A Constructive Proof and Algorithm for the 2D Brouwer Fixed-Point Theorem with Surjective Mapping

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Abstract

This article investigates the two-dimensional Brouwer Fixed-Point Theorem within the context of a surjective continuous transformation function $f(x)$. This function can be interpreted as defining a continuous vector field. In this framework, each point in the disk is mapped to another point via a specific vector associated with the continuous transformation, thereby establishing a coherent vector field. This function can be interpreted as defining a continuous vector field. In this framework, the vector field can be decomposed two vector fields. Instead of proving the existence of fixed point directly, the article aim to focus on prove the vector fields always has intersection where at this point, the vector fields has opposite directiond and same norm.The paper also provide the programming experiment which further verifies the proof.

1 Introduction

Let $f : D_1 \to D_2$ be a surjective mapping on a disk D_1, where$f$ can be represented by a continuous vector field. Assume this vector field can be decomposed into the sum of a smooth deformation vector field $\vec{D}(x)$ and a rotation vector field $\vec{R}(x)$, where:

1. $\vec{D}(x)$ is an arbitrary continuous deformation map with boundary values zero,
2. $\vec{R}(x)$ is a continuous rotation map with a single rotation center.

Then, either:

1. There exists a point in the disk where the vectors $\vec{D}(x)$ and $\vec{R}(x)$ have the same magnitude but opposite directions (resulting in a zero vector), or
2. There exists a point where the vector field is zero, indicating a fixed point.

In the case where the map must be decomposed into a rotation map with two or more centers of rotation, the disk must contain fixed points due to the differing orientations of the rotation vectors, which create opposing forces, ensuring the existence of fixed points by symmetry. In the subsequent sections, Section 2 and Section 3 outline the fundamental properties of the continuous map f. Sections 4 and 5 examine the characteristics of the lines of equal length and the opposite directions of the vectors $\vec{D}(x)$ and $\vec{R}(x)$. Section 6 explores the methods for locating fixed points under various conditions of the continuous map f. Section 7 presents a summary of the findings. Finally, Section 8 discusses the experiments conducted using programming to validate the proof.

1.1 Background

The fixed-point theorem is a fundamental result in mathematics that asserts the existence of points that remain invariant under a given function. One of the most well-known fixed-point theorems is Brouwer's Fixed-Point Theorem[2], which states that any continuous function mapping a compact convex set to itself has at least one fixed point. This theorem is particularly significant in topology and has applications across various fields, including economics, game theory, and differential equations.
Theorem 1.1 (Brouwer's Fixed-Point Theorem)**.** *Let D be a compact convex subset of $\mathbf{R}^n$. If $f : D \to D$ is a continuous function, then there exists at least one point $x \in D$ such that $f(x) = x$.*

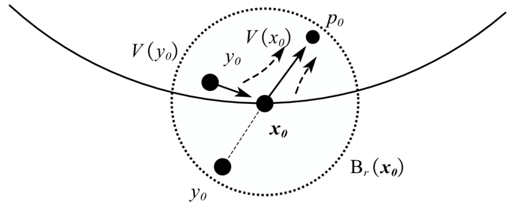

Figure 1: Neighbourhood of x_0

Original Proof of Brouwer's Fixed-Point Theorem: Brouwer's original proof employs topological arguments to demonstrate the existence of a fixed point for continuous mappings. The essence of the proof relies on the assumption that if a continuous mapping f does not have a fixed point, then a retraction r can be constructed to project the entire disk onto its boundary. However, it is known that there exists a homotopy f_0 in D^2 that can be continuously deformed to a constant loop. If we compose this homotopy with the retraction r, it implies the existence of a homotopy from S^1 (the boundary of the disk) to a constant loop, which is a contradiction in topology. Therefore, the assumption that f has no fixed point must be false, confirming that at least one fixed point exists within the disk.

2 Continuous Vector Field Map

This section will prove the case when the vector field map has only one rotation center.

Lemma 2.1. *Let D_1 and D_2 be two 2D disks in metric spaces. If $f : D_1 \to D_2$ is a surjective continuous mapping, then the boundary points on ∂D_1 are mapped to the boundary points on ∂D_2.*

Proof. Let D_1 be a closed disk in $\mathbf{R}^2$ and D_2 be another closed disk. Since f is a continuous surjective mapping, it is possible to construct a homotopy $h_t : D_1 \to D_2$ for $t \in [0, 1]$ such that:

$$h_0(x) = x \quad \text{for } x \in D_1,$$

and

$$h_1(D_1) = D_2.$$

This homotopy represents a continuous deformation from the disk D_1 to the disk D_2.

For any point $x_0 \in \partial D_1$, let $f(x_0) = p_0 \in D_2$. By the continuity of the mapping f, there exists a point $y_0 \in D_1$ close to x_0 such that $h_{\Delta t}(y_0)$ maps to x_0 as $\Delta t \to 0$. This implies the existence of a neighborhood $B_r(x_0)$ such that:

$$\forall x_0 \in \partial D_1, \exists y_0 \in D_1 \cap B_r(x_0).$$

As $\Delta t \to 0$, the mappings are given by:

$$h_{\Delta t}(x_0) = p_0, \quad h_{\Delta t}(y_0) = x_0.$$

The velocity at x_0 is defined as:

$$V(x_0) = \lim_{\Delta t \to 0} \frac{p_0 - x_0}{\Delta t}.$$

Similarly, the velocity at y_0 is:

$$V(y_0) = \lim_{\Delta t \to 0} \frac{x_0 - y_0}{\Delta t}.$$

Since the velocity is continuous in the neighborhood of x_0, it follows that:

$$V\left(\lim_{r\to 0} D_1 \cap B_r(x_0)\right) = V(x_0).$$

Given $y_0 \in D_1 \cap B_r(x_0)$, it can be stated that:

$$V(y_0) = \lim_{\Delta t\to 0} \frac{x_0 - y_0}{\Delta t} = V\left(\lim_{r\to 0} D_1 \cap B_r(x_0)\right) = V(x_0) = \lim_{\Delta t\to 0} \frac{p_0 - x_0}{\Delta t}.$$

Thus, it can be concluded that:

$$\lim_{\Delta t\to 0} \frac{x_0 - y_0}{\Delta t} = \lim_{\Delta t\to 0} \frac{p_0 - x_0}{\Delta t}.$$

From this, the relationship holds:

$$y_0 = 2x_0 - p_0.$$

By considering x_0 as the origin, it is evident that y_0 must lie in the opposite direction of p_0 and outside the boundary. Therefore, there must exist some x_0 for which no corresponding y_0 exists under $h_{\Delta t}$, leading to a contradiction.

This contradiction implies that the initial assumption—that boundary points of D_1 do not map to the boundary points of D_2—must be false. Hence, it is concluded that boundary points on ∂D_1 are indeed mapped to boundary points on ∂D_2 under the continuous surjective mapping f. □

3 Composition of Vector Fields

Lemma 3.1. *Let $f : D_1 \to D_2$ be a surjective mapping from a disk D_1 to a disk D_2. Then, f can be expressed as a composition of a continuous deformation vector field $\vec{D}(x)$, which is zero at the boundary, and multiple rotation center vector fields $\vec{R}(x)$.*

Proof. By Lemma 2.1, the boundary points must be mapped to other boundary points under the continuous mapping f. Additionally, the interior points are transformed to other interior points by some continuous transformation. The composition of these transformations can be expressed as:

$$h_{\Delta t}(x_0) = x_0 + \vec{D}_{\Delta t}(x_0) + \vec{R}_{\Delta t}(x_0), \quad t \in [0, 1],$$

where $\vec{D}_{\Delta t}(x_0)$ and $\vec{R}_{\Delta t}(x_0)$ are vectors pointing from x_0 to the final position $h_{\Delta t}(x_0)$ at time Δt. To clarify, it holds that $h_0(x_0) = x_0$ and $h_1(x_0) = f(x_0)$, which maps $x_0 \in D_1$ to D_2 (see Figure 2). If $f(x_0)$ is a fixed point, then:

$$f(x_0) = x_0 = h_1(x_0),$$

leading to:

$$\vec{D}_{\Delta t=1}(x_0) = -\vec{R}_{\Delta t=1}(x_0).$$

This relationship implies that the magnitudes of the vectors are equal and they point in opposite directions, effectively canceling out any movement within D_1, thereby ensuring that the points remain fixed during the transformation.

Alternatively, using projections in polar coordinates may not yield valid deformations, particularly if the deformation vectors are always perpendicular to the rotation vectors. Such a configuration would introduce discontinuities at the center of the disk. Under this composition, the center point remains stationary because its neighborhood contains deformation vectors directed in all possible orientations.

For conciseness, the parameter Δt can be eliminated by considering only the final positions after the transformation, since the functions are valid for any arbitrary time. Let the deformation and rotation vector fields be denoted as $\vec{D}(x_0)$ and $\vec{R}(x_0)$, respectively.

□

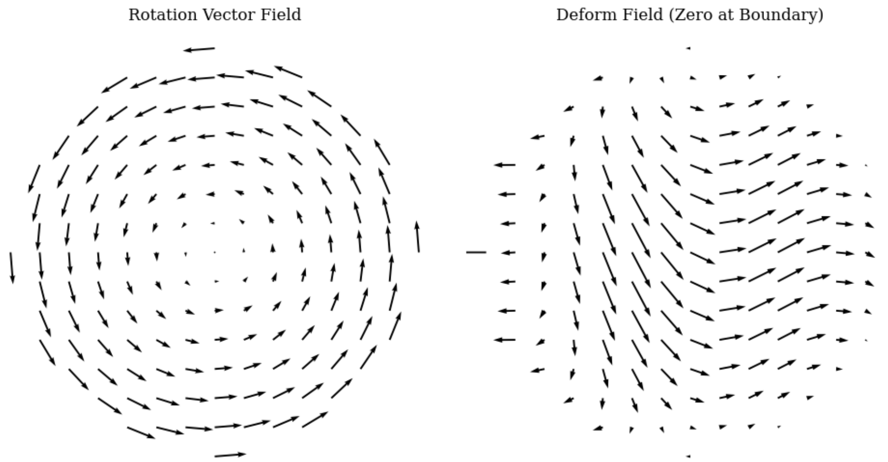

Figure 2: $\vec{R}(x_0)$ and $\vec{D}(x_0)$

4 Contours of equal Norm

Theorem 4.1. *In every direction there must be exists one or more points that at these points the norm of $\vec{D}(x_0)$ are equal to its norm of $\vec{R}(x_0)$.*

$$|\vec{D}(x_0)| = |\vec{R}(x_0)|$$

Proof. The article will first address the case of a single center of rotation. The more general case involving multiple centers will be discussed later.

Given a fixed direction, we use it as an axis and define the norms of the deformation vector $|\vec{D}(x_0)|$ and the rotation vector $|\vec{R}(x_0)|$ as continuous functions along a section of that direction. These functions are expressed as follows:

$$|\vec{D}_\theta(r)| = \begin{cases} 0, & \text{if } r = -R, \\ u_\theta(r), & \text{if } 0 \leq u_\theta(r) < 2R, \\ 0, & \text{if } r = R, \end{cases}$$

$$|\vec{R}_\theta(r)| = \begin{cases} b_1, & \text{if } r = -R, \quad b_1 \geq 0, \\ v_{\theta;1}(r), & \text{if } -R < r < 0, \quad v_{\theta;1}(r) \geq 0, \\ 0, & \text{if } r = 0, \\ v_{\theta;2}(r), & \text{if } 0 < r < R, \quad v_{\theta;2}(r) \geq 0, \\ b_2, & \text{if } r = R, \quad b_2 \geq 0, \end{cases}$$

where the norms $|\vec{D}(x_0)|$ and $|\vec{R}(x_0)|$ depend on the variables r and θ, with $|\vec{D}_\theta(r)|$ representing the magnitude of the deformation vector in a given direction θ, and r representing the distance from the center of the disk. Here, θ is the angle between the specified direction and the horizontal axis, measured counterclockwise. The parameter $r \in [-R, R]$, where R is the radius of the disk.

In this formulation, $u_\theta(r)$ is a continuous function that vanishes at the boundary of the disk. The function $v_{\theta;1}(r)$ starts from an initial value b_1 and decreases to zero at $r = 0$, where the rotation ceases. Meanwhile, $v_{\theta;2}(r)$ begins at zero and increases to b_2 as r approaches R, where the magnitude of the rotation vector is determined by the rotation function.

There are three distinct cases in which $|\vec{D}(x_0)| = |\vec{R}(x_0)|$.

Case 1: Solutions occur exactly at the boundary or center when $|\vec{D}_\theta(r)| = |\vec{R}_\theta(r)| = 0$. In this case, fixed points are immediately obtained at $\theta = \theta$ and $r = 0, -R$, or R. Notably, if the solution occurs at the center, then for every direction, there will be a zero point at the center.

Example: This situation arises when $u_\theta(r) = 0$, while $v_{\theta;1}(r) \neq 0$ and $v_{\theta;2}(r) \neq 0$.

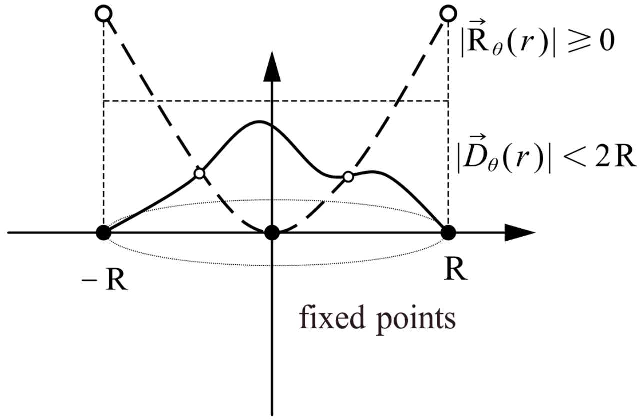

Figure 3: norm of $|\vec{D}_\theta(r)|$ and$|\vec{R}_\theta(r)|$ in direction θ

Case 2: There are two non-zero solutions. This occurs when $u_\theta(r) \neq 0$ and there are no zero points at $r = 0$.

Proof: By the *Intermediate Value Theorem* ,[1] in the interval $[-R, 0]$,

$$|\vec{D}_\theta(r)| - |\vec{R}_\theta(r)| < 0, \quad \text{at}\, r = -R,$$

$$|\vec{D}_\theta(r)| - |\vec{R}_\theta(r)| > 0, \quad \text{at}\, r = 0.$$

Thus, there must exist an intersection point c_1 where the two functions are equal. Similarly, there is another solution c_2 in the interval $[0, R]$, where

$$|\vec{D}_\theta(c_1)| = |\vec{R}_\theta(c_1)|.$$

This is illustrated visually in Figure 3.

Case 3: There are more than two or infinitely many solutions. This can occur when there are multiple loops, with at least one loop around the center.

□

Corollary 4.2. *Because the vectors field map $|\vec{D}_\theta(r)|$ and $|\vec{R}_\theta(r)|$are all continuous on θ and r, the two intersection points must form a closed loop $\mathcal{C}$ around the center.*

5 Contours of direction

Lemma 5.1. *If $\vec{R}(x)$ is only zero at the center and $\vec{D}(x)$ does not have any zero vectors, then the solutions are non-trivial. In this scenario, there exists a line $\mathcal{L}$ that extends from the zero point. On this line, the deformation vector $\vec{D}(\mathcal{P})$ and the rotation vector $\vec{R}(\mathcal{P})$ point in opposite directions. This relationship can be expressed mathematically as:*

$$\Theta(\vec{D}(\mathcal{P})) + \pi = \Theta(\vec{R}(\mathcal{P})),$$

or equivalently,

$$\vec{D}(\mathcal{P}) \cdot \vec{R}(\mathcal{P}) = -1,$$

indicating that the two vectors have opposite direction.

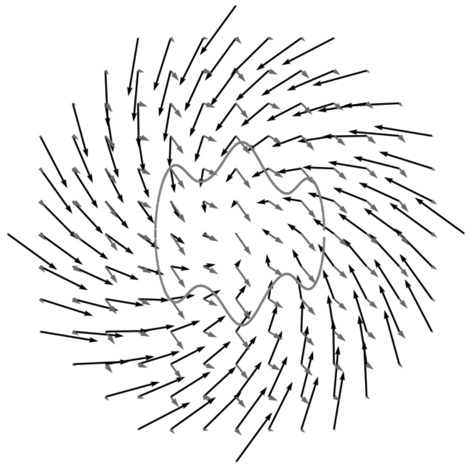

Figure 4: Closed loop $\mathcal{C}$

Proof. There are some special situations that can be discussed initially. If $\vec{D}(0) = 0$ at the center, then a fixed point exists at the center.

Next, we will prove the existence of a point in the opposite direction near the center. If $|\vec{D}(0)| \neq 0$ and $|\vec{D}(x)|$ is continuous at zero, then there exists a ball $\mathrm{B}_\epsilon(0)$ with radius ϵ. For all $\delta > 0$, the function measuring the change in the deformation vector around the boundary $\partial\mathrm{B}_\epsilon(0)$ satisfies:

$$-\delta < \Delta\Theta(\vec{D}(x_0)) < \delta,$$

for all $x_0 \in \partial\mathrm{B}_\epsilon(0)$. Here, $\Delta\Theta(\vec{D}(x_0))$ measures the relative angle of $\vec{D}(x_0)$ with respect to $\vec{R}(x_0)$ at any base point θ_0 chosen on $\mathrm{B}_\epsilon(0)$.

Additionally, since $\Theta(\vec{R}(x_0))$ is a surjective continuous function on $\partial\mathrm{B}_\epsilon(0)$ for all $\epsilon > 0$, mapping onto $[0, 2\pi]$, there must exist an intersection point $\mathcal{P}$ between the two functions $\Theta(\vec{R}(x_0))$ and $\Theta(\vec{D}(x_0))$. It is important to note that $\vec{R}(x_0)$ is a monotonically increasing function, as the points can only rotate along the loop without returning.

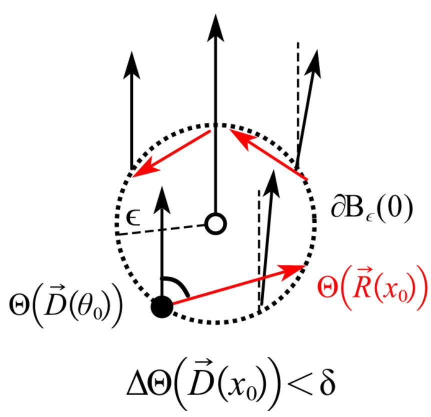

Figure 5: neighbourhood of $x_0 = 0$

To be simple,set $\vec{R}(x_0)$ at θ_0 equals 0, and denoted the point return to θ_0 after it finished a loop as θ_0' , noted that $\Theta(\vec{R}(\theta_0'))$ must equal to $2k\pi$ for $k = 1, 2, 3,$ because of continuity.Also,

$$0 \leq \Theta(\vec{D}(\theta_0)) \leq 2\pi, \Theta(\vec{D}(\theta_0)) = 0 \Rightarrow \Theta(\vec{D}(\theta_0)) - \Theta(\vec{R}(\theta_0)) \leq 0$$

Because

$$\Theta(\vec{R}(\theta_0')) \geq 2\pi + \Theta(\vec{R}(\theta_0)),$$

$$
\begin{aligned}
\Theta(\vec{D}(\theta_0')) - \Theta(\vec{R}(\theta_0')) &\leq \Theta(\vec{D}(\theta_0)) + \delta - \Theta(\vec{R}(\theta_0')) \\
&\leq \Theta(\vec{D}(\theta_0)) + \delta - (2\pi + \Theta(\vec{R}(\theta_0))) \\
&= \Theta(\vec{D}(\theta_0)) - \Theta(\vec{R}(\theta_0)) - 2\pi + \delta \leq 0 + \delta
\end{aligned}
$$

for all $\delta > 0$. When the equality holds, it is evident that there is an intersection point. If the equality does not hold, then, according to the Intermediate Value Theorem, there must exist another intersection point $\mathcal{P}$. To illustrate this concept, a figure follows.

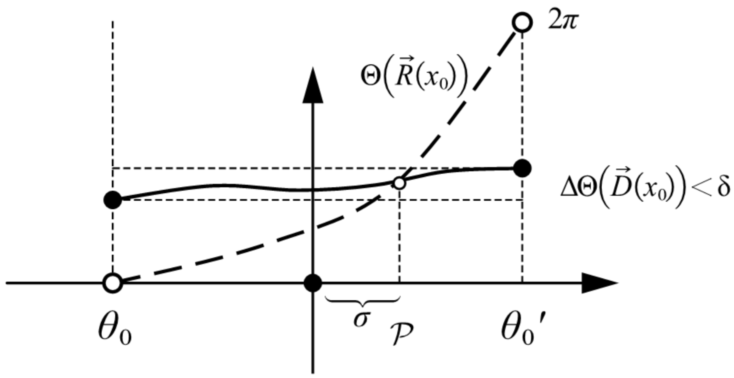

Figure 6: $\Theta(\vec{D}(x_0))$ and$\Theta(\vec{R}(x_0))$

At this point, we have $\Theta(\vec{D}(\mathcal{P})) = \Theta(\vec{R}(\mathcal{P}))$. Similarly, $\Theta(\vec{D}(\mathcal{P})) + \pi$ represents a parallel transport of the curve upward, and at θ_0', this can be extended further because θ_0' and θ_0 are connected. Thus, we can conclude that there exists a point $x_0 \in \partial \mathrm{B}_\epsilon(0)$ such that $\Theta(\vec{D}(\mathcal{P})) = \Theta(\vec{R}(\mathcal{P}))$. □

Lemma 5.2. *The lines $\mathcal{L}$ where $\Theta(\vec{D}(\mathcal{P})) + \pi = \Theta(\vec{R}(\mathcal{P}))$ and $\Theta(\vec{D}(\mathcal{P})) = \Theta(\vec{R}(\mathcal{P}))$ do not vanish unless either of the vector fields has a zero vector.*

Proof. When the point x_0 is identified, there exists a ball $\mathrm{B}_\epsilon(x_0)$ such that $y_0 \in \partial \mathrm{B}_\epsilon(x_0)$ and $\Theta(\vec{D}(y_0)) = \Theta(\vec{R}(y_0))$.

This can be shown as follows: For any given $\delta > 0$, there exists a ball $\mathrm{B}_\epsilon(x_0)$ where $y_0 \in \partial \mathrm{B}_\epsilon(x_0)$ satisfies

$$-\delta < \Theta(\vec{D}(y_0)) - \Theta(\vec{D}(x_0)) < \delta.$$

This indicates that $\Theta(\vec{D}(y_0))$ is bounded and centered at $\Theta(\vec{D}(x_0))$. However, for any $\mathrm{B}_\epsilon(x_0)$, it follows that $\Delta\Theta(\vec{R}(y_0)) > 0$. Therefore, for any $c > 0$, a $\mathrm{B}_\epsilon(x_0)$ can be identified such that $\delta < c$ and

$$\Delta\Theta(\vec{D}(y_0)) < \Delta\Theta(\vec{R}(y_0)).$$

Proving the same direction is generally more concise than proving the opposite direction. For the proof regarding the opposite direction, one can simply replace $\Theta(\vec{D}(p))$ with $\Theta(\vec{D}(p)) + \pi$ throughout.

Since $\vec{R}(y_0)$ also has a point on $\partial \mathrm{B}_\epsilon(x_0)$ where $\Theta(\vec{R}(y_0)) = \Theta(\vec{D}(x_0)) = \Theta(\vec{R}(x_0))$, the two functions must intersect at some point $\mathcal{P}$.

The next step involves applying the Intermediate Value Theorem once more. Let c be a positive constant and θ_0 be a base point, such that

$$\Theta(\vec{R}(\theta_0')) - \Theta(\vec{R}(x_0)) \geq c,$$

and

$$\Theta(\vec{D}(\theta_0')) - \Theta(\vec{D}(x_0)) \leq \delta.$$

From these inequalities, the following can be deduced:

$$\Theta(\vec{R}(\theta_0')) \geq \Theta(\vec{R}(x_0)) + c,$$
$$\Theta(\vec{D}(\theta_0')) \leq \Theta(\vec{D}(x_0)) + \delta.$$

Additionally, since $c > \delta$ and $\Theta(\vec{R}(x_0)) = \Theta(\vec{D}(x_0))$, it follows that:

$$\Theta(\vec{R}(\theta_0')) - \Theta(\vec{D}(\theta_0')) \geq \Theta(\vec{R}(x_0)) + c - \Theta(\vec{D}(x_0)) - \delta > 0.$$

Similarly, it can be concluded that:

$$\Theta(\vec{R}(\theta_0)) - \Theta(\vec{D}(\theta_0)) < 0.$$

This result can also be visualized in the following figure. □

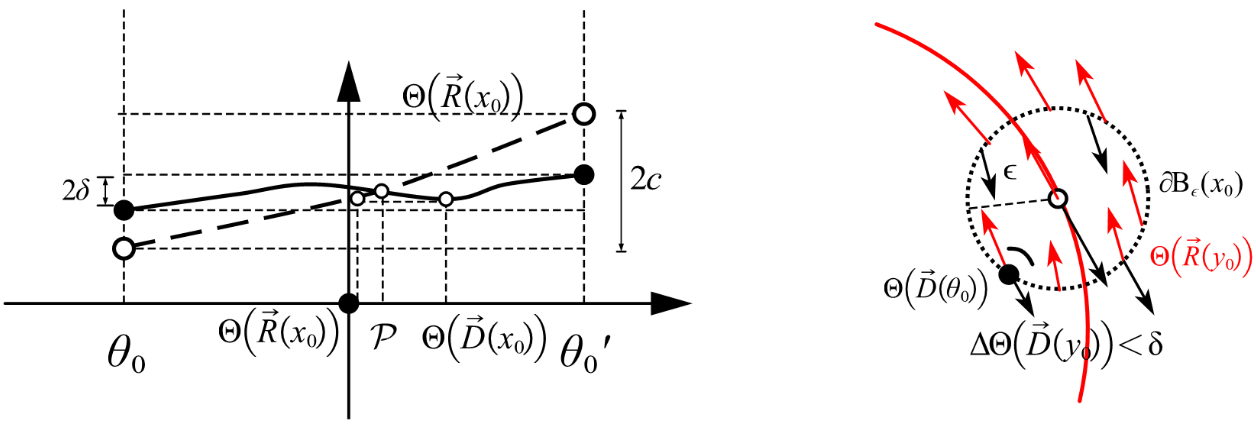

Figure 7: Right: y_0 in $\partial B_\epsilon(x_0)$; Left: It is a plot of functions $\Theta(\vec{D}(y_0))$ and $\Theta(\vec{R}(y_0))$ along $\partial B_\epsilon(x_0)$ with base point θ_0 . The two functions must have intersections because in a convex set $\Delta\Theta(\vec{R}(y_0)) > 0$ everywhere but $\lim_{\epsilon\to 0} \Delta\Theta(\vec{D}(y_0)) = 0$.

6 Analysis of Fixed Points

This chapter aims to prove the existence of fixed points under different situations.

6.1 Non-Zero Vectors on Both Maps

Corollary 6.1. *When $|\vec{D}(x_0)| \neq 0$, $|\partial\vec{D}(x_0)| = 0$, $|\vec{R}(x_0)| \neq 0$, and $|\vec{R}(0)| = 0$, the lines of opposite directions $\mathcal{L}$ and lines of equal length $\mathcal{C}$ must have at least one intersection, and the intersections are fixed points.*

Proof. Assume that the line $\mathcal{L}$ does not exit the area enclosed by $\mathcal{C}$. However, if it does, the lines will always intersect.

Since $\mathcal{L}$ will not vanish until it reaches a zero vector (as per Lemma 5.2), and given that $|\vec{D}(x_0)| \neq 0$ and $|\partial\vec{D}(x_0)| = 0$, in the region enclosed by $\mathcal{C}$, $\mathcal{L}$ can either remain non-vanishing or terminate at zero.

However, when $\epsilon \to 0$, $\partial B_\epsilon(0)$ has only one solution. Therefore, the line will not turn back to zero, which results in the following relation for any point $\mathcal{P}$ in the closed area enclosed by $\mathcal{C}$:

$$\Theta(\vec{D}(\mathcal{P})) + \pi = \Theta(\vec{R}(\mathcal{P})).$$

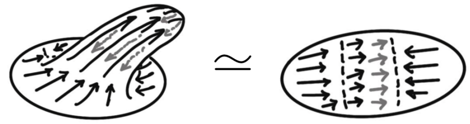

Figure 8: $\eta_t(\mathcal{M}, \vec{D}(x_0))$

It is also known that there exists a line $\mathcal{L}'$ where $\Theta(\vec{D}(\mathcal{P})) = \Theta(\vec{R}(\mathcal{P}))$ and a rotation zero vector at 0. Therefore, the assumption that $\mathcal{L}$ does not exit the region was incorrect. Hence, $\mathcal{L}$ must intersect $\mathcal{C}$, leading to the existence of fixed points at these intersections. □

6.2 Zero vectors on the Deformation map only:

In this case,it is much easier to prove that there exists fixed points. The process is very similar to Collorary 6.1.

Claim 6.2. *If $|\vec{R}(x_0)| \neq 0, |\vec{R}(0)| = 0$, there exists fixed points if $|\vec{D}(y_0)| = 0$ for some $\vec{D}(y_0)$, $y_0 \in D$.*

Proof. If $|\vec{D}(0)| = 0$, it is obvious that the fixed point are at 0. If $|\vec{D}(0)| \neq 0$, then using Intermediate Value Theorem again, $|\vec{D}_\theta(0)| - |\vec{R}_\theta(0)| > 0$, $x_0 \in (0, R], |\vec{D}_\theta(x_0)| - |\vec{R}_\theta(x_0)| < 0$, so clearly there are still intersections in the interval$(0, R]$. Therefore, the singularity point always are outside the loop. Using the same approach as **Collorary 6.1**,the prove can be done. □

6.3 Zero vectors on both maps

If $|\vec{D}(x_0)| = 0$, and simultaneously$|\vec{R}(x_0)| = 0$, then solutions of x_0 are the fixed points.

6.4 Zero vectors on the Rotation map only

There are two cases: First case is rotation map have zero areas in the center.

$$|\vec{R}(\mathcal{A})| = 0$$

In this case, If the rotation map's zero boundary forms an closed area for deformation map, which means, on $\partial\vec{R}(\mathcal{A})$, $\vec{D}(x_0)$ always points into this area.

According to Poincaré-Bendixson Theorem[3] ,If a continuous vector field is defined on a two-dimensional manifold, and if a trajectory (solution) is either closed or infinite and remains in a compact region (i.e., its flow is bounded), then the limit set of this trajectory must be either a periodic orbit or a fixed point.

$\vec{D}(x_0)$ satisfied this situations—it is closed for it always points into this closed area and never leaves, even if it is a folding map with some points stretching further and overlap another points outside the map.

Imagine the overlap part is lifted up a little bit into a 3D spaces, the vector $\vec{D}(x_0)$ are projected onto the lifted manifold, noted $\widetilde{\vec{D}(x_0)}$. Then this new two-dimensional manifold is homotopic to a scaled 2D manifold of the original area shape. The detail are presented here.

Define a homotopy $h_t(\mathcal{M}, \vec{D}(x_0))$ which sends the overlapped manifold to a lift up manifold $\widetilde{\mathcal{M}}$ and a lift up $\widetilde{\vec{D}(x_0)}$on $\widetilde{\mathcal{M}}$.

$$h_t(\mathcal{M}, \vec{D}(x_0)) = \widetilde{\mathcal{M}}, \widetilde{\vec{D}(x_0)}, t \in [0, 1]$$

The deformation vector are projected like following illustrations.

Define another homotopy η_t, which deforms the lifted-up part of $\widetilde{\mathcal{M}}$ back until no overlapping points remain when projected onto the original area $\mathcal{A}$. The projected homotopy $\eta_t(\widetilde{\vec{D}(x_0)})$, denoted as $p(\eta_t(\widetilde{\vec{D}(x_0)}))$, is homotopic to $\vec{D}(x_0)$ under $p\eta h$, while maintaining the condition that zero vectors

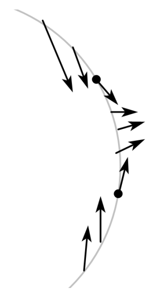

Figure 9: flow out of a section of $\partial\mathcal{A}$, two points on the boundary

remain zero throughout the deformation. In this new region, by the Poincaré-Bendixson Theorem, there must exist a fixed point, since there are no closed loops. Similarly, it can also be shown that when $\vec{D}(x_0)$ always points outward along the boundary, the vector field forms a closed region outside the boundary.

Claim 6.3. *If $\vec{D}(x_0)$ is not closed at the boundary of zero rotation area $\mathcal{A}$ when $\mathcal{A}$ covers the rotation center , there exists a line $\mathcal{L}$ expanded from the boundary point of $\mathcal{A}$, denoted as $\partial\mathcal{A}$, where $\Theta(\vec{D}(\mathcal{P}))+\pi = \Theta(\vec{R}(\mathcal{P}))$.*

Proof. This proof follows a similar argument to **Lemma 5.1**. Since $\vec{D}(x_0)$ is not closed, there must exist a vector tangent to $\partial\mathcal{A}$ at some points.

If at this point $\mathcal{P}$, we have the relation $\Theta(\vec{D}(\mathcal{P})) + \pi = \Theta(\vec{R}(\mathcal{P}))$, then the claim is immediately correct, as it implies an opposite direction between the two vector fields, guaranteeing the existence of a fixed point.

Now, suppose that on $\partial\mathcal{A}$, there is only one vector tangent to $\mathcal{P}$ such that $\Theta(\vec{D}(\mathcal{P})) = \Theta(\vec{R}(\mathcal{P}))$. We know that in the neighborhood of $\mathcal{P}$, the following condition holds:

$$\forall y_0 \in D_1 \cap \mathrm{B}_r(\mathcal{P}), \quad \text{as} \quad r \to 0, \quad \Theta(\vec{D}(y_0)) = \Theta(\vec{D}(\mathcal{P})).$$

Since the region is not closed and contains no loops, the limit set of all trajectories starting from $\partial\mathcal{A}$ must "flow" out of the area through the tangent points on $\partial\mathcal{A}$. This leads to a contradiction with the earlier statement that the only vector tangent at $\mathcal{P}$ satisfies $\Theta(\vec{D}(\mathcal{P})) = \Theta(\vec{R}(\mathcal{P}))$, because there always exist interior points in $\mathrm{B}_r(\mathcal{P})$ whose direction points exactly towards $\mathcal{P}$.

Thus, this contradiction leaves us with two possible explanations: 1.The point $\mathcal{P}$ is the zero vector, and therefore it is a fixed point. 2.The flow will exit through a section of $\partial\mathcal{A}$. Furthermore, consider another boundary point on this section of $\partial\mathcal{A}$. If $\Theta(\vec{D}(\mathcal{P})) = \Theta(\vec{R}(\mathcal{P}))$ holds at this boundary point as well, the same reasoning applies: there must exist interior points with directions pointing towards $\mathcal{P}$ that have nearly opposite values of $\Theta(\vec{D}(\mathcal{P}))$, which again leads to the conclusion that $\mathcal{P}$ must be a fixed point. (Figure 9)

Therefore, at the boundary of the section of $\partial\mathcal{A}$, one point satisfies $\Theta(\vec{D}(y_0)) = \Theta(\vec{R}(\mathcal{P}))$ and another point satisfies $\Theta(\vec{D}(y_0)) + \pi = \Theta(\vec{R}(\mathcal{P}))$.

Once these points are found, there exist two lines, denoted $\mathcal{L}$, extended from these two points according to **Lemma 5.2**. By applying a similar argument as in **Claim 6.2**, the loop will lie outside the area $\mathcal{A}$.

Furthermore, by the proof of **Corollary 6.1**, the line $\mathcal{L}$ must extend beyond the loop $\mathcal{C}$, since it cannot remain within a closed area. This extension ensures that the lines do not form a closed region, thereby reinforcing the existence of a fixed point at the intersection.

Other case is the rotation map have zero areas. Under this case, because the $\Theta(\vec{R}(x_0)) = 0$ on $\partial\mathcal{A}$, it would still be continuous to construct a new deformation map in this area thus

$$\vec{D}(x_0) = \widetilde{\vec{D}(x_0)} + \widetilde{\vec{R}(x_0)}$$

where $\widetilde{\vec{R}(x_0)} \neq 0$ but very small. Then because only on $\partial\mathcal{A}$, $\widetilde{\vec{R}(x_0)} = 0$, there still $\exists \mathrm{B}_\epsilon(x_0)$,$y_0 \in \partial\mathrm{B}_\epsilon(x_0)$,$\Theta(\vec{D}(y_0)) = \Theta(\vec{R}(y_0))$. The line $\updownarrow$ still can be extended.

$\square$

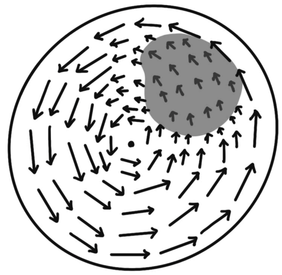

Figure 10: Rotation map have zero areas.

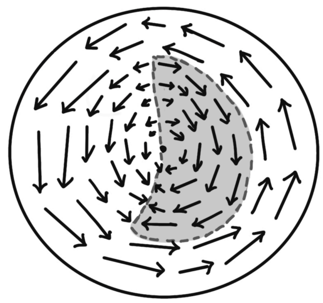

Figure 11: Rotation map have zero areas.

6.5 Inversed rotation area

First of all, it is evident that if there are fixed points on the boundary ∂D where the rotation has changed direction, it becomes necessary to explore the scenario where rotation only changes direction within a specific area.

In this case, if the rotation changes direction, there must be a region bounded by vectors with zero rotation. By employing a similar approach as in Claim 6.3, we can construct a modified vector field $\widetilde{\vec{D}(x_0)}$:

$$\vec{D}(x_0) = \widetilde{\vec{D}(x_0)} + \widetilde{\vec{R}(x_0)},$$

which yields a non-inverted rotation map.

6.6 Multiple Rotation Centers

When a rotation map has two centers of rotation, it must contain fixed points. This is because there will always be a point where the rotation vectors have different directions at the intersection line of the two rotation vector fields, and on the boundary ∂D, the rotation vector must be zero.

6.7 The Proof is Invalid for Non-Convex Areas

If the shape is non-convex, the second part regarding opposite directions fails. There will not exist a $\mathrm{B}_\epsilon(x_0)$ such that for $y_0 \in \partial \mathrm{B}_\epsilon(x_0)$, $\Theta(\vec{D}(y_0)) = \Theta(\vec{R}(y_0))$. This is because there may be a straight line in the loop where the rotation vector remains unchanged.

6.8 Summary

This sections discusses the proof of the 2D Brouwer Fixed-Point Theorem under the mentioned conditions before.

7 Experiments

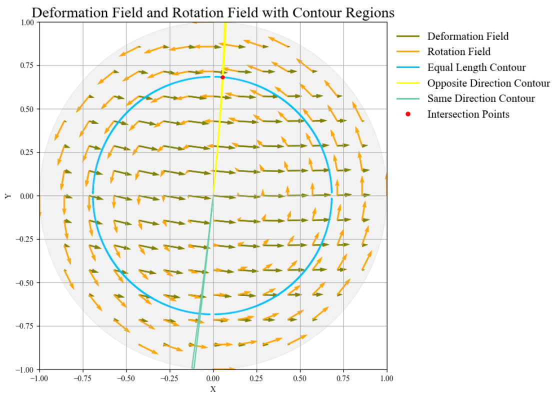

Figure 12: $\vec{D}(x_0)$ has no non zero vector

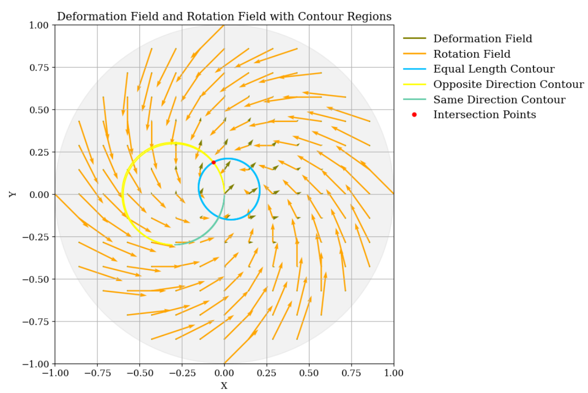

Figure 13: $\vec{D}(x_0)$ has single zero point

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