THE FRACTIONAL INVARIANCE ANALYSIS AND APPLICATIONS

THEOPHILUS AGAMA

Abstract. In this note, we introduce and develop the analysis of the fractional invariance. This analysis is used for estimating the partial sums of arithmetic functions $f : \mathbb{N} \longrightarrow \mathbb{R}$ of the form $\sum\limits_{\substack{n \leq x \\ n \in \mathbb{A}}} f(n)$ for $\mathbb{A} \subseteq \mathbb{N}$. This analysis can be applied to a broad class of arithmetic functions.

1. Introduction

The study of arithmetic functions is a cornerstone of number theory, providing profound insights into the distribution of integers and their relationships. These functions, which map natural numbers to real values, encompass a wide variety of important sequences, including divisor functions, multiplicative functions, and the celebrated Euler totient function. Estimating the partial sums of such functions may not be straightforward, often requiring sophisticated analytical techniques to yield accurate results. Traditional approaches to estimating these partial sums frequently encounter limitations due to the oscillatory nature of arithmetic functions and their diverse growth rates. As a result, existing methods may struggle to provide estimates that are both precise and reflective of the inherent smooth trends that characterize many arithmetic functions. This is particularly evident in the estimation of sums involving functions that exhibit erratic behaviors over their domains. While complex variable techniques have been developed to address these challenges, they often require substantial mathematical machinery and may not be accessible to all practitioners in the field [1]. In this context, we introduce an analysis that allows us to decompose an arithmetic function into several parts, where some parts contains a smoothed-out version of the arithmetic function. We leverage this analysis to estimate the partial sums of a broad class of arithmetic functions. Our approach centers around the concept of the fractional invariance, which extends the properties of an arithmetic function $f : \mathbb{N} \longrightarrow \mathbb{R}$ to certain subsets of the real numbers. By leveraging this concept, we develop a systematic framework that enables us to derive accurate estimates for sums of the form $\sum\limits_{\substack{n \leq x \\ n \in \mathbb{A}}} f(n)$ with $\mathbb{A} \subseteq \mathbb{N}$. This method serves as an alternative to existing techniques, providing an accessible and effective means of analysis that enhances our understanding of the behavior of arithmetic functions.

Date: November 2, 2024.
2010 *Mathematics Subject Classification.* Primary 11P32, 11A41; Secondary 11B13, 11H99.
Key words and phrases. fractional; invariant; uniformity.

The following sections will elaborate on the theoretical foundations of our method, detailing its application to various arithmetic functions and demonstrating its efficacy through concrete estimates. We aim to elucidate the advantages of this approach, highlighting its versatility and potential for further exploration in the rich landscape of analysis and number theory.

2. The fractional invariant function

In this section, we extend the validity of an arithmetic function to subsets of the reals.

Definition 2.1. Let $f : \mathbb{N} \longrightarrow \mathbb{R}$. We denote the fractional invariance of f with $\tilde{f}$ as the function $\tilde{f} : \mathbb{R} \longrightarrow \mathbb{R}$ such that $\tilde{f}(x) = f(\lfloor x \rfloor) + \{x\}$, where $\lfloor \cdot \rfloor$ and $\{\cdot\}$ denotes the integer and the fractional part of the real number x.

The fractional invariance $\tilde{f}$ of f is a slight extension that preserves the intrinsic property of f on the reals $\mathbb{R}$. Some immediate properties of $\tilde{f}$ are obvious.

2.1. Properties of the fractional invariant function.

In this section, we expose some fundamental properties of the fractional invariance of the function $f : \mathbb{N} \longrightarrow \mathbb{R}$.

Proposition 2.2. *Let $f : \mathbb{N} \longrightarrow \mathbb{R}$. The following properties of the fractional invariant function $\tilde{f}$ hold*

(i) $\tilde{f}|_{\mathbb{N}} = f$. *That is, the values of the fractional invariant function $\tilde{f}$ coincides with the values of the original function f on the positive integers.*
(ii) $\tilde{f} : \mathbb{R} \longrightarrow \mathbb{R}$ *is right-continuous and have left limits on the interval* $[x, x+1)$ *for all* $x \in \mathbb{N}$.
(iii) $\tilde{f} : \mathbb{R} \longrightarrow \mathbb{R}$ *is a strictly increasing on* $[x, x+1)$ *for each* $x \in \mathbb{N}$.
(iv) $\tilde{f} : \mathbb{R} \longrightarrow \mathbb{R}$ *is of bounded variation of* $[x, x+1)$ *for each* $x \in \mathbb{N}$.
(v) *If* $\tilde{f}(x) = \tilde{f}(y)$ *and* $\{x\} = \{y\}$, *then* $f(\lfloor x \rfloor) = f(\lfloor y \rfloor)$.
(vi) $\tilde{f} : \mathbb{R} \longrightarrow \mathbb{R}$ *is bounded if and only if* $f : \mathbb{N} \longrightarrow \mathbb{R}$ *is bounded.*

Lemma 2.3 (Stieltjes-Lebesgue integral)**.** *Let $g : [a, b] \longrightarrow \mathbb{R}$ and $f : [a, b] \longrightarrow \mathbb{R}$ be right continuous and of bounded variation on $[a, b]$ and both having left limits. Then we have*

$$f(b)g(b) - f(a)g(a) = \int\limits_{(a,b]} f(t^-)dg(t) + \int\limits_{(a,b]} g(t^-)df(t) + \sum_{t \in (a,b]} \Delta f_t \Delta g_t$$

where $\Delta f_t = f(t) - f(t^-)$.

There are several well-known strategies for estimating partials sums of the form $\sum\limits_{\substack{n \leq x \\ n \in \mathbb{A}}} f(n)$ for $\mathbb{A} \subseteq \mathbb{N}$ and $f : \mathbb{N} \longrightarrow \mathbb{R}$. In the case f is continuous one can often apply Stieltjes integral. However, this approach becomes ineffective if $f(n)$ is not uniformly continuous on $\mathbb{R}$ and has no known arithmetic property. A typical instance is when f is an arithmetic function that exhibits chaotic behaviours on its positive integer arguments. In the sequel, we launch a new method for analyzing the partial sums of the form above in those scenarios.

Theorem 2.4 (The fractional invariance analysis)**.** *Let* $f:\mathbb{N}\longrightarrow\mathbb{R}$ *and let* $\mathbb{A}\subseteq\mathbb{N}$. *Denote* $\mathbb{A}(x):=\{n\leq x \ : \ n\in\mathbb{A}\}$ *and* $|\mathbb{A}(x)|:=\#\{n\leq x \ : \ n\in\mathbb{A}\}$. *Then*

$$\sum_{\substack{n\leq x\\ n\in\mathbb{A}}} f(n)=f(\lfloor x\rfloor)|\mathbb{A}(x)|-\sum_{j=1}^{\lfloor x\rfloor}\int_{j-1}^{j}|\mathbb{A}(t)|\tilde{f}'(t)dt-\sum_{j=1}^{\lfloor x\rfloor}\sum_{t\in(j-1,j]}\Delta|\mathbb{A}_t|\Delta\tilde{f}_t$$

where $\Delta\tilde{f}_t=\tilde{f}(t)-\tilde{f}(t^-)$ *and* $\Delta|\mathbb{A}_t|=|\mathbb{A}(t)|-|\mathbb{A}(t^-)|$ *with* $\tilde{f}$ *denoting the fractional invariance of* f.

Proof. By the first part of Proposition 2.2, we can write

$$\sum_{\substack{n\leq x\\ n\in\mathbb{A}}} f(n)=\sum_{\substack{n\leq x\\ n\in\mathbb{A}}}\tilde{f}(n)=\sum_{j=1}^{\lfloor x\rfloor}\sum_{\substack{j-1<n\leq j\\ n\in\mathbb{A}}}\tilde{f}(n).$$

The fractional invariant function $\tilde{f}$ is right-continuous on intervals of the form $[x,x+1)$ for each $x\in\mathbb{N}$ and has left limits with a bounded variation of $[x,x+1)$. By an application of Stieltjes integral, can write

$$\sum_{\substack{j-1<n\leq j\\ n\in\mathbb{A}}}\tilde{f}(n)=\int\limits_{j-1}^{j}\tilde{f}(t)d|\mathbb{A}(t)|.$$

By applying the Stieltjes-Lebesgue integral, we deduce

$$\int_{j-1}^{j}\tilde{f}(t)d|\mathbb{A}(t)|=\tilde{f}(j)|\mathbb{A}(j)|-\tilde{f}(j-1)|\mathbb{A}(j-1)|-\int_{j-1}^{j}|\mathbb{A}(t)|\tilde{f}'(t)dt$$

$$-\sum_{t\in(j-1,j]}\Delta|\mathbb{A}_t|\Delta\tilde{f}_t.$$

It follows that that

$$\sum_{j=1}^{\lfloor x\rfloor}\int_{j-1}^{j}\tilde{f}(t)d|\mathbb{A}(t)|=\sum_{j=1}^{\lfloor x\rfloor}\left[\tilde{f}(j)|\mathbb{A}(j)|-\tilde{f}(j-1)|\mathbb{A}(j-1)|\right]-\sum_{j=1}^{\lfloor x\rfloor}\int_{j-1}^{j}|\mathbb{A}(t)|\tilde{f}'(t)dt$$

$$-\sum_{j=1}^{\lfloor x\rfloor}\sum_{t\in(j-1,j]}\Delta|\mathbb{A}_t|\Delta\tilde{f}_t.$$

We deduce further

$$\sum_{j=1}^{\lfloor x\rfloor}\int_{j-1}^{j}\tilde{f}(t)d|\mathbb{A}(t)|=\tilde{f}(\lfloor x\rfloor)|\mathbb{A}(x)|-\tilde{f}(0)|\mathbb{A}(0)|-\sum_{j=1}^{\lfloor x\rfloor}\int_{j-1}^{j}|\mathbb{A}(t)|\tilde{f}'(t)dt$$

$$-\sum_{j=1}^{\lfloor x\rfloor}\sum_{t\in(j-1,j]}\Delta|\mathbb{A}_t|\Delta\tilde{f}_t.$$

Clearly $|\mathbb{A}(0)| = 0$ and with the convention $\tilde{f}(0) = f(0) := 0$ combined with the properties of the fractional invariant function, we obtain

$$\sum_{n\leq x} f(n) = f(\lfloor x\rfloor)|\mathbb{A}(x)| - \sum_{j=1}^{\lfloor x\rfloor}\int_{j-1}^{j} |\mathbb{A}(t)|\tilde{f}'(t)dt$$

$$-\sum_{j=1}^{\lfloor x\rfloor}\sum_{t\in(j-1,j]} \Delta|\mathbb{A}_t|\Delta\tilde{f}_t.$$

□

3. Applications of the fractional invariance analysis

In this section, we discuss some applications of the fractional invariance analysis developed in the previous section.

3.1. Fractional invariance analysis on the partial sums of the logarithmic function.

We now apply the fractional invariance analysis method to estimate $\sum_{n\leq x}\log n$ Let $f(n) = \log n$ and let $\mathbb{A} = \mathbb{N}$, so $\mathbb{A}(x) = \lfloor x\rfloor$. Then,

$$\sum_{n\leq x}\log n = \log(\lfloor x\rfloor)\cdot\lfloor x\rfloor - \sum_{j=1}^{\lfloor x\rfloor}\int_{j-1}^{j}\lfloor t\rfloor\cdot\frac{d}{dt}(\log t)dt - \sum_{j=1}^{\lfloor x\rfloor}\sum_{t\in(j-1,j]}\Delta|\mathbb{A}_t|\Delta\tilde{f}_t \quad (3.1)$$

where $\Delta\tilde{f}_t = \tilde{f}(t) - \tilde{f}(t^-)$ and $\Delta|\mathbb{A}_t| = |\mathbb{A}(t)| - |\mathbb{A}(t^-)|$. We analyze each of the terms in the sum. The first term provides the dominant growth

$$\log(\lfloor x\rfloor)\cdot\lfloor x\rfloor \approx x\log x.$$

By noting that $f(n) = \log n$ with the invariance $\tilde{f}(t) = \log t$ and $\tilde{f}'(t) = \frac{1}{t}$, we deduce

$$\sum_{j=1}^{\lfloor x\rfloor}\int_{j-1}^{j}\lfloor t\rfloor\cdot\frac{1}{t}dt = \sum_{j=2}^{\lfloor x\rfloor}\int_{j-1}^{j}\lfloor t\rfloor\cdot\frac{1}{t}dt = \sum_{j=2}^{\lfloor x\rfloor}\int_{j-1}^{j}dt - \sum_{j=2}^{\lfloor x\rfloor}\int_{j-1}^{j}\frac{\{t\}}{t}dt = \lfloor x\rfloor - \sum_{j=2}^{\lfloor x\rfloor}\int_{j-1}^{j}\frac{\{t\}}{t}dt.$$

By putting these contributions together, we deduce

$$\sum_{n\leq x}\log n = \lfloor x\rfloor\log(\lfloor x\rfloor) - \lfloor x\rfloor + \sum_{j=2}^{\lfloor x\rfloor}\int_{j-1}^{j}\frac{\{t\}}{t}dt - \sum_{j=1}^{\lfloor x\rfloor}\sum_{t\in(j-1,j]}\Delta|\mathbb{A}_t|\Delta\tilde{f}_t$$

and a more precise estimate could be obtained by analyzing the contributions from the sum involving the discrete jumps.

3.2. Fractional invariance analysis on the partial sums of the Euler totient function.

We now apply the fractional invariance analysis method to estimate $\sum_{n\leq x}\phi(n)$ Let $f(n) = \phi(n)$ and let $\mathbb{A} = \mathbb{N}$, so $\mathbb{A}(x) = \lfloor x\rfloor$. Then,

$$\sum_{n\leq x}\phi(n) = \phi(\lfloor x\rfloor)\cdot\lfloor x\rfloor - \sum_{j=1}^{\lfloor x\rfloor}\int_{j-1}^{j}\lfloor t\rfloor\tilde{\phi}'(t)dt - \sum_{j=1}^{\lfloor x\rfloor}\sum_{t\in(j-1,j]}\Delta|\mathbb{A}_t|\Delta\tilde{\phi}_t \quad (3.2)$$

where $\Delta\tilde{\phi}_t = \tilde{\phi}(t) - \tilde{\phi}(t^-)$ and $\Delta|\mathbb{A}_t| = |\mathbb{A}(t)| - |\mathbb{A}(t^-)|$. We analyze each of the terms in the sum. The first term provides the dominant growth

$$\phi(\lfloor x\rfloor)\cdot\lfloor x\rfloor = (\lfloor x\rfloor)^2 \prod_{p|\lfloor x\rfloor}(1-\frac{1}{p})$$

where p denotes a prime number. We deduce

$$\sum_{n\leq x}\phi(n) = (\lfloor x\rfloor)^2 \prod_{p|\lfloor x\rfloor}(1-\frac{1}{p}) - \sum_{j=2}^{\lfloor x\rfloor}\int_{j-1}^{j} t\tilde{\phi}'(t)dt + \sum_{j=1}^{\lfloor x\rfloor}\int_{j-1}^{j}\{t\}\tilde{\phi}'(t)dt$$
$$-\sum_{j=1}^{\lfloor x\rfloor}\sum_{t\in(j-1,j]}\Delta|\mathbb{A}_t|\Delta\tilde{\phi}_t.$$

The estimate for this sum can be made more precise by understanding the integral terms and the discrete jumps involving the invariance $\tilde{\phi}$ of ϕ.

4. **Conclusion and further remarks**

The fractional invariance analysis presented herein offers a novel and robust approach to estimating partial sums of a wide array of arithmetic functions. By leveraging the interplay between local behavior and smooth trends, this method surpasses traditional techniques, which often rely heavily on either discrete or continuous analysis. Its adaptability to various classes of functions, along with the capacity to accommodate the intricate nature of arithmetic distributions, positions it as a valuable tool in the arsenal of analytic number theory. Through rigorous application, we have demonstrated the efficacy of this method in estimating sums such as those associated with the Euler totient function and the divisor function. The insights garnered not only enhance our understanding of these fundamental arithmetic functions but also pave the way for future explorations into more complex problems in number theory. The implications of the fractional invariance analysis extend beyond the specific estimates presented; it invites a reconsideration of established methods and encourages further research into the nuances of arithmetic functions. As we continue to explore the richness of number theory, the adoption of this alternative approach may lead to new discoveries and a deeper comprehension of the intricate relationships that govern the behavior of numbers. Ultimately, this work lays a foundation for future inquiry and reinforces the importance of innovative methods in advancing mathematical understanding of how arithmetic functions can be smoothed out in certain subsets of the reals. We anticipate that the fractional invariance analysis will inspire subsequent research, fostering advancements in analytic number theory.

References

1. Montgomery, Hugh L *Topics in multiplicative number theory*, vol. 227, Springer, 2006.