot K(a_0, \pi) \mp n_0 \cdot D \cdot K(\pi)] \pm m_0 \cdot K(a_0, \pi) \cdot K(\pi) \mp n_0 \cdot K^2(a_0, \pi) \\ &= K(\pi) \cdot m'_1 + K(a_0, \pi) \cdot n'_1. \end{aligned} \tag{39}$$

Comparing Equations (38) and (39) with previously obtained m_1 and n_1 (Equations (32) and (33)), we see that m_1' and n_1' are analogues to $K(\rho,\omega)$ and $K(\omega)$ values. Therefore functional mechanism in both conjugated classes C and C' is the same. We illustrate this by conjugates to Examples 6.2 – 6.5.

Example 6.6. $D = 21, N = 4$. We use m_1' and n_1' as analogues to $K(\rho,\omega)$ and $K(\omega)$.

$$\frac{m_1'}{n_1'} = \frac{K(4,1,1,1,1)}{K(1,1,1,1)}; \quad \frac{m_2'}{n_2'} = \frac{K(4,\boxed{1,1,2,1,1},4+4,1,1,1,1)}{K(\boxed{1,1,2,1,1},4+4,1,1,1,1)}.$$

Again an odd length π unit $1,1,2,1,1$ is highlighted in the box. All higher roots for this example in continuant form will end by $8,1,1,2$.

Example 6.7. $D = 13, N = 4$. We use m_1' and n_1' as analogues to $K(\rho,\omega)$ and $K(\omega)$.

$$\frac{m_1'}{n_1'} = \frac{K(3,1,1,1,1,6)}{K(1,1,1,1,6)}; \quad \frac{m_2'}{n_2'} = \frac{K(3,\boxed{1,1,1,1},6,\boxed{1,1,1,1},3+3,1,1,1,1,6)}{K(\boxed{1,1,1,1},6,\boxed{1,1,1,1},3+3,1,1,1,1,6)}.$$

For even length π unit two of these units are necessary.

Example 6.8. $D = 5, N = 4$. We use m_1' and n_1' as analogues to $K(\rho,\omega)$ and $K(\omega)$.

$$\frac{m_1'}{n_1'} = \frac{K(2,3)}{K(3)}; \quad \frac{m_2'}{n_2'} = \frac{K(2,\square,4,\square,2+2,3)}{K(\square,4,\square,2+2,3)}.$$

Empty set $\square$ is an even length palindrome π.

Example 6.9. $D = 2, N = -343$. We use m_1' and n_1' as analogues to $K(\rho,\omega)$ and $K(\omega)$.

$$\frac{m_1'}{n_1'} = \frac{K(1,7,3)}{K(7,3)}; \quad \frac{m_2'}{n_2'} = \frac{K(1,\square,2,\square,1+1,7,3)}{K(\square,2,\square,1+1,7,3)}.$$

Empty set $\square$ is an even length palindrome π. All higher roots for this example in continuant form will have a long sequence of two's, ending by 7, 3.

Both roots in the generalized Pell's equation (29) are squared, therefore their signs do not impact their divisibility and *abc*-properties of the resulting equation.

Example 6.10. $D = 13, N = 4. K(\rho,\omega) = 11; K(\omega) = 3; K(a_0,\pi) = 649; K(\pi) = 180.$
For class C:

$$[K(a_0,\pi) + K(\pi)\cdot\sqrt{D}]\cdot[+K(\rho,\omega) + K(\omega)\cdot\sqrt{D}] \to$$
$$\to m_1 = +K(a_0,\pi)\cdot K(\rho,\omega) + K(\pi)\cdot K(\omega)\cdot D = 7139 + 7020 = 14159;$$
$$\to n_1 = +K(\pi)\cdot K(\rho,\omega) + K(a_0,\pi)\cdot K(\omega) = 1980 + 1947 = 3927.$$

For class C':

$$[K(a_0,\pi) + K(\pi)\cdot\sqrt{D}]\cdot[-K(\rho,\omega) + K(\omega)\cdot\sqrt{D}] \to$$
$$\to m_1' = -K(a_0,\pi)\cdot K(\rho,\omega) + K(\pi)\cdot K(\omega)\cdot D = -7139 + 7020 = -119;$$
$$\to n_1' = -K(\pi)\cdot K(\rho,\omega) + K(a_0,\pi)\cdot K(\omega) = -1980 + 1947 = -33.$$

We can take absolute values of m_1' and n_1':

$$119^2 - 13\cdot 33^2 = 14161 - 14157 = 4.$$

7 Genuine ambiguity

In the previous section we accented that coprimality requirements of components in *abc*-equation result in discarding a lot of fundamental solutions for Equation (29), which was illustrated by Tables 12 and 13. This is applied to ambiguous solutions, too. Therefore in this section we will discuss the remaining ones of ambiguous fundamental solutions for Equation (29), calling them *genuine ambiguous* (maybe anyone can offer a better term).

7.1 Necessary and sufficient conditions

Ambiguity implies coincidence of solution classes C and C'.

$$K(a_0,\pi)\cdot K(\rho,\omega)+K(a_0,\pi,a_0)\cdot K(\omega)=K(a_0,\pi,a_0+\rho,\omega)=m_1. \tag{40}$$

We equalize conjugated one to class C:

$$K(a_0,\pi)\cdot K(\rho,\omega)-K(a_0,\pi,a_0)\cdot K(\omega)=K(\rho,\omega)=m_0. \tag{41}$$

We add Equations (40) and (41):

$$K(a_0,\pi,a_0+\rho,\omega)+K(\rho,\omega)=2K(a_0,\pi)\cdot K(\rho,\omega);$$

$$K(a_0,\pi,a_0+\rho,\omega)=2K(a_0,\pi)\cdot K(\rho,\omega)-K(\rho,\omega) \text{ or } m_1=2x_0\cdot m_0-m_0.$$

We elongate sequence by $\pi, 2a_0, \pi$ instead of π.

$$K(a_0,\pi,2a_0,\pi)\cdot K(\rho,\omega)+K(a_0,\pi,2a_0,\pi,a_0)\cdot K(\omega)=K(a_0,\pi,2a_0,\pi,a_0+\rho,\omega)=m_2;$$

$$K(a_0,\pi,2a_0,\pi)\cdot K(\rho,\omega)-K(a_0,\pi,2a_0,\pi,a_0)\cdot K(\omega)=K(a_0,\pi,a_0+\rho,\omega)=m_1.$$

We add them, etc. and get $m_2=2x_1\cdot m_0-m_1$, then inductively:

$$m_k=2x_{k-1}\cdot m_0-m_{k-1}\quad (k=1,2,...). \tag{42}$$

Analogously from

$$K(\pi)\cdot K(\rho,\omega)+K(a_0,\pi)\cdot K(\omega)=K(\pi,a_0+\rho,\omega)=n_1 \tag{43}$$

$$\text{and}\qquad K(\pi)\cdot K(\rho,\omega)-K(a_0,\pi)\cdot K(\omega)=K(\omega)=n_0 \tag{44}$$

we get $n_1=2x_0\cdot n_0+n_0$, etc., and finally inductively:

$$n_k=2x_{k-1}\cdot n_0+n_{k-1}\quad (k=1,2,...). \tag{45}$$

Previously we proposed two versions for conjugate class (Equations (34) and (35)), so from

$$K(a_0,\pi)\cdot K(\rho,\omega)+K(a_0,\pi,a_0)\cdot K(\omega)=K(a_0,\pi,a_0+\rho,\omega)=m_1 \tag{46}$$

$$\text{and}\qquad -K(a_0,\pi)\cdot K(\rho,\omega)+K(a_0,\pi,a_0)\cdot K(\omega)=K(\rho,\omega)=m_0 \tag{47}$$

we get for ambiguity

$$m_k = 2x_{k-1} \cdot m_0 + m_{k-1} \quad (k = 1, 2, ...) \tag{48}$$

and, in an identical way,

$$n_k = 2x_{k-1} \cdot n_0 - n_{k-1} \quad (k = 1, 2, ...). \tag{49}$$

Both versions for ambiguity can be combined:

$$\begin{cases} m_k = 2x_{k-1} \cdot m_0 \mp m_{k-1}, \\ n_k = 2x_{k-1} \cdot n_0 \pm n_{k-1} \quad (k = 1, 2, ...). \end{cases} \tag{50}$$

Now we write once more the first version of ambiguity:

$$K(\rho, \omega) = K(a_0, \pi) \cdot K(\rho, \omega) - K(a_0, \pi, a_0) \cdot K(\omega); \tag{51}$$

$$K(\omega) = K(\pi) \cdot K(\rho, \omega) - K(a_0, \pi) \cdot K(\omega). \tag{52}$$

As $K(\rho, \omega) \perp K(\omega)$, then

- from Equation (51) we have $K(\rho, \omega) | K(a_0, \pi, a_0)$ – condition (A);
- from Equation (52) we have $K(\omega) | K(\pi)$ – condition (B).

The restriction for genuine ambiguity is component coprimality in generalized Pell's equation

$$K^2(\rho, \omega) - \frac{K^2(a_0, \pi) - 1}{K^2(\pi)} \cdot K^2(\omega) = N.$$

As $K(\rho, \omega) \perp K(\omega)$, this restriction also gives

$$K(\rho, \omega) \perp \frac{K^2(a_0, \pi) - 1}{K^2(\pi)} = \frac{K(a_0, \pi, a_0) \cdot K(\pi)}{K^2(\pi)} = \frac{K(a_0, \pi, a_0)}{K(\pi)}.$$

Together with condition (A) this gives $K(\rho, \omega) | K(\pi)$, then we take into account condition (B) and obtain $K(\rho, \omega) \cdot K(\omega) | K(\pi)$.

From Equations (51) and (52) we get the following.

$$\begin{aligned} K(\rho, \omega) \cdot K(\omega) = {} & K(a_0, \pi) \cdot K(\pi) \cdot K^2(\rho, \omega) + K(a_0, \pi, a_0) \cdot K(a_0, \pi) \cdot K^2(\omega) \\ & - K(a_0, \pi, a_0) \cdot K(\pi) \cdot K(\rho, \omega) \cdot K(\omega) - K^2(a_0, \pi) \cdot K(\rho, \omega) \cdot K(\omega) \end{aligned}$$

$$= K(a_0, \pi) \cdot K(\pi) \cdot [K^2(\rho, \omega) + \frac{K(a_0, \pi, a_0)}{K(\pi)} \cdot K^2(\omega)] - K(\rho, \omega) \cdot K(\omega) \cdot [2K^2(a_0, \pi) - 1].$$

From this

$$K(\rho, \omega) \cdot K(\omega) = \frac{K(a_0, \pi) \cdot K(\pi)}{2K^2(a_0, \pi)} \cdot [K^2(\rho, \omega) + \frac{K(a_0, \pi, a_0)}{K(\pi)} \cdot K^2(\omega)]$$

$$= \frac{K(\pi)}{2K(a_0, \pi)} \cdot [K^2(\rho, \omega) + \frac{K(a_0, \pi, a_0)}{K(\pi)} \cdot K^2(\omega)].$$

Then

$$[K(\rho,\omega)\cdot K(\omega)]\cdot K(a_0,\pi)=K(\pi)\cdot\frac{1}{2}\cdot[K^2(\rho,\omega)+\frac{K(a_0,\pi,a_0)}{K(\pi)}\cdot K^2(\omega)]. \tag{53}$$

We have $K(a_0,\pi)\perp K(\pi)$ and $K(\rho,\omega)\cdot K(\omega)|K(\pi)$. In square brackets of Equation (53) is a sum of two coprime summands, one of them divides by $K(\omega)$, the other by $K(\rho,\omega)$, therefore their sum is coprime to $K(\rho,\omega)\cdot K(\omega)$. Now we have two possibilities.

1. This works for positive/negative Pell's equations, because $N=\pm 1$ gives an ambiguous class.

$$\begin{cases} K(\pi) &= 2K(\rho,\omega)\cdot K(\omega), \\ K(a_0,\pi) &= K^2(\rho,\omega)+\dfrac{K(a_0,\pi,a_0)}{K(\pi)}\cdot K^2(\omega). \end{cases} \tag{54}$$

2. This is a situation with genuine ambiguous solutions to generalized Pell's equation (29).

$$\begin{cases} K(\pi) &= K(\rho,\omega)\cdot K(\omega), \\ K(a_0,\pi) &= \dfrac{1}{2}\cdot[K^2(\rho,\omega)+\dfrac{K(a_0,\pi,a_0)}{K(\pi)}\cdot K^2(\omega)]. \end{cases} \tag{55}$$

From the second version of ambiguity

$$K(\rho,\omega)=-K(a_0,\pi)\cdot K(\rho,\omega)+K(a_0,\pi,a_0)\cdot K(\omega) \tag{56}$$

$$K(\omega)=-K(\pi)\cdot K(\rho,\omega)+K(a_0,\pi)\cdot K(\omega) \tag{57}$$

we can obtain the same results (Eq. systems (54) and (55)).

We equalize m_1 values from Equations (40) and (42):

$$2K(a_0,\pi)\cdot K(\rho,\omega)-K(\rho,\omega)=K(a_0,\pi)\cdot K(\rho,\omega)+K(a_0,\pi,a_0)\cdot K(\omega);$$

$$K(\rho,\omega)\cdot[K(a_0,\pi)-1]=\frac{K^2(a_0,\pi)-1}{K(\pi)}\cdot K(\omega);$$

$$\frac{K(\rho,\omega)}{K(\omega)}=\frac{K(a_0,\pi)+1}{K(\pi)}. \tag{58}$$

Expression (58) can be obtained also from Equations (43) and (45).

As we recently obtained $K(\pi)=K(\rho,\omega)\cdot K(\omega)$ for genuine ambiguity (Eq. system (55)), from Equation (58) we get:

$$\frac{K(\rho,\omega)}{K(\omega)}=\frac{K(a_0,\pi)+1}{K(\rho,\omega)\cdot K(\omega)};$$

$$K^2(\rho,\omega)=K(a_0,\pi)+1.$$

$$N=K(a_0,\pi)+1-\frac{K^2(a_0,\pi)-1}{K^2(\pi)}\cdot K^2(\omega)=K(a_0,\pi)+1-\frac{K^2(a_0,\pi)-1}{K^2(\rho,\omega)}$$

$$=K(a_0,\pi)+1-\frac{K^2(a_0,\pi)-1}{K(a_0,\pi)+1}=K(a_0,\pi)+1-K(a_0,\pi)+1=2.$$

Analogously from the second version of ambiguity (Equations (48) and (49)) we obtain

$$\frac{K(\rho,\omega)}{K(\omega)}=\frac{K(a_0,\pi)-1}{K(\pi)} \tag{59}$$

and $N=-2$.

Remark. By employing $K(\pi) = 2K(\rho,\omega) \cdot K(\omega)$ instead of $K(\pi) = K(\rho,\omega) \cdot K(\omega)$ we will obtain $N = \pm 1$ – standard values for positive/negative Pell's equations.

Formula $K(\pi) = K(\rho,\omega) \cdot K(\omega)$ also means that genuine ambiguous fundamental solution is single. If $K(\rho,\omega), K(\omega)$ and $K(\rho',\omega'), K(\omega')$ will be two different partitions of $K(\pi)$ in coprime parts, then $K(\rho,\omega) \cdot K(\omega) = K(\rho',\omega') \cdot K(\omega')$. Then, for example, $K(\rho,\omega) > K(\rho',\omega')$, which means $K(\omega) < K(\omega')$ and, taking $N = 2$, we get $D \cdot K^2(\omega) + 2 < D \cdot K^2(\omega') + 2$, which gives $K^2(\rho,\omega) < K^2(\rho',\omega')$ – a contradiction.

We rewrite our generalized Pell's equation $K^2(\rho,\omega) - D \cdot K^2(\omega) = N$, for which we assume existence of genuine ambiguous fundamental solutions. From recently obtained $N = \pm 2$ we conclude that $K^2(\rho,\omega)$ and $D \cdot K^2(\omega)$ are odd numbers. As squares $K^2(\rho,\omega) \equiv 1 \pmod 4$ and $K^2(\omega) \equiv 1 \pmod 4$, but $N \equiv \pm 2 \pmod 4$, that means $D \equiv 3 \pmod 4$. Then $K(\pi)$ is an odd number, but $K(a_0,\pi)$ is an even number.

From equations $K^2(\rho,\omega) - D \cdot K^2(\omega) = N$ and $K^2(a_0,\pi) - D \cdot K^2(\pi) = 1$ we express D and equalize:

$$\frac{K^2(\rho,\omega) - N}{K^2(\omega)} = \frac{K^2 a_0,\pi) - 1}{K^2(\pi)} = \frac{K^2 a_0,\pi) - 1}{K^2(\rho,\omega) \cdot K^2(\omega)}.$$

Thus $K^2(\rho,\omega) \cdot [K^2(\rho,\omega) \pm 2] = K^2 a_0,\pi) - 1 = [K(a_0,\pi) - 1] \cdot [K(a_0,\pi) + 1]$. As $K(a_0,\pi)$ is even, $K(a_0,\pi) - 1$ or $K(a_0,\pi) + 1$ must be perfect square – we obtained this previously in calculations of $N = \pm 2$ values. Then, substituing $K^2(\rho,\omega)$ by $K(a_0,\pi) \pm 1$ in our equation $K^2(\rho,\omega) - D \cdot K^2(\omega) = N$, we get $D \cdot K^2(\omega) = K(a_0,\pi) \pm 1$.

All conclusions from current subsection can be united in the

Theorem 7.1. ***1.*** *If natural non-square* $D = \dfrac{K(a_0,\pi,a_0)}{K(\pi)} \equiv 3 \pmod 4$ *is the discriminant of the positive Pell's equation* $K^2(a_0,\pi) - D \cdot K^2(\pi) = 1$*, if* $D | K(a_0,\pi) - 1$*, and if* $K(a_0,\pi) + 1$ *is perfect square, then one and only one pair* $K(\rho,\omega), K(\omega)$ *of coprime divisors of* $K(\pi)$*, which satisfy expressions* $K(\pi) = K(\rho,\omega) \cdot K(\omega)$ *and* $K(a_0,\pi) = \dfrac{1}{2} \cdot [K^2(\rho,\omega) + D \cdot K^2(\omega)]$*, are genuine ambiguous fundamental roots of generalized Pell's equation* $K^2(\rho,\omega) - D \cdot K^2(\omega) = 2$*. There are not solutions for the corresponding equation* $K^2(\rho,\omega) - D \cdot K^2(\omega) = -2$*.*

2. *If natural non-square* $D = \dfrac{K(a_0,\pi,a_0)}{K(\pi)} \equiv 3 \pmod 4$ *is the discriminant of the positive Pell's equation* $K^2(a_0,\pi) - D \cdot K^2(\pi) = 1$*, if* $D | K(a_0,\pi) + 1$*, and if* $K(a_0,\pi) - 1$ *is perfect square, then one and only one pair* $K(\rho,\omega), K(\omega)$ *of coprime divisors of* $K(\pi)$*, which satisfy expressions* $K(\pi) = K(\rho,\omega) \cdot K(\omega)$ *and* $K(a_0,\pi) = \dfrac{1}{2} \cdot [K^2(\rho,\omega) + D \cdot K^2(\omega)]$*, are genuine ambiguous fundamental roots of generalized Pell's equation* $K^2(\rho,\omega) - D \cdot K^2(\omega) = -2$*. There are not solutions for the corresponding equation* $K^2(\rho,\omega) - D \cdot K^2(\omega) = 2$*.*

The sequence of $D \equiv 3 \pmod 4$ values for equation $K^2(\rho,\omega) - D \cdot K^2(\omega) = 2$ with genuine ambiguous solutions is the following.

$D = 7, 23, 31, 47, 71, 79, 103, 119, 127, 151, 167, 191, 199, 223, 239, 263, 271, ...$

Table 14 shows links with the corresponding Pell's resolvent.

Table 14. Equation $K^2(\rho,\omega) - D \cdot K^2(\omega) = 2$.

D	$K(a_0,\pi)/K(\pi)$	$K(a_0,\pi)-1$	$K(a_0,\pi)+1$	$K(\rho,\omega)$	$K(\omega)$
7	8/3	7	9	3	1
23	24/5	23	25	5	1
31	1520/273	$1519 = 31 \cdot 49$	$1521 = 39^2$	39	7
47	48/7	47	49	7	1
71	3480/413	$3479 = 71 \cdot 49$	$3481 = 59^2$	59	7
79	80/9	79	81	9	1
103	227528/22419	$227527 = 103 \cdot 2209$	$227529 = 477^2$	477	47

The sequence of $D \equiv 3 \pmod 4$ values for equation $K^2(\rho,\omega) - D \cdot K^2(\omega) = -2$ with genuine ambiguous solutions is the following.

$D =$ 3, 11, 19, 27, 43, 51, 59, 67, 83, 107, 123, 131, 139, 163, 171, 179, 187, 211, 227, 243, 251, 267, ...

Table 15 shows links with the corresponding Pell's resolvent.

Table 15. Equation $K^2(\rho,\omega) - D \cdot K^2(\omega) = -2$.

D	$K(a_0,\pi)/K(\pi)$	$K(a_0,\pi)+1$	$K(a_0,\pi)-1$	$K(\rho,\omega)$	$K(\omega)$
3	2/1	3	1	1	1
11	10/3	11	9	3	1
19	170/39	$171 = 19 \cdot 9$	$169 = 13^2$	13	3
27	26/5	27	25	5	1
43	3482/531	$3483 = 43 \cdot 81$	$3481 = 59^2$	59	9
51	50/7	51	49	7	1
59	530/69	$531 = 59 \cdot 9$	$529 = 23^2$	23	3

7.2 Higher roots, divisibility and *abc*-properties

Consequently, if $K(\rho,\omega), K(\omega)$ correspond to requirements of Theorem 7.1, then they are genuine ambiguous fundamental roots m_0, n_0, but all higher roots are obtained by standard method:

$$\frac{m_0}{n_0} = \frac{K(\rho,\omega)}{K(\omega)}; \quad \frac{m_1}{n_1} = \frac{K(a_0,\pi,a_0+\rho,\omega)}{K(\pi,a_0+\rho,\omega)}; \quad \frac{m_2}{n_2} = \frac{K(a_0,\pi,2a_0,\pi,a_0+\rho,\omega)}{K(\pi,2a_0,\pi,a_0+\rho,\omega)}; \quad \text{etc.}$$

($D \equiv 3 \pmod 4$ from Theorem 7.1 means π is odd length.)

We illustrate divisibility properties of these higher roots by their origin.

Theorem 7.2. *If $K(\rho,\omega), K(\omega)$ are genuine ambiguous fundamental roots m_0, n_0 of the generalized Pell's equation $K^2(\rho,\omega) - D \cdot K^2(\omega) = \pm 2$ and $K(a_0,\pi), K(\pi)$ are fundamental roots of the corresponding Pell's resolvent, then their higher roots meet the following two relations:*

$$K(\underbrace{\pi,2a_0}_{2k\text{-times}},\pi,2a_0,\pi,2a_0,\pi) = K(a_0,\pi,\underbrace{2a_0,\pi}_{k\text{-times}},a_0+\rho,\omega) \cdot K(\pi,\underbrace{2a_0,\pi}_{k\text{-times}},a_0+\rho,\omega); \tag{60}$$

$$K(\underbrace{\pi,2a_0}_{2k\text{-times}},\pi,2a_0,\pi,2a_0,\pi,a_0) = \frac{1}{2} \cdot [K^2(a_0,\pi,\underbrace{2a_0,\pi}_{k\text{-times}},a_0+\rho,\omega) + D \cdot K^2(\pi,\underbrace{2a_0,\pi}_{k\text{-times}},a_0+\rho,\omega). \tag{61}$$

Here $k = 0, 1, 2, ...$

Expressions (60) and (61) are analogues to Equations (55).

Proof. **1.** $k=0$ and we must justify $K(a_0,\pi,a_0+\rho,\omega)\cdot K(\pi,a_0+\rho,\omega)=K(3\pi)$.

$$K(3\pi)=K(\pi,2a_0,\pi)\cdot K(a_0,\pi)+K(a_0,\pi,2a_0,\pi)\cdot K(\pi)=3K^2(a_0,\pi)\cdot K(\pi)+K(a_0,\pi,a_0)\cdot K^2(\pi).$$

$$\begin{aligned}
&K(a_0,\pi,a_0+\rho,\omega)\cdot K(\pi,a_0+\rho,\omega)\\
&\quad=[K(a_0,\pi)\cdot K(\rho,\omega)+K(a_0,\pi,a_0)\cdot K(\omega)]\cdot[K(\pi)\cdot K(\rho,\omega)+K(a_0,\pi)\cdot K(\omega)]\\
&=K(a_0,\pi)\cdot K(\pi)\cdot K^2(\rho,\omega)+K(a_0,\pi,a_0)\cdot K(\pi)\cdot K(\rho,\omega)\cdot K(\omega)+K^2(a_0,\pi)\cdot K(\rho,\omega)\cdot K(\omega)\\
&\qquad+K(a_0,\pi,a_0)\cdot K(a_0,\pi)\cdot K^2(\omega)\\
&=K(a_0,\pi)\cdot K(\pi)\cdot[K^2(\rho,\omega)+\frac{K(a_0,\pi,a_0)}{K(\pi)}\cdot K^2(\omega)]+K(a_0,\pi,a_0)\cdot K^2(\pi)+K^2(a_0,\pi)\cdot K(\pi)
\end{aligned}$$

$=3K^2(a_0,\pi)\cdot K(\pi)+K(a_0,\pi,a_0)\cdot K^2(\pi)$.
We obtain the same, this confirms Equation (60) for $k=0$.
2. From binomial form (Section 4.3) we obtain:

$$\begin{aligned}
&K(a_0,2n\pi)+\sqrt{D}\cdot K(2n\pi)=[K(a_0,\pi)+\sqrt{D}\cdot K(\pi)]^{2n}\\
&\quad=[K(a_0,n\pi)+\sqrt{D}\cdot K(n\pi)]^2=K^2(a_0,n\pi)+2\sqrt{D}\cdot K(a_0,n\pi)\cdot K(n\pi)+D\cdot K^2(n\pi).
\end{aligned}$$

That means

$$\left\{\begin{aligned} K(a_0,2n\pi)&=K^2(a_0,n\pi)+D\cdot K^2(n\pi),\\ K(2n\pi)&=2K(a_0,n\pi)\cdot K(n\pi).\end{aligned}\right.\tag{62}$$

3. For $k>0$ we at first split off unnecessary fragment to obtain even number of π units, then use relations (62).

$$K(\underbrace{\pi,2a_0}_{2k\text{-times}},\pi,2a_0,\pi,2a_0,\pi)=K(\underbrace{\pi,2a_0}_{2k\text{-times}},\pi,2a_0,\pi)\cdot K(a_0,\pi)+K(\underbrace{\pi,2a_0}_{2k\text{-times}},\pi,2a_0,\pi,a_0)\cdot K(\pi)$$

$$=2K(\underbrace{\pi,2a_0}_{k\text{-times}},\pi,a_0)\cdot K(\underbrace{\pi,2a_0}_{k\text{-times}},\pi)\cdot K(a_0,\pi)+[K^2(\underbrace{\pi,2a_0}_{k\text{-times}},\pi,a_0)+\frac{K(a_0,\pi,a_0)}{K(\pi)}\cdot K^2(\underbrace{\pi,2a_0}_{k\text{-times}},\pi)]\cdot K(\pi).$$

Now we calculate the right side of Equation (60).

$$\begin{aligned}
&K(a_0,\pi,\underbrace{2a_0,\pi}_{k\text{-times}},a_0+\rho,\omega)\cdot K(\pi,\underbrace{2a_0,\pi}_{k\text{-times}},a_0+\rho,\omega)\\
&=[K(\rho,\omega)\cdot K(a_0,\pi,\underbrace{2a_0,\pi}_{k\text{-times}})+\frac{K(a_0,\pi,a_0)}{K(\pi)}\cdot K(\underbrace{\pi,2a_0}_{k\text{-times}},\pi)\cdot K(\omega)]\cdot[K(\rho,\omega)\cdot K(\pi,\underbrace{2a_0,\pi}_{k\text{-times}})\\
&\qquad+K(\omega)\cdot K(a_0,\pi,\underbrace{2a_0,\pi}_{k\text{-times}})]\\
&=K^2(\rho,\omega)\cdot K(a_0,\pi,\underbrace{2a_0,\pi}_{k\text{-times}})\cdot K(\underbrace{\pi,2a_0}_{k\text{-times}},\pi)+K(\rho,\omega)\cdot K(\omega)\cdot\frac{K(a_0,\pi,a_0)}{K(\pi)}\cdot K^2(\underbrace{\pi,2a_0}_{k\text{-times}},\pi)
\end{aligned}$$

$$+K(\rho,\omega)\cdot K(\omega)\cdot K^2(a_0,\pi,\underbrace{2a_0,\pi}_{k\text{-times}})+\frac{K(a_0,\pi,a_0)}{K(\pi)}\cdot K(\underbrace{\pi,2a_0}_{k\text{-times}},\pi)\cdot K^2(\omega)\cdot K(a_0,\pi,\underbrace{2a_0,\pi}_{k\text{-times}})$$

$$=K(a_0,\pi,\underbrace{2a_0,\pi}_{k\text{-times}})\cdot K(\underbrace{\pi,2a_0}_{k\text{-times}},\pi)\cdot[K^2(\rho,\omega)+\frac{K(a_0,\pi,a_0)}{K(\pi)}\cdot K^2(\omega)]$$

$$+K(\rho,\omega)\cdot K(\omega)\cdot[K^2(a_0,\pi,\underbrace{2a_0,\pi}_{k\text{-times}})+\frac{K(a_0,\pi,a_0)}{K(\pi)}\cdot K^2(\underbrace{\pi,2a_0}_{k\text{-times}},\pi)]$$

$$=2K(\underbrace{\pi,2a_0}_{k\text{-times}},\pi,a_0)\cdot K(\underbrace{\pi,2a_0}_{k\text{-times}},\pi)\cdot K(a_0,\pi)+[K^2(\underbrace{\pi,2a_0}_{k\text{-times}},\pi,a_0)+\frac{K(a_0,\pi,a_0)}{K(\pi)}\cdot K^2(\underbrace{\pi,2a_0}_{k\text{-times}},\pi)]\cdot K(\pi).$$

This confirms Equation (60). The proof of Equation (61), closely analogous to recent one, is left to the reader. □

In view of Equations (55) and (60), we can employ the corresponding rows and columns of Table 1 for describing $m_i\cdot n_i|y_j$ divisibility in $K(\pi)|K(\pi)$ terms.

Table 16.

Divisor $m_i\cdot n_i$	Dividends (trivial cases, as $1\pi\|1\pi$, are omitted) y_j				
1π	3π	5π	7π	9π	...
3π	9π	15π	21π	27π	...
5π	15π	25π	35π	45π	...
7π	21π	35π	49π	63π	...
9π	27π	45π	63π	81π	...
...	...	...	...	...	...

Together with $m_i\perp n_i$, $m_0|m_k$ and $n_0|n_k$ for all $k=0,1,2,...$, as well as Equations (55) and (60), the divisibility table $n_i|n_j$ coincides with the Table 2, but the divisibility table $m_i|m_j$ coincides with the Table 5 and we do not repeat them here. As the result for genuine ambiguous solutions of equation $m^2-D\cdot n^2=\pm2$ each m_i with k units of π is a divisor of all m_j with $k+n(2k+1)$ units of π. Here $k=0,1,2,...$ and $n=0,1,2,...$ The same for $n_i|n_j$ divisibility.

Theorem 7.3. *Now π is an odd length palindromic unit. If roots (m_i,n_i) of the genuine ambiguous generalized Pell's equation $m^2-D\cdot n^2=\pm2$, having k palindromic units π, produce an abc-triple, then abc-triples are produced by all roots (m_j,n_j) of this equation, having $k+n\cdot(2k+1)$ palindromic units. Here $k=0,1,2,...$ and $n=0,1,2,...$*

Proof. Our *abc*-equation (for $N=2$) is $2+[K^2(\rho,\omega)-2]=K^2(\rho,\omega)$, which gives

$$2\cdot R[K^2(\rho,\omega)]\cdot R[K^2(\rho,\omega)-2]<K^2(\rho,\omega)$$

as starting point. Radical expressions (in square brackets) are two coprime odd numbers.

According to obtained divisibility data,

$$\frac{K^2(a_0, \pi, a_0 + \rho, \omega) - 2}{K^2(\rho, \omega) - 2} = \frac{K^2(\pi, a_0 + \rho, \omega)}{K^2(\omega)} = A^2 \quad \text{and} \quad \frac{K^2(a_0, \pi, a_0 + \rho, \omega)}{K^2(\rho, \omega)} = B^2,$$

where A and B are natural numbers. So we can work analogously to the corresponding parts of Theorems 5.1 and 5.2. Detailed outline is left to concerned reader. $\square$

Theorem 7.3 suggests that for genuine ambiguous generalized Pell's equations each primary *abc*-triple with the number of palindromes specified in the left column of the Table 17 induces an infinite sequence of secondary *abc*-triples, specified in the right columns of the Table 17.

Table 17.

Primary	Secondary				
0π	1π	2π	3π	4π	...
1π	4π	7π	10π	13π	...
2π	7π	12π	17π	22π	...
3π	10π	17π	24π	31π	...
4π	13π	22π	31π	40π	...
...	...	...	...	...	...

The following experimental Table 18 illustrates emerging of *abc*-triples from higher roots of genuine ambiguous Pell's equations with specified number of palindromes (T means "True" – we get an *abc*-triple; F means "False").

Table 18.

		Number of π units																	
D	N	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
3	-2	F	F	F	F	T	F	F	T	F	F	F	F	F	T	F	F	F	F
7	2	F	F	F	T	T	F	F	T	F	F	T	F	F	T	F	F	F	T
11	-2	F	F	F	F	T	T	F	F	F	F	T	F	F	T	F	F	T	F
19	-2	F	F	F	F	T	F	T	F	F	T	F	F	F	T	F	F	T	F
23	2	F	F	F	F	F	F	F	T	F	F	T	T	T	F	F	F	F	F
27	-2	F	T	T	F	T	F	F	T	F	F	T	F	T	T	F	F	T	T

For $D = 3, N = -2$ here is a longer experimental sequence (Table 19), limited by my laptop's performance.

Table 19.

	Number of π units																	
D	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
3	F	F	F	F	T	F	F	T	F	F	F	F	F	T	F	F	F	F
	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35
	F	F	F	F	T	F	F	F	F	T	F	F	F	T	F	F	F	F
	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50			
	F	T	F	F	T	F	F	F	F	F	F	T	F	T	F			

In Table 19 for $D = 3, N = -2$ we see the first appearance of *abc*-triple at 4π. This primary *abc*-triple gives further secondary *abc*-triples at 13π, 22π, 31π, 40π and 49π. The next primary at 7π gives secondary triples at 22π and 37π; secondary triples from different primary sources can overlap. Remains two primary at 27π and 47π.

As genuine ambiguous fundamental roots (m_0, n_0) of the generalized Pell's equation (29) with $N = 2$ produce an *abc*-triples for $D = 343$ and $D = 511$, all higher roots for these D values will produce *abc*-triples.

As genuine ambiguous fundamental roots (m_0, n_0) of the generalized Pell's equation (29) with $N = -2$ produce an *abc*-triples for $D = 243, 251, 307, 867, 1107, 1331$, all higher roots for these D values will produce *abc*-triples.

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IMPORTANT REMARK

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