Unscented Kalman filtering method without the matrix square root to estimate the satellite attitude using magnetometer

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ABSTRACT

In satellite mission, attitude control system plays an important role, and precise attitude control presents high attitude determination requirements. The TRIAD (TRIaxial Attitude Determination) method, which is widely used for satellite attitude determination, requires two sensor signals. However, when the reference vector direction to be observed in these sensors is close, the attitude determination error increases. Thus, in this case, attitude estimation is required, and the state estimator of nonlinear objects is widely used for extended Kalman and unscented Kalman filters. In this paper, we propose a method for determining satellite attitude using an unscented Kalman filter with high estimation accuracy compared to an extended Kalman filter. To reduce the amount of computation in the unscented Kalman filter and to ensure the real-time of the estimation, we use the unscented Kalman filtering method with a new sigma point selection. Compared with the traditional unscented Kalman filter, it ensures better real-time and higher accuracy.

Keywords: sigma points, Kalman filter, state estimation, magnetometer, satellite attitude

1. Introduction

Satellite attitude determination is mainly done by static methods such as TRIAD and QUEST (QUaternion ESTimation), and dynamic methods using state estimators [1,2]. Static attitude determination methods have the advantage of high real-time and non-divergent, but more than two sensors must be used, and the reliability of attitude determination can be reduced if the sensor signal is insufficient in various operating environments or the direction of the reference vector is close.

Therefore, the extended Kalman filtering methods for estimating satellite attitude in a dynamic way have been investigated. Extended Kalman filter is a theory that extends linear Kalman filtering theory to nonlinear objects, which linearizes nonlinear objects near the operating point [3,4]. Therefore, there is necessarily a linearization error in the system model, which affects the state estimation accuracy of the system [4,5]. The extended Kalman filter estimates the state with a first degree of accuracy and, importantly, it makes the estimation possible to vary greatly depending on how the initial value is set.

To overcome these drawbacks and improve the estimation accuracy and reliability, the unscented Kalman filtering theory has been developed [3,6-9].

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The unscented Kalman filter can estimate nonlinear objects with three degrees of accuracy by computing the predicted and covariance of the next time in a statistical manner without linearizing nonlinear systems [10-13].

In general, the unscented Kalman filter uses two times or more sigma points of the state number[14-18]. And to compute these sigma points, matrix square root operations are performed, and this matrix square root computation is often computationally expensive, and moreover, the regularity of the matrix is destroyed and the operation cannot be done [19-21].

Many applications as satellite attitude sensors are solar sensors and magnetometers [1].

The solar sensor has a relatively high measurement accuracy compared to the magnetometer, but it is not available in the shaded area. In many cases, attitude determination using solar sensors and magnetometers simultaneously is required, but in this case, attitude determination using magnetometer alone is necessary [6].

Therefore, we propose a new method to improve the reliability of low-cost and sigma point selection and apply it to satellite attitude estimation using magnetometer signals only to verify its effectiveness.

This paper is organized as follows.

First, a new sigma point selection method is introduced. Then, a method for estimating satellite attitude using magnetometer signals only is proposed. Finally, the simulation results of the proposed method are presented.

2. Selecting method of sigma points

This section describes the selecting method of sigma points using the solution of simultaneous equation.

Let the sigma points set as $\boldsymbol{\chi}_i$ whose expected value is $\hat{\mathbf{x}}_{k-1}$ and co-variance matrix is $\mathbf{P}_{k-1}$.

Assume that the co-variance matrix $\mathbf{P}_{k-1}$ is

$$\mathbf{P}_{k-1} = \begin{pmatrix} P_{11} & P_{12} & \cdots & P_{1n} \\ P_{21} & P_{22} & \cdots & P_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ P_{n1} & P_{n2} & \cdots & P_{nn} \end{pmatrix}$$

Here, the detailed expression of P_{ij} is

$$P_{ij} = \frac{1}{n}\sum_{m=1}^{n}\left(\chi_{m,i} - \hat{x}_{k-1,i}\right)\left(\chi_{m,j} - \hat{x}_{k-1,j}\right) \tag{2.1}$$

By the permutation of $a_{ij} = \chi_{ij} - \hat{x}_{k-1,j}$, the Eq.(2.1) is given below.

$$P_{ij} = \frac{1}{n}\sum_{m=1}^{n} a_{mi}a_{mj} \tag{2.2}$$

Theorem 1: Given the mean and covariance, $2n$ sigma points are selected as follows

$$\begin{aligned} \chi_{ij} &= |a_{ij}| + \hat{x}_{k-1,j}, & i &= 1,\cdots,n \\ \chi_{ij} &= -|a_{(i-n)j}| + \hat{x}_{k-1,j}, & i &= n+1,\cdots,2n \end{aligned} \tag{2.3}$$

Where $\quad a_{ii} = (nP_{ii} - \sum_{k=1}^{i-1} a_{ki})^{\frac{1}{2}}$, $a_{ij} = (nP_{ji} - \sum_{k=1}^{i-1} a_{ki}a_{kj}) / a_{ii}$

We use the following Lemmas to prove theorem 1.

Lemma 1.1: If $n = 2$, that is, in the case of 2 order system, using the solution $\{a_{ij}\}$ of the simultaneous equation

$$\begin{bmatrix} a_{11}^2 = 2P_{11} & a_{21} = 0 \\ a_{12} = \dfrac{2P_{12}}{a_{11}} & a_{22}^2 = 2P_{22} - a_{12}^2 \end{bmatrix} \tag{2.4}$$

and equation $\chi_{ij} - \hat{x}_{k-1,j} = a_{ij}$, we obtain the sigma points set $\{\chi_{ij}\}$.

Proof: Consider covariance $\mathbf{P}_{k-1}$ of time $k-1$ as

$$\mathbf{P}_{k-1} = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}$$

Where, $P_{12} = P_{21}$.

From Eq.(2.1) and the fact that the covariance matrix is symmetric, the following 3 simultaneous equations can be obtained.

$$\begin{cases} \dfrac{1}{2}\left(a_{11}^2 + a_{21}^2\right) = P_{11} \\ \dfrac{1}{2}\left(a_{12}a_{11} + a_{22}a_{21}\right) = P_{21} \\ \dfrac{1}{2}\left(a_{12}^2 + a_{22}^2\right) = P_{22} \end{cases} \tag{2.5}$$

In this simultaneous equation, the number of equations is three, and the number of the variables is four, so the one of them is assumed to be free. Consider the free variable as $a_{21} = 0$,

$$\begin{bmatrix} a_{11}^2 = 2P_{11} & a_{21} = 0 \\ a_{12} = \dfrac{2P_{12}}{a_{11}} & a_{22}^2 = 2P_{22} - a_{12}^2 \end{bmatrix}$$

Then, obtain $\{a_{ij}\}$ and replace $\chi_{ij} - \hat{x}_{k-1,j} = a_{ij}$ to get $\{\chi_{ij}\}$.

Lemma 1.2: If $n=3$, that is, in the case of 3 order system, using the solution $\{a_{ij}\}$ of the simultaneous equation

$$\begin{bmatrix} a_{11}^2 = 3P_{11} & a_{21} = 0 & a_{31} = 0 \\ a_{12} = \dfrac{3P_{21}}{a_{11}} & a_{22}^2 = 3P_{22} - a_{12}^2 & a_{32} = 0 \\ a_{13} = \dfrac{3P_{31}}{a_{11}} & a_{23} = \dfrac{3P_{32} - a_{13}a_{12}}{a_{22}} & a_{33}^2 = 3P_{33} - a_{13}^2 - a_{23}^2 \end{bmatrix} \tag{2.6}$$

and equation $\chi_{ij} - \hat{x}_{k-1,j} = a_{ij}$, we obtain the sigma points set $\{\chi_{ij}\}$.

Proof: Consider covariance $\mathbf{P}_{k-1}$ of time $k-1$ as

$$\mathbf{P}_{k-1} = \begin{pmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{pmatrix}$$

Where, $P_{ij} = P_{ji}$. Using Eq.(2.1) and the covariance matrix is symmetric, the following simultaneous equations are obtained

$$\begin{cases} a_{11}^2 + a_{21}^2 + a_{31}^2 = 3P_{11} \\ a_{12}a_{11} + a_{22}a_{21} + a_{32}a_{31} = 3P_{21} \\ a_{12}^2 + a_{22}^2 + a_{32}^2 = 3P_{22} \\ a_{13}a_{11} + a_{23}a_{21} + a_{33}a_{31} = 3P_{31} \\ a_{13}a_{12} + a_{23}a_{22} + a_{33}a_{32} = 3P_{32} \\ a_{13}^2 + a_{23}^2 + a_{33}^2 = 3P_{33} \end{cases} \tag{2.7}$$

In this simultaneous equation, the number of equation is six, and the variable is nine, so we assume that three of them are free.

If the free variable is $a_{21} = a_{31} = a_{32} = 0$,

$$\begin{bmatrix} a_{11}^2 = 3P_{11} & a_{21} = 0 & a_{31} = 0 \\ a_{12} = \dfrac{3P_{21}}{a_{11}} & a_{22}^2 = 3P_{22} - a_{12}^2 & a_{32} = 0 \\ a_{13} = \dfrac{3P_{31}}{a_{11}} & a_{23} = \dfrac{3P_{32} - a_{13}a_{12}}{a_{22}} & a_{33}^2 = 3P_{33} - a_{13}^2 - a_{23}^2 \end{bmatrix}$$

Then, obtain $\{a_{ij}\}$ and replace $\chi_{ij} - \hat{x}_{k-1,j} = a_{ij}$ to obtain $\{\chi_{ij}\}$.

Lemma 1.4: If $n=4$, that is, in the case of 4 order system, using the solution $\{a_{ij}\}$ of the simultaneous equation

$$\begin{bmatrix} a_{11}^2 = 4P_{11} & a_{21} = 0 & a_{31} = 0 & a_{41} = 0 \\ a_{12} = \dfrac{4P_{21}}{a_{11}} & a_{22}^2 = 4P_{22} - a_{12}^2 & a_{32} = 0 & a_{42} = 0 \\ a_{13} = \dfrac{4P_{31}}{a_{11}} & a_{23} = \dfrac{4P_{32} - a_{13}a_{12}}{a_{22}} & a_{33}^2 = 4P_{33} - a_{13}^2 - a_{23}^2 & a_{43} = 0 \\ a_{14} = \dfrac{4P_{41}}{a_{11}} & a_{24} = \dfrac{4P_{42} - a_{14}a_{12}}{a_{22}} & a_{34} = \dfrac{4P_{43} - a_{14}a_{13} - a_{24}a_{23}}{a_{33}} & a_{44}^2 = 4P_{44} - a_{14}^2 - a_{24}^2 - a_{34}^2 \end{bmatrix} \quad (2.8)$$

and equation $\chi_{ij} - \hat{x}_{k-1,j} = a_{ij}$, we obtain the sigma points set $\{\chi_{ij}\}$.

We can prove theorem 1 in the same way as the above for a certain natural number from lemma 1.1, 1.2 and 1.3.

Theorem 2: The amount of calculation for selecting sigma points using the solution of the simultaneous equations is smaller than that using the matrix square roots.

Proof: The previous UKF gets the sigma points set using matrix square root of covariance matrix, i.e. Cholesky decomposition.

First, Cholesky decomposition is given as the following equation through the LDL decomposition of the covariance matrix.

$$\mathbf{P} = \mathbf{LDL}^T = \mathbf{L}\sqrt{\mathbf{D}}\sqrt{\mathbf{D}}\mathbf{L}^T = \left(\mathbf{L}\sqrt{\mathbf{D}}\right)\left(\sqrt{\mathbf{D}}\mathbf{L}\right)^T \quad (2.9)$$

Here, the calculation amount is given as follows.

In LDL decomposition, the calculation is $n^3/6$ times of multiplication, addition, and subtraction (n times of division). Then, calculate square root of diagonal matrix D and multiply on the both sides – it results in n times of square root, $2n^2$ times of multiplication. In total, there are $n^3/3$ times of addition and subtraction, $\frac{n^3}{6} + 2n^2 + n$ times of multiplication and division, and n times of square root calculation.

Then, the calculation amount to get solution of simultaneous equation is as follows.

There is no addition, $n^3/6 + n^2 - n$ times of subtraction, and $n^3/6 + n^2 - n/6$ times of multiplications and division, n times of square root.

The amount of calculation for selecting sigma points using the matrix square roots and the solution of simultaneous equations is shown in Table 1.

Table 1: Comparison of the amount of calculation for selecting sigma points using the matrix square roots and the solution of simultaneous equations

<table>
<tr><td rowspan="2">Method</td><td>Addition</td><td rowspan="2">Total</td><td>Multiplication</td><td rowspan="2">Total</td><td rowspan="2">Square root</td></tr>
<tr><td>Subtraction</td><td>division</td></tr>
<tr><td rowspan="2">Cholesky decomposition method</td><td>$\frac{n^3}{6}$</td><td rowspan="2">$\frac{n^3}{3}$</td><td>$\frac{n^3}{6} + 2n^2$</td><td rowspan="2">$\frac{n^3}{6} + 2n^2 + n$</td><td rowspan="2">n</td></tr>
<tr><td>$\frac{n^3}{6}$</td><td>n</td></tr>
</table>

<table>
<tr><td rowspan="2">Method of using solution of simultaneous equation</td><td>0</td><td rowspan="2">$\frac{n^3}{6}+n^2-\frac{n}{6}$</td><td>$\frac{n^3}{6}+\frac{n^2}{2}+\frac{n}{3}$</td><td rowspan="2">$\frac{n^3}{6}+n^2-\frac{n}{6}$</td><td rowspan="2">$n$</td></tr>
<tr><td>$\frac{n^3}{6}+n^2-\frac{n}{6}$</td><td>$\frac{n^2}{2}-\frac{n}{2}$</td></tr>
</table>

As you can see in Table 1, the amount of calculation of the new method for selecting sigma points is smaller than that of the previous methods.

Theorem 3: In general, in the simultaneous equation, the number of variables shouldn't be smaller than the number of equations, so the number of sigma points should be at least $2\cdot\text{int}\left(\frac{n+2}{2}\right)$.

Proof: If the number of the simultaneous equations is *m*,

$$n\cdot m \geq \frac{n(n+1)}{2} \Rightarrow m \geq \frac{n+1}{2}\text{, then } m_{\min} = \text{int}\left(\frac{n+2}{2}\right)$$

Therefore, the minimum number of sigma points should be $2\cdot\left[\frac{n+2}{2}\right]$.

At this point, the amount of calculation is given in Table 2.

Table 2: The amount of calculation for selecting sigma points using the solution of minimum simultaneous equations

<table>
<tr><td rowspan="2">Method</td><td>Addition</td><td rowspan="2">Total</td><td>Multiplication</td><td rowspan="2">Total</td><td rowspan="2">Root</td></tr>
<tr><td>Subtraction</td><td>Division</td></tr>
<tr><td rowspan="2">Optimization method</td><td>0</td><td rowspan="2">$\frac{m(m-1)}{2}\left(n-\frac{2m-1}{3}\right)$</td><td>$\frac{m}{2}\left(n+nm-m+1-\frac{(m-1)(2m-1)}{3}\right)$</td><td rowspan="2">$\frac{m}{2}\left(n+nm-m+1-\frac{(m-1)(2m-1)}{3}\right)+\frac{m(2n-m-1)}{2}$</td><td rowspan="2">$m$</td></tr>
<tr><td>$\frac{m(m-1)}{2}\left(n-\frac{2m-1}{3}\right)$</td><td>$\frac{m(2n-m-1)}{2}$</td></tr>
</table>

3. A method to determine the satellite attitude using only magnetometer signals

3.1. Satellite attitude equation of motion

The satellite attitude equation of motion consists of satellite attitude dynamic equation and kinematic equation.

The satellite attitude dynamic equation describes the relation between the torque and the angular velocity of the satellite and is expressed as follows.

$$\mathbf{J}\dot{\boldsymbol{\omega}} + [\boldsymbol{\omega}\times]\mathbf{J}\boldsymbol{\omega} = \mathbf{T} \tag{3.1}$$

Where $\mathbf{J}$ -inertial matrix of the satellite

$\boldsymbol{\omega}$ - angular velocity vector in ECI frame.

$\mathbf{T}$ - external torque to the satellite.

$[\boldsymbol{\omega}\times]$ - skew-symmetric operator as

$$[\boldsymbol{\omega}\times] = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$

Satellite kinematic equation describes the relation between the attitude kinematic parameters and is expressed as

$$\begin{aligned} \dot{\mathbf{q}}_{13} &= \frac{1}{2}(q_4\mathbf{I} - [\mathbf{q}_{13}\times])\boldsymbol{\omega} \\ \dot{q}_4 &= -\frac{1}{2}\mathbf{q}_{13}^T\boldsymbol{\omega} \end{aligned} \tag{3.2}$$

Combining Eq.(3.1) and Eq.(3.2), the state vector is taken to be $\mathbf{X} = [\mathbf{q}\ \boldsymbol{\omega}]^T =$ $= [q_1\ q_2\ q_3\ q_4\ \omega_x\ \omega_y\ \omega_z]^{\mathrm{T}}$ and the satellite attitude equation of motion is given as follows.

$$\dot{\mathbf{X}} = \begin{bmatrix} \dot{\mathbf{q}}_{13} \\ \dot{q}_4 \\ \dot{\boldsymbol{\omega}} \end{bmatrix} = \mathbf{f}(\mathbf{X}, T, t) = \begin{bmatrix} \frac{1}{2}(q_4\mathbf{I} - [\mathbf{q}_{13}\times])\boldsymbol{\omega} \\ -\frac{1}{2}\mathbf{q}_{13}^T\boldsymbol{\omega} \\ \mathbf{J}^{-1}(-[\boldsymbol{\omega}\times]\mathbf{J}\boldsymbol{\omega} + \mathbf{T}) \end{bmatrix} \tag{3.3}$$

3.2. Measurement equation of magnetometer

To determine the satellite attitude using magnetometer, the intensity of earth magnetic field is measured and its equation is expressed below.

$$\mathbf{B}_b = \mathbf{R}_i^b\mathbf{B}_i + \mathbf{n}_b \tag{3.4}$$

Where $\mathbf{B}_b$ - intensity vector of earth magnetic field for satellite body frame.

$\mathbf{B}_i$ - intensity vector of earth magnetic field for inertial frame. (It can be given using IGRF model).

$\mathbf{n}_b$ - noise vector for measuring intensity of earth magnetic field.

$\mathbf{R}_i^b$ - attitude matrix of satellite body frame for inertial frame.

$$\mathbf{R}_i^b = (\mathbf{R}_b^i)^{\mathrm{T}} = \begin{pmatrix} q_1^2 - q_2^2 - q_3^2 + q_4^2 & 2(q_1q_2 + q_4q_3) & 2(q_1q_3 - q_4q_2) \\ 2(q_1q_2 - q_4q_3) & -q_1^2 + q_2^2 - q_3^2 + q_4^2 & 2(q_2q_3 + q_4q_1) \\ 2(q_1q_3 + q_4q_2) & 2(q_2q_3 - q_4q_1) & -q_1^2 - q_2^2 + q_3^2 + q_4^2 \end{pmatrix}$$

3.1. Design of UKF

To design the discrete Kalman Filter, the continuous satellite attitude model should be converted to the discrete model.

The discrete equation of motion of satellite attitude is given as follows. [Appendix A]

$$\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k,\ \ \mathbf{T}_k,\ \ \mathbf{w}_k) = \begin{bmatrix} \overline{\boldsymbol{\Omega}}(\boldsymbol{\omega}_k)\mathbf{q}_k \\ \boldsymbol{\omega}_k + \Delta t\,(-\mathbf{J}^{-1}[\boldsymbol{\omega}_k \times](\mathbf{J}\boldsymbol{\omega}_k) + \mathbf{J}^{-1}\mathbf{T}_k) \end{bmatrix} + \mathbf{w}_k \tag{3.5}$$

Where

$$\mathbf{x}_k = [\,\mathbf{q}_k^T \quad \boldsymbol{\omega}_k^T\,]^T$$

$$\overline{\boldsymbol{\Omega}}(\boldsymbol{\omega}_k) = \begin{bmatrix} \cos\left(\frac{1}{2}\|\boldsymbol{\omega}_k\|\Delta t\right)\mathbf{I} + [\boldsymbol{\Psi}_k \times] & \boldsymbol{\Psi}_k \\ -\boldsymbol{\Psi}_k^T & \cos\left(\frac{1}{2}\|\boldsymbol{\omega}_k\|\Delta t\right) \end{bmatrix}$$

$$\boldsymbol{\Psi}_k = \frac{\sin\left(\frac{1}{2}\|\boldsymbol{\omega}_k\|\Delta t\right)\boldsymbol{\omega}_k}{\|\boldsymbol{\omega}_k\|}$$

$\mathbf{w}_k$ - Noise vector of the system.

The covariance of $\mathbf{w}_k$ is $\mathbf{Q}_k$ and is given as follows.

$$\mathbf{Q}_k = \{Q_{ij}\} = \mathbf{E}(\mathbf{w}_k,\ \mathbf{w}_k^T) = \begin{cases} Q_k & (i = j) \\ 0 & (i \neq j) \end{cases}$$

Then, the measurement equation of three-axis magnetometer is expressed as follows.

$$\mathbf{y}_k = \mathbf{h}(\mathbf{x}_k, \mathbf{v}_k) = \mathbf{R}_{ib}(\mathbf{q}_k)\mathbf{B}_{i,k} + \mathbf{v}_k \tag{3.6}$$

Where, $\mathbf{B}_{i,\,k}$ is the intensity vector of earth magnetic field for the ECI frame at a given position of the orbit at time k and $\mathbf{v}_k$ is known as the covariance matrix of the measurement noise.

$$\mathbf{R}_k = \{R_{ij}\} = \mathbf{E}(\mathbf{v}_k,\ \mathbf{v}_k^T) = \begin{cases} R_k & (i = j) \\ 0 & (i \neq j) \end{cases} \tag{3.7}$$

As a result, nonlinear discrete state equation and observation equation of Eq.(3.5) and Eq.(3.6) can be expressed as follows.

$$\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k,\ \ \mathbf{w}_k) \tag{3.8}$$

$$\mathbf{y}_k = \mathbf{h}(\mathbf{x}_k,\ \ \mathbf{v}_k) \tag{3.9}$$

We can estimate the attitude of satellite by applying UKF algorithm (Appendix B) into the discrete nonlinear system using Eq.(3.8) and Eq.(3.9).

4. Simulation examples

In this paper, the parameters are given as follows.

First, the satellite orbit elements are as follows.

Semi-major axis: $a = 7214.1\text{km}$

Eccentricity: $e = 7.8 \times 10^{-3}$

Inclination: $i = 97.4^{\circ}$

Right ascension of ascending node: $\Omega = 324.96^{\circ}$

Argument of perigee $\omega = 155.74^{\circ}$

Time of passing perigee: 10h 0m 0s, Sep 1, 2022.

Second, the satellite attitude parameter - inertial matrix is given as follows.

$$\mathbf{J} = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 15 & 0 \\ 0 & 0 & 12 \end{bmatrix} \text{ kg} \cdot \text{m}^2$$

Third, IGRF-2020 is used as the earth magnetic field model.

Fourth, the intensity noise of earth magnetic field is assumed to be in the range of $\pm 50 \text{ nT}$.

Fifth, sampling period is 1s and the initial value of UKF is set as follows.

$$\mathbf{x}(0) = [0\ 0\ 0\ 1\ 0\ 0\ 0]^T$$

$$\mathbf{P}(0) = \text{diag}([10^{-1}\ 10^{-1}\ 10^{-1}\ 10^{-1}\ (0.5 \times \text{pi}/180)\text{^}2\ (0.5 \times \text{pi}/180)\text{^}2\ (0.5 \times \text{pi}/180)\text{^}2])$$

Finally, attitude estimation procedure of the UKF using matrix square root is compared with that of the UKF using the solution of simultaneous equation and its result is shown in figure 1 and 2.

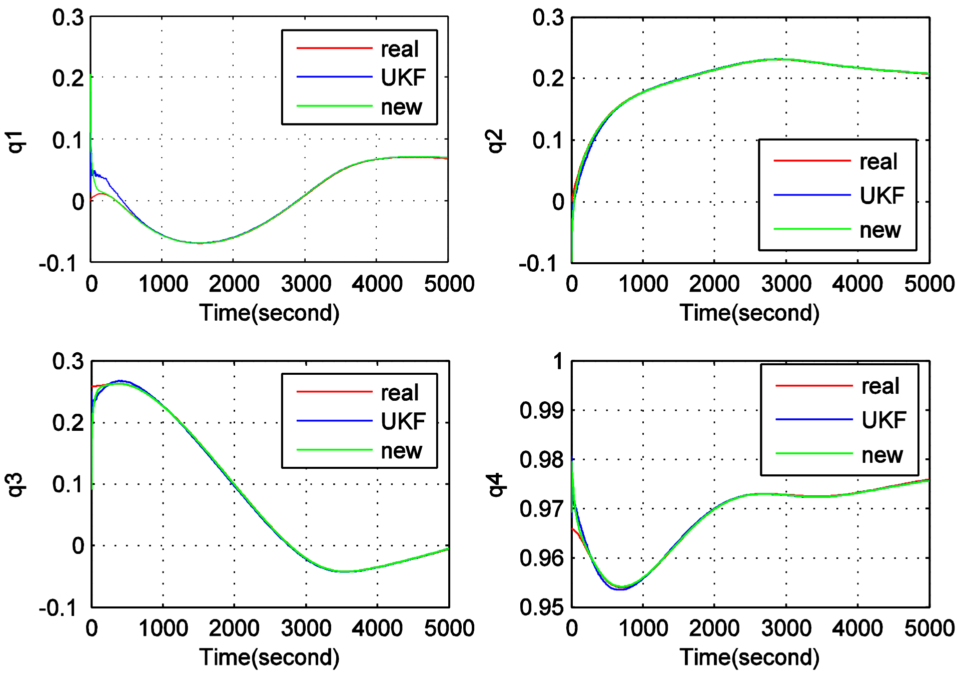

Figure1. Estimation process of attitude quaternion in 0-5000s

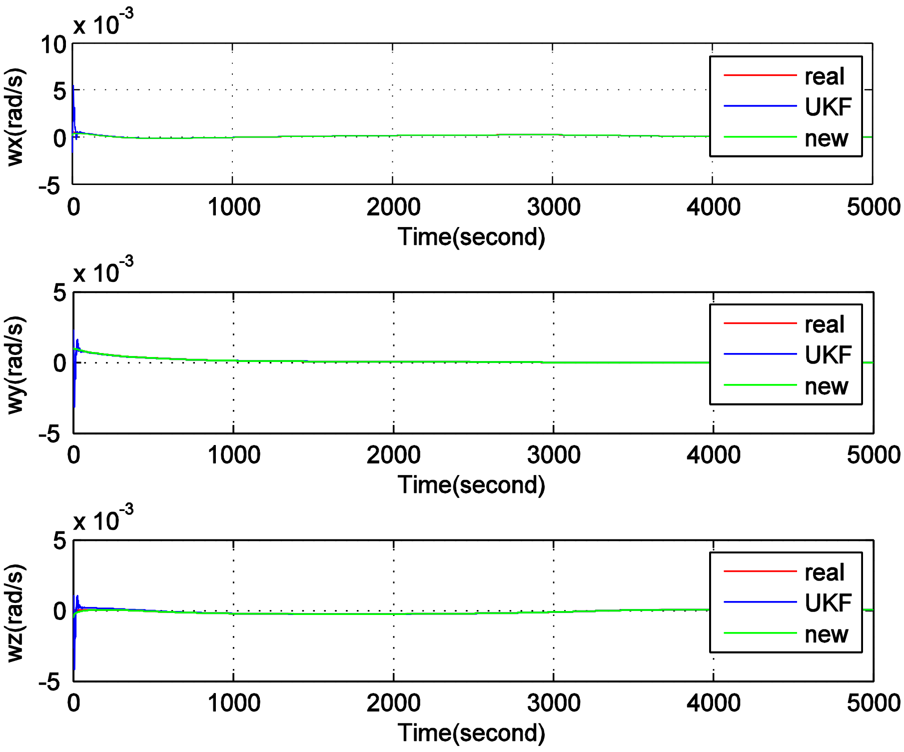

Figure2. Estimation process of attitude angular velocity in 0-5000s

To analyse the estimation speed, the following are the estimation result of the attitude quaternion and attitude angular velocity in 0-500s.

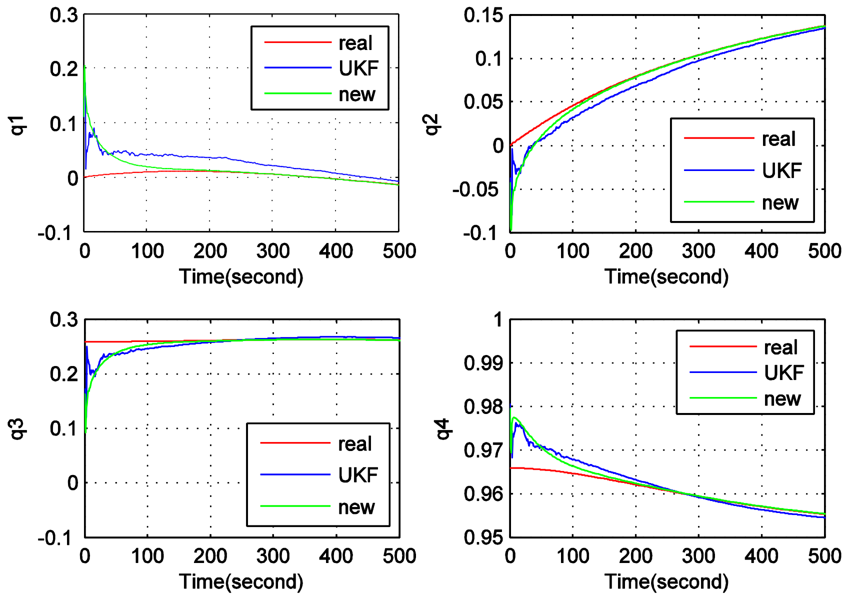

Figure3. Estimation process of attitude quaternion in 0-500s

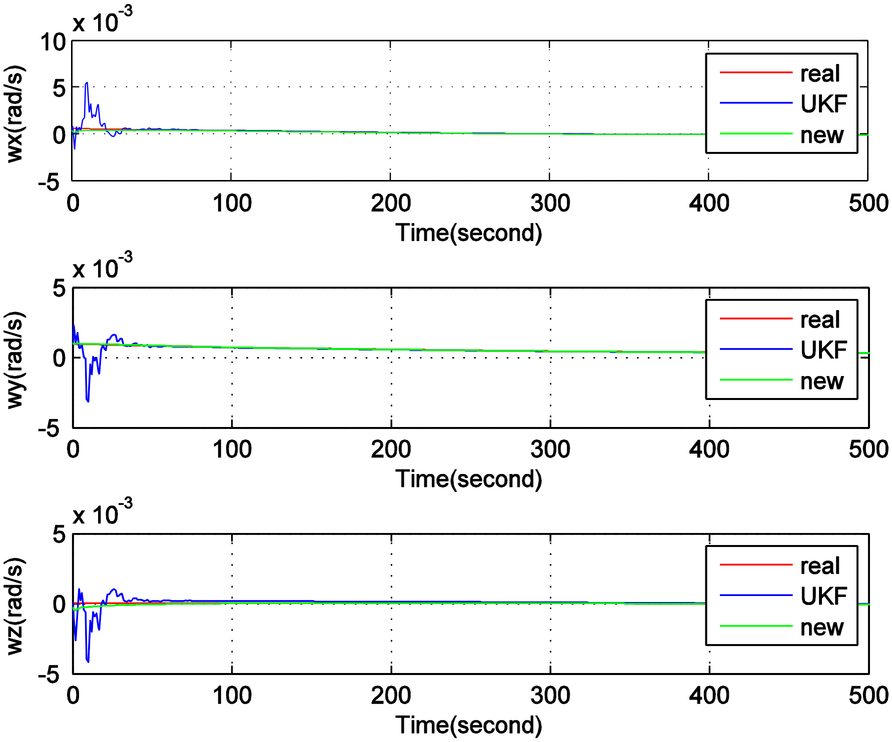

Figure4.Estimation process of attitude angular velocity in 0-500s

To analyse the accuracy, the following estimation results of the attitude quaternion and attitude angular velocity in 450-500s are shown below.

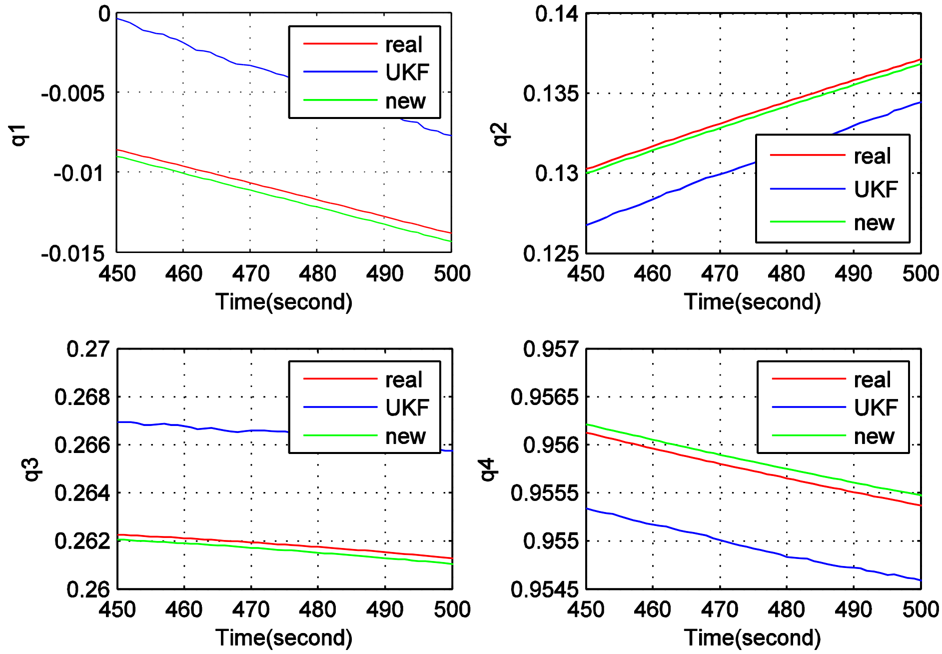

Figure5.Estimation process of attitude quaternion in 450-500s

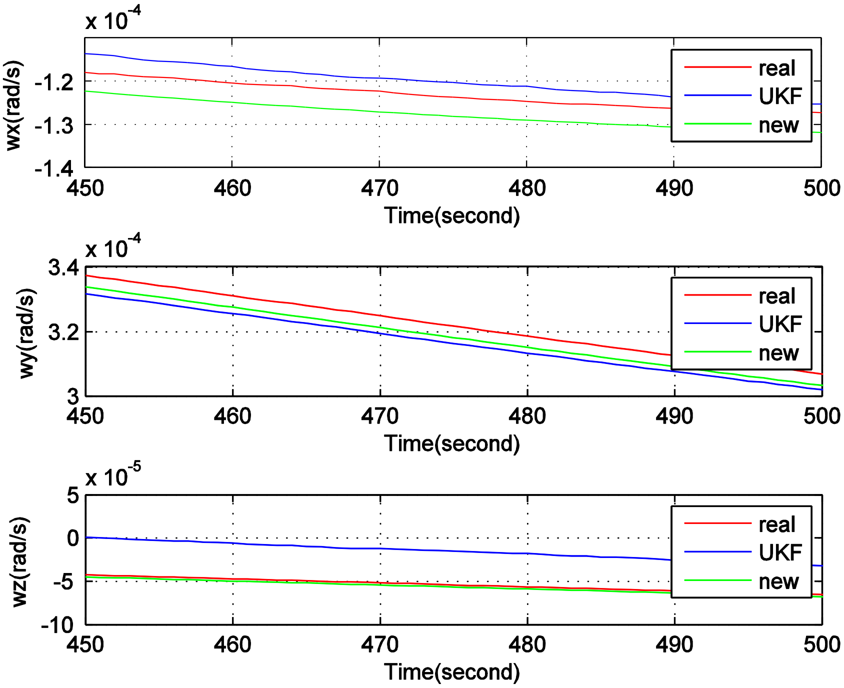

Figure6. Estimation process of attitude angular velocity in 450-500s

As we can be see from the simulation results, this new UKF method is proved to be very effective not only in calculation amount but also in time and accuracy of estimation.

5. Conclusion

This paper proposed a new method for selecting sigma points without calculating matrix square roots.

The amount of calculation for selecting sigma points proposed in this paper is smaller than that using the matrix square root. This method is proved to be efficient in the estimation of satellite attitude using magnetometer. This can be efficiently used in the attitude estimation of low-orbit and low cost satellite because of high accuracy and reliability.

Appendix A: Discretization of satellite attitude equation of motion

First, consider discretization of satellite attitude dynamic equation Eq.(3.1).

The discretization of dynamic equation can be performed by the definition of derived function- $\dot{\boldsymbol{\omega}}_{ib} = \lim_{\Delta t \to 0} \frac{\boldsymbol{\omega}_{ib}(t+\Delta t) - \boldsymbol{\omega}_{ib}(t)}{\Delta t}$

$$\boldsymbol{\omega}_{k+1} = \boldsymbol{\omega}_k + \Delta t[-\mathbf{J}^{-1}[\boldsymbol{\omega}_k \times](\mathbf{J}\boldsymbol{\omega}_k) + \mathbf{J}^{-1}\mathbf{T}_k] \tag{A.1}$$

Then, consider discretization of the satellite attitude kinematic equation Eq.(3.2).

Kinematic equation can be expressed as follows.

$$\dot{\mathbf{q}} = -\frac{1}{2}\begin{bmatrix} q_4\mathbf{I} + [\mathbf{q}_{13}\times] \\ \mathbf{q}_{13}^T \end{bmatrix}\boldsymbol{\omega} = \frac{1}{2}\boldsymbol{\Omega}(\boldsymbol{\omega})\mathbf{q} \tag{A.2}$$

Where,

$$\boldsymbol{\Omega}(\boldsymbol{\omega}) = \begin{bmatrix} 0 & \omega_z & -\omega_y & \omega_x \\ -\omega_z & 0 & \omega_x & \omega_y \\ \omega_y & -\omega_x & 0 & \omega_z \\ -\omega_x & -\omega_y & -\omega_z & 0 \end{bmatrix} = \begin{bmatrix} -[\boldsymbol{\omega}\times] & \boldsymbol{\omega} \\ -\boldsymbol{\omega}^T & 0 \end{bmatrix}$$

From the formula of linear differential equation, the solution to Eq.(A.2) is:

$$\mathbf{q}(t) = \exp\left(\frac{1}{2}\boldsymbol{\Omega}(\boldsymbol{\omega})t\right)\mathbf{q}_0 \tag{A.3}$$

The exponential function of the equation can be expanded into a Taylor series.

$$\exp\left(\frac{1}{2}\boldsymbol{\Omega}(\boldsymbol{\omega})t\right) = \sum_{j=0}^{\infty}\frac{\left(\frac{1}{2}\boldsymbol{\Omega}(\boldsymbol{\omega})t\right)^j}{j!} = \sum_{k=0}^{\infty}\left\{\frac{\left(\frac{1}{2}\boldsymbol{\Omega}(\boldsymbol{\omega})t\right)^{2k}}{(2k)!} + \frac{\left(\frac{1}{2}\boldsymbol{\Omega}(\boldsymbol{\omega})t\right)^{2k+1}}{(2k+1)!}\right\} \tag{A.4}$$

Now, we substitute the function matrix $\boldsymbol{\Omega}(\boldsymbol{\omega})$, then the following relation is given.

$$\left.\begin{aligned} \boldsymbol{\Omega}^{2k}(\boldsymbol{\omega}) &= (-1)^k \,\|\boldsymbol{\omega}\|^{2k}\, \mathbf{I} \\ \boldsymbol{\Omega}^{2k+1}(\boldsymbol{\omega}) &= (-1)^k \,\|\boldsymbol{\omega}\|^{2k}\, \boldsymbol{\Omega}(\boldsymbol{\omega}) \end{aligned}\right\} \tag{A.5}$$

Substituting Eq.(A.5) into Eq.(A.4) , the following result is obtained:

$$\begin{aligned} \exp\left(\frac{1}{2}\boldsymbol{\Omega}(\boldsymbol{\omega})t\right) &= \mathbf{I}\sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{1}{2}\|\boldsymbol{\omega}\|t\right)^{2k}}{(2k)!} + \|\boldsymbol{\omega}\|^{-1}\boldsymbol{\Omega}(\boldsymbol{\omega})\sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{1}{2}\|\boldsymbol{\omega}\|t\right)^{2k+1}}{(2k+1)!} = \\ &= \mathbf{I}\cos\left(\frac{1}{2}\|\boldsymbol{\omega}\|t\right) + \boldsymbol{\Omega}(\boldsymbol{\omega})\frac{\sin\left(\frac{1}{2}\|\boldsymbol{\omega}\|t\right)}{\|\boldsymbol{\omega}\|} \end{aligned} \tag{A.6}$$

Eq (A.3) is sampled with a sampling period $t \approx \Delta t$.

$$\boldsymbol{q}_{k+1} = \overline{\boldsymbol{\Omega}}(\boldsymbol{\omega}_k)\boldsymbol{q}_k \tag{A.7}$$

Where

$$\overline{\boldsymbol{\Omega}}(\boldsymbol{\omega}_k) = \begin{bmatrix} \cos\left(\frac{1}{2}\|\boldsymbol{\omega}_k\|\Delta t\right)\boldsymbol{I} + [\boldsymbol{\Psi}_k\times] & \boldsymbol{\Psi}_k \\ -\boldsymbol{\Psi}_k^T & \cos\left(\frac{1}{2}\|\boldsymbol{\omega}_k\|\Delta t\right) \end{bmatrix}$$

$$\boldsymbol{\Psi}_k = \frac{\sin\left(\frac{1}{2}\|\boldsymbol{\omega}_k\|\Delta t\right)\boldsymbol{\omega}_k}{\|\boldsymbol{\omega}_k\|}$$

Combining Eq.(A.1) and Eq.(A.7), the discrete equation of satellite attitude is as follows.

$$\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k,\ \mathbf{T}_k,\ \mathbf{w}_k) = \begin{bmatrix} \overline{\boldsymbol{\Omega}}(\boldsymbol{\omega}_k)\mathbf{q}_k \\ \boldsymbol{\omega}_k + \Delta t(-\mathbf{J}^{-1}[\boldsymbol{\omega}_k \times](\mathbf{J}\boldsymbol{\omega}_k) + \mathbf{J}^{-1}\mathbf{T}_k) \end{bmatrix} + \mathbf{w}_k \tag{A.8}$$

Appendix B: UKF algorithm using the new method for selecting sigma points

Step1: Initialize the state and covariance matrix.

$$\hat{\mathbf{x}}_0 = \mathbf{x}(0), \quad \mathbf{P}_0 = \mathbf{P}(0) \tag{B.1}$$

Step2: Calculate the sigma points using the state estimation value and covariance matrix of the previous time. Select the sigma points using theorem 1.

$$\begin{aligned} \chi_{ij} &= \left|a_{ij}\right| + \hat{x}_{k-1,j}\,, & i &= 1,\cdots,n \\ \chi_{ij} &= -\left|a_{(i-n)j}\right| + \hat{x}_{k-1,j}\,, & i &= n+1,\cdots,2n \end{aligned} \tag{B.2}$$

Step3: Substituting the given sigma points into the nonlinear discrete plant model, get the another sigma points that determine the posterior statistic. Then, using them, calculate posterior mean value and covariance matrix.

$$\left.\begin{aligned} \boldsymbol{\chi}_{k/k-1} &= \mathbf{f}(\boldsymbol{\chi}_{k-1},\ \mathbf{u}_{k-1}) \\ \hat{\mathbf{x}}_{k/k-1} &= \frac{1}{2n}\sum_{i=1}^{2n} \boldsymbol{\chi}_{k/k-1,\,i} \\ \mathbf{P}_{k/k-1} &= \frac{1}{2n}\sum_{i=1}^{2n} (\boldsymbol{\chi}_{k|k-1,\,i} - \hat{\mathbf{x}}_{k/k-1})(\boldsymbol{\chi}_{k|k-1,\,i} - \hat{\mathbf{x}}_{k/k-1})^T \end{aligned}\right\} \tag{B.3}$$

Step4: The calculated sigma points disseminate through the nonlinear measured model and using these disseminated sigma points, calculate the measuring prediction value and covariance matrix.

$$\mathbf{y}_{k/k-1} = \mathbf{h}(\boldsymbol{\chi}_{k/k-1}) \tag{B.4}$$

$$\hat{\mathbf{y}}_k = \frac{1}{2n}\sum_{i=0}^{2n} \mathbf{y}_{k/k-1,\,i} \tag{B.5}$$

$$\left.\begin{aligned} \mathbf{P}_y &= \frac{1}{2n}\sum_{i=1}^{2n} (\mathbf{y}_{k/k-1,\,i} - \hat{\mathbf{y}}_k)(\mathbf{y}_{k/k-1,\,i} - \hat{\mathbf{y}}_k)^T \\ \mathbf{P}_{xy} &= \frac{1}{2n}\sum_{i=1}^{2n} (\boldsymbol{\chi}_{k/k-1,\,i} - \hat{\mathbf{x}}_{k/k-1})(\mathbf{y}_{k/k-1,\,i} - \hat{\mathbf{y}}_k)^T \end{aligned}\right\} \tag{B.6}$$

Step 5: Using the calculated covariance matrix, calculate the Kalman gain and update the state variable and covariance matrix.

$$\mathbf{K}_k = \mathbf{P}_{xy}\mathbf{P}_y^{-1} \tag{B.7}$$

$$\left.\begin{aligned} \hat{\mathbf{x}}_k &= \hat{\mathbf{x}}_{k/k-1} + \mathbf{K}_k(\mathbf{y}_k - \hat{\mathbf{y}}_k) \\ \mathbf{P}_k &= \mathbf{P}_{k/k-1} - \mathbf{K}_k\mathbf{P}_y\mathbf{K}_k^T \end{aligned}\right\} \tag{B.8}$$

Step6: Change $k := k - 1$ and repeat the steps from 2 to 5.

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