ANALYSIS ON THE TOPOLOGY OF PROBLEM SPACES

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Abstract. We develop the analysis of the theory of problem and their solution spaces. We adapt some classical concepts in functional analysis to study problems and their corresponding solution spaces. We introduce the notion of **compactness**, **density**, **convexity**, **boundedness**, **amenability** and the **interior**. We examine the overall interplay among these concepts in theory.

1. **Introduction and background**

In [2], [3], [1], we introduced and systematically studied the theory of problems and their corresponding solution spaces.
Let X denotes a solution (resp. answer) to problem Y (resp. question). Then we call the collection of all problems to be solved to provide solution X to problem Y the problem space induced by providing solution X to problem Y. We denote this space with $\mathcal{P}_Y(X)$. If K is any subspace of the space $\mathcal{P}_Y(X)$, then we denote this relation with $K \subseteq \mathcal{P}_Y(X)$. If the space K is a subspace of the space $\mathcal{P}_Y(X)$ with $K \neq \mathcal{P}_Y(X)$, then we write $K \subset \mathcal{P}_Y(X)$. We say problem V is a sub-problem of problem Y if providing a solution to problem Y furnishes a solution to problem V. If V is a sub-problem of the problem Y, then we write $V \leq Y$. If V is a sub-problem of the problem Y and $V \neq Y$, then we write $V < Y$ and we call V a proper sub-problem of Y.
Let $\mathcal{P}_Y(X)$ be the problem space induced by providing the solution X to problem Y. Then we call the number of problems in the space (size) the **complexity** of the space and denote by $\mathbb{C}[\mathcal{P}_Y(X)]$ the complexity of the space. We make the assignment $Z \in \mathcal{P}_Y(X)$ if problem Z is also a problem in this space.
Let X denotes a solution (resp. answer) to problem Y (resp. question). Then we call the collection of all solutions to problems obtained as a result of providing the solution X to problem Y the solution space induced by providing solution X to problem Y. We denote this space with $\mathcal{S}_Y(X)$. If K is any subspace of the space $\mathcal{S}_Y(X)$, then we denote this relation with $K \subset \mathcal{S}_Y(X)$. We make the assignment $T \in \mathcal{S}_Y(X)$ if solution T is also a solution in this space.
Let $\mathcal{S}_Y(X)$ be the solution space induced by providing the solution X to problem Y. Then we call the number of solutions in the space (size) the **index** of the space and denote by $\mathbb{I}[\mathcal{S}_Y(X)]$ the index of this space.
Let Y and V be any two problems. Then we say problem Y is equivalent to problem V if providing solution to problem Y also provides a solution to problem V and conversely providing a solution to problem V also provides a solution to problem Y. We denote the equivalence with $V \equiv Y$.

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Let V be a problem and Y a proper sub-problem of V. Then we say Y is the maximal sub-problem of V if all other proper sub-problems of V are sub-problems of Y. We say it is the minimal sub-problem of V if it is a sub-problem of all other sub-problems of V.
Let $\mathcal{P}_Y(X)$ be a problem space. Then we say $\mathcal{P}_Y(X)$ is separable if and only there exist some $\mathcal{P}_V(U) \subset \mathcal{P}_Y(X)$ and $\mathcal{P}_K(L) \subset \mathcal{P}_Y(X)$ such that

$$\mathcal{P}_V(U) \cup \mathcal{P}_K(L) = \mathcal{P}_Y(X)$$

with

$$\mathcal{P}_V(U) \cap \mathcal{P}_K(L) = \emptyset$$

and $F \not\equiv G$ for any $F \in \mathcal{P}_V(U)$ and $G \in \mathcal{P}_K(L)$. Otherwise, we say the problem space is inseparable. Similarly, we say a solution space $\mathcal{S}_Y(X)$ is separable if and only if there exist some $\mathcal{S}_V(U) \subset \mathcal{S}_Y(X)$ and $\mathcal{S}_K(L) \subset \mathcal{S}_Y(X)$ such that

$$\mathcal{S}_V(U) \cup \mathcal{S}_K(L) = \mathcal{S}_Y(X)$$

with

$$\mathcal{S}_V(U) \cap \mathcal{S}_K(L) = \emptyset$$

and $R \not\perp W$ for any $R \in \mathcal{S}_V(U)$ and $W \in \mathcal{S}_K(L)$. Otherwise, we say the solution space is inseparable.
Let $\mathcal{P}_Y(X), \mathcal{P}_V(U)$ be problem spaces with

$$\mathcal{P}_V(U) \subset \mathcal{P}_Y(X).$$

Then we say the quotient space induced by $\mathcal{P}_V(U)$ in $\mathcal{P}_Y(X)$ regulated by a fixed $T \in \mathcal{P}_Y(X)$, denoted by $\mathcal{P}_Y(X)/_T\mathcal{P}_V(U)$, is the collection of problems

$$\mathcal{P}_Y(X)/_T\mathcal{P}_V(U) := \{T\} \cup \mathcal{P}_V(U).$$

If $\mathcal{P}_Y(X)/_T\mathcal{P}_V(U) := \{T\} \cup \mathcal{P}_V(U) = \mathcal{P}_Y(X)$ for some $T \in \mathcal{P}_Y(X)$ then we say $\mathcal{P}_V(U)$ is a principal subspace of the space $\mathcal{P}_Y(X)$. On the other hand, if $\mathcal{P}_Y(X)/_T\mathcal{P}_V(U) := \{T\} \cup \mathcal{P}_V(U) = \mathcal{P}_V(U)$ for all $T \in \mathcal{P}_Y(X)$ $(T \neq Y)$ then we say $\mathcal{P}_V(U)$ is an ideal sub-space of the problem space $\mathcal{P}_Y(X)$.

2. **Compact problems and solutions**

In this section we study the notion of compactness of problems and their corresponding solutions.

Definition 2.1. Let $\mathcal{P}_X(Y)$ and $\mathcal{S}_X(Y)$ denotes the problem and solutions spaces, respectively, induced by providing solution X to problem Y. We say the problem space $\mathcal{P}_X(Y)$ is *compact* if and only if there exists a finite number of problem spaces $\mathcal{P}_{U_1}(V_1), \mathcal{P}_{U_2}(V_2), \ldots, \mathcal{P}_{U_k}(V_k)$ such that

$$\mathcal{P}_X(Y) \subset \mathcal{P}_{U_1}(V_1) \cup \mathcal{P}_{U_2}(V_2) \cup \cdots \cup \mathcal{P}_{U_k}(V_k).$$

Similarly, we say the solution space $\mathcal{S}_X(Y)$ is *compact* if and only if there exists a finite number of solution spaces $\mathcal{S}_{U_1}(V_1), \mathcal{S}_{U_2}(V_2), \ldots, \mathcal{S}_{U_k}(V_k)$ such that

$$\mathcal{S}_X(Y) \subset \mathcal{S}_{U_1}(V_1) \cup \mathcal{S}_{U_2}(V_2) \cup \cdots \cup \mathcal{S}_{U_k}(V_k).$$

Proposition 2.1. *Let $\mathcal{P}_X(Y)$ be a problem space induced by providing solution Y to problem X. If $\mathcal{P}_X(Y)$ is compact, then the problem space $\mathcal{P}_{X_i}(Y_i)$ with $\mathcal{P}_{X_i}(Y_i)$ is also compact.*

Proof. Suppose $\mathcal{P}_X(Y)$ is compact, then it follows that for a finite $k \in \mathbb{N}$ there exists problems spaces $\mathcal{P}_{U_1}(V_1), \mathcal{P}_{U_2}(V_2), \ldots, \mathcal{P}_{U_k}(V_k)$ such that

$$\mathcal{P}_X(Y) \subset \mathcal{P}_{U_1}(V_1) \cup \mathcal{P}_{U_2}(V_2) \cup \cdots \cup \mathcal{P}_{U_k}(V_k).$$

The compactness of $\mathcal{P}_{X_i}(Y_i)$ follows trivially since $\mathcal{P}_{X_i}(Y_i) \subset \mathcal{P}_X(Y)$. □

Proposition 2.2. *Let $\mathcal{P}_X(Y)$ be the problem space induced by providing solution Y to problem X and let $\mathcal{P}_{X_i}(Y_i) \subset \mathcal{P}_X(Y)$. If $\mathcal{P}_{X_i}(Y_i)$ is compact and principal, then $\mathcal{P}_X(Y)$ is compact.*

Proof. Let $\mathcal{P}_{X_i}(Y_i) \subset \mathcal{P}_X(Y)$ and suppose that $\mathcal{P}_{X_i}(Y_i)$, then there exists a sub-problem $X_j \leq X$ such that we can write $\mathcal{P}_X(Y) = \mathcal{P}_{X_i}(Y_i) \cup \{X_j\}$. Under the requirement that $\mathcal{P}_{X_i}(Y_i)$ is compact, it follows that for a finite $k \in \mathbb{N}$ there exists problems spaces $\mathcal{P}_{U_1}(V_1), \mathcal{P}_{U_2}(V_2), \ldots, \mathcal{P}_{U_k}(V_k)$ such that

$$\mathcal{P}_{X_i}(Y_i) \subset \mathcal{P}_{U_1}(V_1) \cup \mathcal{P}_{U_2}(V_2) \cup \cdots \cup \mathcal{P}_{U_k}(V_k)$$

and we have

$$\mathcal{P}_X(Y) \subset \{X_j\} \cup \mathcal{P}_{U_1}(V_1) \cup \mathcal{P}_{U_2}(V_2) \cup \cdots \cup \mathcal{P}_{U_k}(V_k).$$

This proves that the space $\mathcal{P}_X(Y)$ is also compact. □

Proposition 2.3. *Let $\mathcal{P}_X(Y)$ be the problem space induced by providing solution Y to problem X, where X is a regular problem. If $X_i < X$ is the maximal proper sub-problem of X and $\mathcal{P}_{X_i}(Y_i)$ is compact, then $\mathcal{P}_X(Y)$ is also compact.*

Proof. Suppose X is regular problem and let X_i be the maximal proper sub-problem of X, then we can write $X > X_j > X_{j+1} > \cdots$ where $X_{j+n} > X_{j+n+1}$ indicates that X_{j+n+1} is the maximal proper sub-problem of X_{n+j} for $n = 1, 2, \ldots,$ by virtue of the regularity of the problem X. The sequence above contains all the sub-problems of X so that we can put $\bigcup_{n\geq 1} \mathcal{P}_{X_{j+n}}(Y_{j+n}) \subseteq \mathcal{P}_{X_j}(Y_j)$. Since a problem is solved by providing a solution to each sub-problem and X_j is the maximal problem sub-problem of X, we deduce that $\bigcup_{n\geq 1} \mathcal{P}_{X_{j+n}}(Y_{j+n}) \cup \{X\} \subseteq \mathcal{P}_{X_j}(Y_j) \cup \{X\} = \mathcal{P}_X(Y)$ and it follows that

$$\mathcal{P}_X(Y) \subset \{X\} \cup \mathcal{P}_{U_1}(V_1) \cup \mathcal{P}_{U_2}(V_2) \cup \cdots \cup \mathcal{P}_{U_k}(V_k)$$

since $\mathcal{P}_{X_j}(Y_j)$ was assumed to be compact. This proves that the space $\mathcal{P}_X(Y)$ is compact. □

3. **Dense problems and solution spaces**

We study the concept of *density* of problems and their corresponding solution spaces in this section.

Definition 3.1. Let $\mathcal{P}_X(Y)$ and $\mathcal{S}_X(Y)$ be the problem and solution spaces, respectively, induced by providing solution Y to problem X. Let $X_i \in \mathcal{P}_X(Y)$ with an induced sub-space $\mathcal{P}_{X_i}(Y_i) \subset \mathcal{P}_X(Y)$ and corresponding solution space $\mathcal{S}_{X_i}(Y_i)$. We say the subspace $\mathcal{P}_{X_i}(Y_i)$ is *dense* in the space $\mathcal{P}_X(Y)$ if and only if for any problem $Z \in \mathcal{P}_X(Y)$ with $Z \neq X$, there exists a proper subspace $\mathcal{P}_{X_j}(Y_j)$ with $Z \in \mathcal{P}_{X_j}(Y_j)$ such that $\mathcal{P}_{X_i}(Y_i) \cap \mathcal{P}_{X_j}(Y_j) \neq \emptyset$. Similarly, we say the subspace $\mathcal{S}_{X_i}(Y_i)$ is *dense* in the space $\mathcal{S}_X(Y)$ if and only if for any solution $W \in \mathcal{P}_X(Y)$ with $W \neq Y$, there exists a proper subspace $\mathcal{S}_{X_j}(Y_j)$ with $W \in \mathcal{S}_{X_j}(Y_j)$ such that $\mathcal{S}_{X_i}(Y_i) \cap \mathcal{S}_{X_j}(Y_j) \neq \emptyset$.

Theorem 3.2 (Characterization theorem)**.** *Let $\mathcal{P}_X(Y)$ be the solution space induced by providing solution Y to problem X. Then $\mathcal{P}_X(Y)$ is separable if and only if it contains no dense subspace.*

Proof. Suppose the problem space $\mathcal{P}_X(Y)$ is separable, then there exists subspaces $\mathcal{P}_{X_i}(Y_i)$ and $\mathcal{P}_{X_j}(Y_j)$ such that $\mathcal{P}_X(Y) = \mathcal{P}_{X_i}(Y_i) \cup \mathcal{P}_{X_j}(Y_j)$ with $\mathcal{P}_{X_i}(Y_i) \cap \mathcal{P}_{X_j}(Y_j) = \emptyset$. Now let $\mathcal{P}_{X_k}(Y_k) \subset \mathcal{P}_X(Y)$ then we must have one of these possibilities: $\mathcal{P}_{X_k}(Y_k) \subset \mathcal{P}_{X_i}(Y_i)$ or $\mathcal{P}_{X_k}(Y_k) \subset \mathcal{P}_{X_i}(Y_i)$. Suppose there exist problems $Z, U \in \mathcal{P}_{X_k}(Y_k)$ such that $Z \in \mathcal{P}_{X_i}(Y_i)$ and $U \in \mathcal{P}_{X_j}(Y_j)$, then we have for their corresponding problem spaces induced with, say, the solutions W and T the following properties $\mathcal{P}_Z(W) \subset \mathcal{P}_{X_i}(Y_i)$ and $\mathcal{P}_U(T) \subset \mathcal{P}_{X_j}(Y_j)$. We know that $\mathcal{P}_U(T) \subseteq \mathcal{P}_{X_k}(Y_k)$ and $\mathcal{P}_Z(W) \subseteq \mathcal{P}_{X_k}(Y_k)$ so that we must have $\mathcal{P}_{X_k}(Y_k) \subseteq \mathcal{P}_{X_i}(Y_i)$ and $\mathcal{P}_{X_k}(Y_k) \subseteq \mathcal{P}_{X_j}(Y_j)$. Suppose without loss of generality that $\mathcal{P}_{X_i}(Y_i) \subset \mathcal{P}_{X_k}(Y_k)$ then we will have

$$\mathcal{P}_{X_i}(Y_i) \cup \mathcal{P}_{X_j}(Y_j) = \mathcal{P}_X(Y) \subset \mathcal{P}_{X_k}(Y_k) \cup \mathcal{P}_{X_j}(Y_j) \subset \mathcal{P}_X(Y)$$

which is absurd. This implies that $\mathcal{P}_{X_i}(Y_i) \cap \mathcal{P}_{X_j}(Y_j) \neq \emptyset$, which violates the requirement that $\mathcal{P}_X(Y)$ is separable. Without loss of generality, we put $\mathcal{P}_{X_k}(Y_k) \subseteq \mathcal{P}_{X_i}(Y_i)$ and choose a problem $V \in \mathcal{P}_{X_j}(Y_j)$ then $\mathcal{P}_V(T) \subseteq \mathcal{P}_{X_j}(Y_j)$. It follows that $\mathcal{P}_{X_k}(Y_k) \cap \mathcal{P}_V(T) = \emptyset$ and since $V \notin \mathcal{P}_{X_l}(Y_l) \subseteq \mathcal{P}_{X_i}(Y_i)$ for subspace $\mathcal{P}_{X_l}(Y_l)$ of $\mathcal{P}_{X_i}(Y_i)$, the problem space $\mathcal{P}_{X_k}(Y_k)$ cannot be dense in $\mathcal{P}(X)(Y)$. Since $\mathcal{P}_{X_k}(Y_k)$ was an arbitrary problem subspace, it follows that the space $\mathcal{P}_X(Y)$ contains no dense sub-problem space. Conversely, suppose that the space $\mathcal{P}_X(Y)$ contains a dense problem sub-space but that the space is separable, then there exists proper sub-spaces $\mathcal{P}_{X_i}(Y_i)$ and $\mathcal{P}_{X_j}(Y_j)$ such that $\mathcal{P}_{X_i}(Y_i) \cup \mathcal{P}_{X_j}(X_j) = \mathcal{P}_X(Y)$ such that $\mathcal{P}_{X_i}(Y_i) \cap \mathcal{P}_{X_j}(Y_j) = \emptyset$. Let $\mathcal{P}_{X_k}(Y_k) \subset \mathcal{P}_X(Y)$ be dense in $\mathcal{P}_X(Y)$ then for $V \in \mathcal{P}_{X_i}(Y_i)$ and $U \in \mathcal{P}_{X_j}(Y_j)$. Since these subspaces are the largest subspaces in the space $\mathcal{P}_X(Y)$ containing the problems V and U, it follows by the density of the subspace $\mathcal{P}_{X_k}(Y_k)$ that $\mathcal{P}_{X_k}(Y_k) \cap \mathcal{P}_{X_i}(Y_i) \neq \emptyset$ and $\mathcal{P}_{X_k}(Y_k) \cap \mathcal{P}_{X_j}(Y_j) \neq \emptyset$. This contradicts the assumption that the space $\mathcal{P}_X(Y)$ is separable. □

4. **Bounded problem and solution spaces**

In this section we study the notion of *bounded* problem and solution spaces.

Definition 4.1. Let $\mathcal{P}_X(Y)$ be a problem space induced by providing solution Y to problem X. We say the space $\mathcal{P}_X(Y)$ is bounded if and only if it has finite complexity. If we denote the complexity of the space with $\mathbb{C}[\mathcal{P}_X(Y)]$, then we say $\mathcal{P}_X(Y)$ is bounded if and only if $\mathbb{C}[\mathcal{P}_X(Y)] < \infty$. Similarly, we say the corresponding solution space $\mathcal{S}_X(Y)$ is bounded if only if it has a finite index. If we denote the index of this space with $\mathbb{I}[\mathcal{S}_X(Y)]$, then $\mathcal{S}_X(Y)$ is bounded if and only if $\mathbb{I}[\mathcal{S}_X(Y)] < \infty$.

Proposition 4.1. *Let $\mathcal{P}_X(Y)$ be the problem space induced by providing solution Y to problem X. If $\mathbb{C}[\mathcal{P}_X(Y)] < \infty$, then $\mathcal{P}_X(Y)$ contains a reducible problem.*

Proof. Suppose each problem $X_i \in \mathcal{P}_X(Y)$ is irreducible, then we can construct the infinite nested sequence of sub-problem spaces $\cdots \subset \mathcal{P}_{X_2}(Y_2) \subset \mathcal{P}_{X_1}(Y_1) \subset \mathcal{P}_X(Y)$ with $X_1 > X_2 > \cdots$, where $X_{j+1} < X_j$ indicates that X_{j+1} is a proper sub-problem of X_j. This implies that the space $\mathcal{P}_X(Y)$ contains infinitely many problems and thus $\mathbb{C}[\mathcal{P}_X(Y)] = \infty$. □

5. **The interior of problem and solution spaces**

In this section we study the topological notion of *interior* of problem and solution spaces.

Definition 5.1. Let $\mathcal{P}_X(Y)$ and $\mathcal{S}_X(Y)$ be the problem and the solutions spaces induced by providing solution Y to problem X. We say a problem $Z \in \mathcal{P}_X(Y)$ is an *interior* problem if there is no problem space $\mathcal{P}_S(T)$ with $\mathcal{P}_S(T) \not\subseteq \mathcal{P}_X(Y)$ such that $Z \in \mathcal{P}_S(T)$. We call the collection of all such problems in $\mathcal{P}_X(Y)$ the interior of $\mathcal{P}_X(Y)$ and denote for this collection $Int[\mathcal{P}_X(Y)]$. We say the interior is non-empty if $Int[\mathcal{P}_X(Y)] \neq \emptyset$; otherwise, we say the interior is empty. Similarly, we say a solution $W \in \mathcal{S}_X(Y)$ is an *interior* solution if there is no solution space $\mathcal{S}_R(T)$ with $\mathcal{S}_R(T) \not\subseteq \mathcal{S}_X(Y)$ such that $W \in \mathcal{P}_S(T)$. We call the collection of all such solutions in $\mathcal{S}_X(Y)$ the interior of $\mathcal{S}_X(Y)$ and denote for this collection $Int[\mathcal{S}_X(Y)]$. We say the interior is non-empty if $Int[\mathcal{S}_X(Y)] \neq \emptyset$; otherwise, we say the interior is empty.

Theorem 5.2. *Let $\mathcal{P}_X(Y)$ be the problem space induced by providing solution Y to problem X. If $Int[\mathcal{P}_X(Y)] = \emptyset$ and $\mathbb{C}[\mathcal{P}_X(Y)] < \infty$, then $\mathcal{P}_X(Y)$ is compact.*

Proof. Suppose $\mathbb{C}[\mathcal{P}_X(Y)] < \infty$, then $\mathcal{P}_X(Y) = \{X, X_1, \ldots, X_k\}$ for a finite $k \in \mathbb{N}$. Since $Int[\mathcal{P}_X(Y)] = \emptyset$, it follows that there exists problem spaces $\mathcal{P}_{T_1}(R_1), \ldots, \mathcal{P}_{T_k}(R_k)$ with $\mathcal{P}_{T_i}(R_i) \not\subseteq \mathcal{P}_X(Y)$ for $i = 1, \ldots, k$ such that $X_i \in \mathcal{P}_{T_i}(R_i)$ for each i. It follows that we can put $\mathcal{P}_X(Y) \subset \bigcup\limits_{i=1}^{k} \mathcal{P}_{T_i}(R_i) \cup \{X\}$. This proves that the problem space $\mathcal{P}_X(Y)$ is compact. □

6. **Convex problem and solution spaces**

We introduce and study the notion of *convexity* of problems and solution spaces in this section.

Definition 6.1. Let $\mathcal{P}_X(Y)$ be the problem space induced by providing solution Y to problem X. We say the space $\mathcal{P}_X(Y)$ is *convex* if for any problem $X_i, X_j \in \mathcal{P}_X(Y)$ $(X_i, X_j \neq X)$, there exist a problem $X_k \in \mathcal{P}_X(Y)$ such that $\{X_i\} \cup \{X_j\} = \{X_k\}$. Similarly, We say the solution space $\mathcal{S}_X(Y)$ is *convex* if for any solution $Y_i, Y_j \in \mathcal{S}_X(Y)$ $(Y_i, Y_j \neq Y)$, there exist a solution $Y_k \in \mathcal{S}_X(Y)$ such that $\{Y_i\} \cup \{Y_j\} = \{Y_k\}$.

The notion of *convexity* of a problem (resp. solution) spaces suggest that each problem in the *convex* problem space is a sub-problem of some problem in the space. It worth noting that convexity of problem and solutions do not unconditionally extend to *convexity* of sub-problem spaces.

Proposition 6.1. *Let $\mathcal{P}_X(Y)$ be the problem space induced by providing solution Y to problem X. If $\mathcal{P}_X(Y)$ is convex and bounded with $\mathbb{C}[\mathcal{P}_X(Y)] \geq 4$, then $\mathcal{P}_X(Y)$ has a principal subspace $\mathcal{P}_{X_k}(Y_k)$ with $\mathbb{C}[\mathcal{P}_{X_k}(Y_k)] \geq 3$.*

Proof. Suppose $\mathcal{P}_X(Y)$ is bounded, then $\mathbb{C}[\mathcal{P}_X(Y)] < \infty$ so that $\mathcal{P}_X(Y)$ contains finitely many problems. Let $X_i, X_j \in \mathcal{P}_X(Y)$ then under the requirement that $\mathcal{P}_X(Y)$ is *convex*, then $\{X_i\} \cup \{X_j\} = \{X_k\}$, where $X_k \in \mathcal{P}_X(Y)$. That is, we can merge to problems in the space to produce another problem in the space. It follows that $X_i \leq X_k$ and $X_j \leq X_k$. That is, X_i and X_j are sub-problems of

X_k. By the minimality of the complexity of the space $\mathbb{C}[\mathcal{P}_X(Y)] \geq 4$, we can repeat this construction by using the newly constructed problems X_k with some $X_s \in \mathcal{P}_X(Y)$ with $X_s \neq X_i, X_j$ to produce a sub-problem space which is principal and has complexity ≥ 3. □

The next result purports that each subspace of a problem space must be *dense* in their mother space.

Theorem 6.2. *Let $\mathcal{P}_X(Y)$ be the problem space induced by providing solution Y to problem X. If $\mathcal{P}_X(Y)$ is convex then every subspace $\mathcal{P}_{X_i}(Y_i) \subset \mathcal{P}_X(Y)$ is dense in $\mathcal{P}_X(Y)$.*

Proof. Suppose the problem space $\mathcal{P}_X(Y)$ is *convex* and put $\mathcal{P}_{X_i}(Y_i) \subset \mathcal{P}_X(Y)$. Next pick a arbitrarily a problem $V \in \mathcal{P}_X(Y)$, then under the *convexity* of the space there exists a problem $W \in \mathcal{P}_X(Y)$ such that $\{X_i\} \cup \{V\} = \{W\}$. This implies that $X_i < W$ and $V < W$; that is, X_i and V are proper sub-problems of W. Since $W \in \mathcal{P}_X(Y)$, it has a solution so let $T \in \mathcal{S}_X(Y)$ be the solution to W and we obtain the induced problem space $\mathcal{P}_W(T) \subset \mathcal{P}_X(Y)$ with $V \in \mathcal{P}_W(T)$. Because $X_i < W$ and is the maximal sub-problem in the space $\mathcal{P}_{X_i}(Y_i)$, it follows that $\mathcal{P}_{X_i}(Y_i) \subset \mathcal{P}_W(T)$. We find that $\mathcal{P}_{X_i}(Y_i) \cap \mathcal{P}_W(T) \neq \emptyset$ with $V \in \mathcal{P}_W(T)$. Since V was chosen arbitrarily in the space $\mathcal{P}_X(Y)$, it follows that $\mathcal{P}_{X_i}(Y_i)$ is *dense* in $\mathcal{P}_X(Y)$. Because the sub-problem space was chosen arbitrarily, it follows that each sub-problem space is dense problem space $\mathcal{P}_X(Y)$. This completes the proof of the claim. □

7. Amenable problem spaces

In this section, we study the notion of *amenability* of problem spaces.

Definition 7.1. Let $\mathcal{P}_X(Y)$ be the problem space induced by providing solution solution Y to problem X. We say the problem space $\mathcal{P}_X(Y)$ is partially *amenable* if there exist proper sub-problem $X_i, X_j \in \mathcal{P}_X(Y)$ such that X_i and X_j are equivalent problems ($X_i \equiv X_j$). We say the space $\mathcal{P}_X(Y)$ is totally *amenable* if for any sub-problem $X_i, X_j \in \mathcal{P}_X(Y)$ then $X_i \equiv X_j$. We say a problem is amenable if it is a problem in some totally *amenable* problem space.

Amenable problems are naturally easily tractable. This notion hold much significance, because if we can identify some totally amenable space that contains a specific problem then finding a solution will reduce to finding a solution to much easier problem in the same space. Subsequent studies will be devoted to a detail and much more specialized study of this important concept and its overall interplay with the theory. Next we launch a result that basically purports the compactness of a space provided one can identify a compact sub-problem space.

Theorem 7.2. *Let $\mathcal{P}_X(Y)$ be a totally amenable problem space. If there exists a sub-problem space $\mathcal{P}_{X_i}(Y_i)$ such that $\mathcal{P}_{X_i}(Y_i)$ is compact, then $\mathcal{P}_X(Y)$ is compact.*

Proof. Put $\mathcal{P}_{X_i}(Y_i) \subset \mathcal{P}_X(Y)$ and suppose $\mathcal{P}_X(Y)$ is an amenable space. This implies that for any problem $X_j \in \mathcal{P}_X(Y)$ then $X_j \equiv X_i$. The induced problem space $\mathcal{P}_{X_j}(Y_j)$ contains the problem X_j and it is the maximal sub-problem of this space. Since $\mathcal{P}_{X_j}(Y_j) \subset \mathcal{P}_X(Y)$, it follows by amenability of the space that we can replace X_j with X_i and Y_j with Y_i, since problem and solution spaces remain invariant on replacement with equivalent problems and alternative solutions, so that under the requirement that $\mathcal{P}_{X_i}(Y_i)$ is compact, we can put

$$\mathcal{P}_{X_j}(Y_j) = \mathcal{P}_{X_i}(Y_i) \subset \bigcup_{s=1}^{k} \mathcal{P}_{S_s}(T_s)$$

for a fixed $k \in \mathbb{N}$. It follows that

$$\bigcup_{i \geq 1} \mathcal{P}_{X_i}(Y_i) \cup \{X\} = \mathcal{P}_X(Y) \subset \bigcup_{s=1}^{k} \mathcal{P}_{S_s}(T_s) \cup \{X\}$$

for a fixed $k \in \mathbb{N}$. This proves that the problem space $\mathcal{P}_X(Y)$ is compact. □

8. **Further discussions**

The current study introduces and studies - in a carefully adaptive manner - some fundamental topological concepts. Although there are some slight variations of these interpretations and meaning of these concepts in functional analysis, most of the notions carry over to the theory of problems and solution spaces. It is our next goal to study various maps between problem and solution spaces spaces and corresponding analogue of a norm - perhaps with a different terminology - in our subsequent studies. The developments of this theory is a long-term endeavour to study the *P vs NP* problem in computer science by laying down a rigorous foundation for future work. We end this discussion by stating a claim which can be easily verified.

Proposition 8.1. *Let $\mathcal{P}_X(Y)$ and $\mathcal{S}_X(Y)$ be the problem and the solution space induced by providing solution Y to problem X. Then the following assertions hold*

(i) *The problem space $\mathcal{P}_X(Y)$ is bounded if and only if the solution space $\mathcal{S}_X(Y)$ is bounded.*

(ii) *The problem space $\mathcal{P}_X(Y)$ is compact if and only if the solution space $\mathcal{S}_X(Y)$ is compact.*

(iii) *A sub-problem space $\mathcal{P}_{X_i}(Y_i) \subset \mathcal{P}_X(Y)$ is dense in $\mathcal{P}_X(Y)$ if and only if the sub-solution space $\mathcal{S}_{X_i}(Y_i) \subset \mathcal{S}_X(Y)$ is dense in $\mathcal{S}_X(Y)$.*

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