Relativistic Form of Newton's Second Law of Motion

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Abstract

The article presents a relativistic extension of Newton's second law of motion, adapted to a four-dimensional spacetime construct where time and space are treated under the framework of special relativity. By introducing a unique concept of "U-space", where time functions as a spatial coordinate with its own metric properties, the study explores Newtonian dynamics through a relativistic lens. This exploration includes the derivation of relativistic force and acceleration equations, considering both constant and variable rest mass scenarios. The derived formulations, based on the principles of special relativity, are presented with modifications that incorporate mass-energy equivalence, enabling a more comprehensive understanding of force interactions within relativistic contexts.

Relativistic forms of physical equations result from the application of special relativity. In special relativity, the concept of spacetime was introduced, where an additional time dimension was added to the spatial dimensions. It is believed that we live in a four-dimensional spacetime, which is why in relativistic physics, two-, three-, and four-dimensional spacetimes are considered. (From a formal point of view, there is nothing to prevent considering spacetimes with more than four dimensions using special relativity).

In special relativity, the concept of time is attributed to two different categories:

1. Time as one of the spacetime coordinates, with particular significance. In this context, time as a coordinate has spatial properties and is relative, because the Lorentz transformation changes the value of the time coordinate of individual points, just as it changes the values of spatial coordinates. The time axis in spacetime is usually denoted by the capital letter $\boldsymbol{T}$.

Note: The time axis $\boldsymbol{T}$ *in diagrams representing spacetime is usually drawn as the vertical axis. However, in all graphs depicting processes occur-*

ring over time, the time axis is always drawn as the horizontal axis. In this context, it seems more natural and intuitive to represent the time axis as horizontal in diagrams representing spacetime.

2. Time as the proper parameter of material points moving in spacetime (denoted by the lowercase letter $\boldsymbol{t}$). The parameter $\boldsymbol{t}$ does not have spatial properties, is a continuously increasing scalar, and is an invariant of the Lorentz transformation.

Due to these differences in the two approaches, it is justified to denote the time axis with a capital letter $\boldsymbol{U}$. Therefore, the time coordinate of points in spacetime will be denoted by the lowercase letter $\boldsymbol{u}$, while the proper time of material points will continue to be denoted by the letter $\boldsymbol{t}$.

Next, for simplicity, we assume that the speed of light is 1 and is dimensionless. Thus, the velocity of massive point objects (vector $\vec{\boldsymbol{v}}$) is also dimensionless, and its absolute value is limited to the right-open interval $\langle 0\,,1)$. In this situation, the time axis $\boldsymbol{U}$ must have a spatial dimension, just like the other axes. This results in a homogeneous metric space, which we will call U-space to distinguish it from spacetime. The parameter $\boldsymbol{c}$ (the speed of light), whose value is determined by the defined second, is in this case a conversion factor for converting seconds to meters, meaning that $\mathbf{299\,792\,458}\,[\boldsymbol{m}]$ on the $\boldsymbol{U}$ axis corresponds to $\mathbf{1}\,[\boldsymbol{s}]$. (One could say that we move along the $\boldsymbol{U}$ axis at the speed of $\boldsymbol{c}$, so the parameter $\boldsymbol{c}$ should be called the speed of life rather than the speed of light).

After these changes, we will consider Newton's second law of motion in the four-dimensional U-space with axes $\boldsymbol{U}$, $\boldsymbol{X}$, $\boldsymbol{Y}$, $\boldsymbol{Z}$. A massive point object traveling through U-space traces its worldline, which we can define with a vector function:

$$\vec{r}(u) = [x(u), y(u), z(u)] \tag{1}$$

The velocity $\vec{\boldsymbol{v}}$ and acceleration $\vec{\boldsymbol{a}}$ of this object are also vector functions:

$$\vec{v}(u) = \frac{d\vec{r}(u)}{du} = \left[\frac{dx(u)}{du}, \frac{dy(u)}{du}, \frac{dz(u)}{du}\right] \tag{2}$$

$$\vec{a}(u) = \frac{d\vec{v}(u)}{du} = \frac{d^2\vec{r}(u)}{du^2} = \left[\frac{d^2x(u)}{du^2}, \frac{d^2y(u)}{du^2}, \frac{d^2z(u)}{du^2}\right] \tag{3}$$

In U-space, velocity is dimensionless because it is the derivative of distance with respect to distance. If we dimension the axes of U-space in meters $[\boldsymbol{m}]$, then acceleration is measured in the inverse of meters $[\mathbf{1}/\boldsymbol{m}]$. If we want to express velocity in $[\boldsymbol{m}/\boldsymbol{s}]$, we need to multiply it by the factor $\boldsymbol{c} = \mathbf{299\,792\,458}\,[\boldsymbol{m}/\boldsymbol{s}]$.

(In U-space, a second is a measure of distance, $\mathbf{1}\,[\boldsymbol{s}] = \mathbf{299\,792\,458}\,[\boldsymbol{m}]$). Similarly, to express acceleration in $[\boldsymbol{m}/\boldsymbol{s}^2]$, we need to multiply it by $\boldsymbol{c}^2$.

Newton's second law of dynamics states that if a body of mass $\boldsymbol{m}$ is subjected to a force $\vec{\boldsymbol{F}}$, the body accelerates with an acceleration $\vec{\boldsymbol{a}}$ that is directly proportional to the force and inversely proportional to the mass of the body. This law is expressed as follows:

$$\vec{\boldsymbol{a}} = \frac{\mathbf{1}}{\boldsymbol{m}} \vec{\boldsymbol{F}} \tag{4}$$

However, following the announcement of the special theory of relativity (STR), it was necessary to create a relativistic version of this principle. The relativistic form of equation (4) looks as follows:

$$\vec{\boldsymbol{a}} = \frac{\sqrt{\mathbf{1} - \frac{|\vec{\boldsymbol{v}}|^2}{\boldsymbol{c}^2}}}{\boldsymbol{m_r}} \left[\vec{\boldsymbol{F}} - \frac{\mathbf{1}}{\boldsymbol{c}^2} \left(\vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{v}} \right) \vec{\boldsymbol{v}} \right] \tag{5}$$

Where $\boldsymbol{m_r}$ denotes the rest mass of the body.

In special relativity (STR), we also have the formula for the total energy $\boldsymbol{E_t}$ of a body with rest mass $\boldsymbol{m_r}$ moving with velocity $\vec{\boldsymbol{v}}$:

$$\boldsymbol{E_t} = \frac{\boldsymbol{m_r c}^2}{\sqrt{\mathbf{1} - \frac{|\vec{\boldsymbol{v}}|^2}{\boldsymbol{c}^2}}} \tag{6}$$

Therefore, we can express formula (5) as:

$$\vec{\boldsymbol{a}} = \frac{\boldsymbol{c}^2}{\boldsymbol{E_t}} \left[\vec{\boldsymbol{F}} - \frac{\mathbf{1}}{\boldsymbol{c}^2} \left(\vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{v}} \right) \vec{\boldsymbol{v}} \right] \tag{7}$$

Formula (7) can be transformed and written as:

$$\vec{\boldsymbol{F}} = \frac{\boldsymbol{E_t}}{\boldsymbol{c}^2} \left(\vec{\boldsymbol{a}} + \frac{\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{v}}}{\boldsymbol{c}^2 - |\vec{\boldsymbol{v}}|^2} \vec{\boldsymbol{v}} \right) \tag{8}$$

Formulas (7) and (8) are the relativistic form of Newton's second law of motion. Formula (7) can also be expressed as follows:

$$\frac{\vec{\boldsymbol{a}}}{\boldsymbol{c}^2} = \frac{\mathbf{1}}{\boldsymbol{E_t}} \left[\vec{\boldsymbol{F}} - \left(\vec{\boldsymbol{F}} \cdot \frac{\vec{\boldsymbol{v}}}{\boldsymbol{c}} \right) \frac{\vec{\boldsymbol{v}}}{\boldsymbol{c}} \right] \tag{9}$$

When we now apply Newton's second law of motion to U-space, where we assume that the speed of light is dimensionless and equals one, and velocity and acceleration are defined by equations (2) and (3) — meaning velocity is dimensionless, constrained within the unit sphere, and acceleration has dimensions of $\left[\frac{1}{m}\right]$ — equation (9) takes the form:

$$\vec{a} = \frac{1}{E_t}\left[\vec{F} - \left(\vec{F}\cdot\vec{v}\right)\vec{v}\right] \tag{10}$$

and correspondingly, the equation for force:

$$\vec{F} = E_t\left(\vec{a} + \frac{\vec{a}\cdot\vec{v}}{1-|\vec{v}|^2}\vec{v}\right) \tag{11}$$

Note: For U-space, where we have assumed $c = 1$*, we write formula (6) as:* $E_t = \frac{m_r}{\sqrt{1-|\vec{v}|^2}}$*. In this formula, the rest mass* m_r *is expressed in the same units as the total energy* E_t*, which can be, for instance, joules.*

Formulas (10) and (11) are the relativistic form of Newton's second law of motion for U-space, assuming that rest mass is a constant parameter.

In the article "What is Mass?" we concluded that the rest mass of an object is its potential energy and depends on the values of the vector field potentials that interact with this object. Therefore, we need to find a more general, relativistic formula for Newton's second law of motion in the case where the rest mass of the object changes during its motion. We will start from the formula for the total energy of the object:

$$E_t = E_k + E_p \tag{12}$$

Where E_k denotes the kinetic energy of the body, and E_p its potential energy. By differentiating (with respect to the coordinate u) both sides of this equation and substituting the symbol for potential energy E_p with the symbol for rest mass m_r, we obtain:

$$\frac{dE_t}{du} = \frac{dE_k}{du} + \frac{dm_r}{du} \tag{13}$$

For the total energy of a massive point object, we have the formula:

$$E_t = \frac{m_r}{\sqrt{1-|\vec{v}|^2}} \tag{14}$$

where the total energy and rest mass are expressed in the same units of energy. Meanwhile, the increase in kinetic energy is determined by the formula:

$$\frac{dE_k}{du} = \vec{F}\cdot\frac{d\vec{r}}{du} = \vec{F}\cdot\vec{v} \tag{15}$$

Differentiating (14), we finally obtain:

$$\boldsymbol{m_r}\frac{\vec{a}\cdot\vec{v}}{\left(\sqrt{1-|\vec{v}|^2}\right)^3}+\frac{d\boldsymbol{m_r}}{du}\frac{1}{\sqrt{1-|\vec{v}|^2}}=\vec{F}\cdot\vec{v}+\frac{d\boldsymbol{m_r}}{du} \tag{16}$$

which can be rewritten as:

$$\vec{F}\cdot\vec{v}=\boldsymbol{m_r}\frac{\vec{a}\cdot\vec{v}}{\left(\sqrt{1-|\vec{v}|^2}\right)^3}+\frac{d\boldsymbol{m_r}}{du}\left(\frac{1}{\sqrt{1-|\vec{v}|^2}}-1\right) \tag{17}$$

Note! When expressing rest mass in joules $[\boldsymbol{J}]=[\boldsymbol{Nm}]$, the derivative $\frac{d\boldsymbol{m_r}}{du}$ has the dimension of force $[\boldsymbol{N}]$.

On the other hand, we know that the relativistic form of Newton's second law of motion looks as follows:

$$\vec{F}=\boldsymbol{E_t}\left(\vec{a}+\boldsymbol{l}\vec{v}\right) \tag{18}$$

where $\boldsymbol{l}$ is the coefficient that we want to determine. Multiplying both sides of the above equation scalarly by the vector $\vec{v}$, we obtain:

$$\vec{F}\cdot\vec{v}=\boldsymbol{E_t}\left(\vec{a}\cdot\vec{v}+\boldsymbol{l}|\vec{v}|^2\right) \tag{19}$$

Now, substitute the right-hand side of equation (17) for $\vec{F}\cdot\vec{v}$ in (19) and calculate the coefficient $\boldsymbol{l}$:

$$\boldsymbol{l}=\frac{d\boldsymbol{m_r}}{du}\frac{1}{\boldsymbol{m_r}\left(1+\sqrt{1-|\vec{v}|^2}\right)}+\frac{\vec{a}\cdot\vec{v}}{1-|\vec{v}|^2} \tag{20}$$

After substituting (20) into (18), we obtain the formula:

$$\vec{F}=\boldsymbol{E_t}\left\{\vec{a}+\left[\frac{d\boldsymbol{m_r}}{du}\frac{1}{\boldsymbol{m_r}\left(1+\sqrt{1-|\vec{v}|^2}\right)}+\frac{\vec{a}\cdot\vec{v}}{1-|\vec{v}|^2}\right]\vec{v}\right\} \tag{21}$$

Since we are dealing with a point object, based on (14), we can write formula (21) as:

$$\vec{F}=\boldsymbol{E_t}\left[\vec{a}+\frac{(\vec{a}\cdot\vec{v})}{1-|\vec{v}|^2}\vec{v}\right]+\frac{d\boldsymbol{m_r}}{du}\frac{1}{1-|\vec{v}|^2+\sqrt{1-|\vec{v}|^2}}\vec{v} \tag{22}$$

Formula (22) can be transformed into the form:

$$\vec{a} = \frac{1}{E_t}\left[\vec{F} - \left(\frac{dm_r}{du}\frac{\sqrt{1-|\vec{v}|^2}}{1+\sqrt{1-|\vec{v}|^2}} + \vec{F}\cdot\vec{v}\right)\vec{v}\right] \tag{23}$$

Formulas (22) and (23) are the relativistic form of Newton's second law of motion.

Here, we can see where formulas (10) and (11) originate from. When the velocity vector $\vec{v}$ of the object is tangent to the equipotential surface of the vector field of forces, the derivative $\frac{dm_r}{du}$ equals zero. (For example, this occurs when a satellite orbits the Earth in a perfectly circular orbit). In this case, formulas (22) and (23) respectively take the form of formulas (11) and (10).

Note: Equation (11) (where we assume that $\frac{dm_r}{du} = 0$*) can also be obtained by differentiating the right-hand side of the formula* $\vec{F} = \frac{d\vec{p}}{du}$*, where the momentum vector* $\vec{p} = \frac{m_r}{\sqrt{1-|\vec{v}|^2}}\vec{v}$*, that is,* $\vec{F} = m_r\frac{d\frac{\vec{v}}{\sqrt{1-|\vec{v}|^2}}}{du}$*. (In U-space, momentum has the dimension of energy. Therefore, if force is given in newtons [N], momentum and rest mass must be given in joules [J].)*

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