Sum of two inverse trigonometric functions

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Abstract: We give some formulas of the type: $y\arcsin(x) + y\arctan(x) = \pi$

I. Introduction

The number Pi is defined as

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \ldots\right) = 3.14159265358979323846264338327950288419716939937 51\ldots \tag{1}$$

In this note we give formulas of the type

$$y\arcsin(x) + y\arctan(x) = \pi \quad , \quad y\arcsin(x) - y\arctan(x) = \pi \tag{2}$$

where $x > 0,\ y > 0$.

II. Formulas

Entry 1.

$$\pi = 4\arcsin(x) + 4\arctan(x) \tag{3}$$

where

$$x = \frac{1}{3}\left(1 + 2^{-2/3}\left(4 + 3\sqrt{78}\right)^{1/3} - 7\left(2\left(4 + 3\sqrt{78}\right)\right)^{-1/3}\right) \tag{4}$$

Entry 2.

$$\pi = 6\arcsin(x) + 6\arctan(x) \tag{5}$$

where

$$x = -\frac{1}{4\sqrt{\frac{3}{-2-\frac{47}{\left(793+48\sqrt{318}\right)^{1/3}}+\left(793+48\sqrt{318}\right)^{1/3}}}} + \frac{1}{2}\left(-\frac{1}{3} + \frac{47}{12\left(793+48\sqrt{318}\right)^{1/3}} - \frac{1}{12}\left(793+48\sqrt{318}\right)^{1/3} + \frac{6}{\sqrt{-2-\frac{47}{\left(793+48\sqrt{318}\right)^{1/3}}+\left(793+48\sqrt{318}\right)^{1/3}}}\right)^{1/2} \tag{6}$$

Entry 3.

$$\pi = 6\arcsin(x) - 6\arctan(x) \tag{7}$$

where

$$x = \frac{1}{4\sqrt{\frac{3}{-2-\frac{47}{\left(793+48\sqrt{318}\right)^{1/3}}+\left(793+48\sqrt{318}\right)^{1/3}}}} + \frac{1}{2}\left(-\frac{1}{3} + \frac{47}{12\left(793+48\sqrt{318}\right)^{1/3}} - \frac{1}{12}\left(793+48\sqrt{318}\right)^{1/3} + \frac{6}{\sqrt{-2-\frac{47}{\left(793+48\sqrt{318}\right)^{1/3}}+\left(793+48\sqrt{318}\right)^{1/3}}}\right)^{1/2} \tag{8}$$

Entry 4.

$$\pi = 3\arcsin(x) + 3\arctan(x) \tag{9}$$

where

$$x = -\frac{1}{4}\sqrt{-2 - \frac{15}{(73 + 16\sqrt{34})^{1/3}} + (73 + 16\sqrt{34})^{1/3}} + \frac{1}{2}\left(-1 + \frac{15}{4(73 + 16\sqrt{34})^{1/3}} - \frac{1}{4}(73 + 16\sqrt{34})^{1/3} + 2\sqrt{\frac{3}{-2 - \frac{15}{(73+16\sqrt{34})^{1/3}} + (73 + 16\sqrt{34})^{1/3}}}\right)^{1/2} \tag{10}$$

Entry 5.

$$\pi = \frac{3}{2}\arcsin(x) + \frac{3}{2}\arctan(x) \tag{11}$$

where

$$x = \frac{1}{4}\sqrt{-2 - \frac{15}{(73 + 16\sqrt{34})^{1/3}} + (73 + 16\sqrt{34})^{1/3}} + \frac{1}{2}\left(-1 + \frac{15}{4(73 + 16\sqrt{34})^{1/3}} - \frac{1}{4}(73 + 16\sqrt{34})^{1/3} + 2\sqrt{\frac{3}{-2 - \frac{15}{(73+16\sqrt{34})^{1/3}} + (73 + 16\sqrt{34})^{1/3}}}\right)^{1/2} \tag{12}$$

Remarks:

$$\arcsin(x) \pm \arctan(x) = 2\sum_{n=0}^{\infty}\frac{(x/2)^{2n+1}}{2n+1}\left(\binom{2n}{n} \pm (-1)^n 2^{2n}\right), \quad |x| < 1 \tag{13}$$

$$a(n) = \binom{2n}{n} + (-1)^n 2^{2n}, \quad a(n) = \{2, -2, 22, -44, 326, -772, 5020, -12952, 78406, \ldots\} \tag{14}$$

$$b(n) = \binom{2n}{n} - (-1)^n 2^{2n}, \quad b(n) = \{0, 6, -10, 84, -186, 1276, -3172, 19816, -52666, \ldots\} \tag{15}$$

$$\arcsin(x) \pm \arctan(x) = \sum_{n=0}^{\infty}\binom{2n}{n}2^{-2n}x^{n+1}\left(\frac{F(1/2, n+1, n+2, -x)}{n+1} \pm (-1)^n\frac{x^n F\left(1/2, n+\frac{1}{2}, n+\frac{3}{2}, -x^2\right)}{2n+1}\right), \quad |x| < 1 \tag{16}$$

where F is the Gauss hypergeometric function.

If $s = \frac{1}{3}\left(1 + 2^{-2/3}(4 + 3\sqrt{78})^{1/3} - 7\left(2(4 + 3\sqrt{78})\right)^{-1/3}\right)$ we have

$$\pi = 2\sqrt{3}\sum_{n=0}^{\infty}(-s)^n\sum_{m=0}^{\lfloor n/3\rfloor}\sum_{k=m}^{\left\lfloor\frac{n-m}{2}\right\rfloor}\frac{(2/3)^k}{2n-2m-2k+1}\binom{k}{m}\binom{n-m-k}{k} \tag{17}$$

where $\lfloor x\rfloor$ is the floor function.

III. References

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