On Born Reciprocal Relativity Theory, The Relativistic Oscillator and the Fulling-Davies-Unruh effect

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Abstract

A continuation of the Born Reciprocal Relativity Theory (BRRT) program in phase space shows that a natural temperature-dependence of mass occurs after recurring to the Fulling-Davies-Unruh effect. The temperature dependence of the mass $m(T)$ resemblances the energy-scale dependence of mass and other physical parameters in the renormalization (group) program of QFT. It is found in a special case that the effective photon mass is no longer zero, which may have far reaching consequences in the resolution of the dark matter problem. The Fulling-Davies-Unruh effect in a $D = 1 + 1$-dim spacetime is analyzed entirely from the perspective of BRRT, and we explain how it may be interpreted in terms of a linear superposition of an infinite number of states resulting from the action of the group $U(1,1)$ on the Lorentz non-invariant vacuum $|\tilde{0}\rangle$ of the relativistic oscillator studied by Bars [8].

1 Novel findings in Born Reciprocal Relativity Theory

Most of the work devoted to Quantum Gravity has been focused on the geometry of spacetime rather than phase space per se. The first indication that phase space should play a role in Quantum Gravity was raised by [1]. The principle behind Born's reciprocal relativity theory [3], [5] was based on the idea proposed long ago by [1] that coordinates and momenta should be unified on the same footing. Consequently, if there is a limiting speed (temporal derivative of the position coordinates) in Nature there should be a maximal force as well, since force is the temporal derivative of the momentum. The principle of maximal acceleration was advocated earlier on by [2]. A *maximal* speed limit (speed of light) must be accompanied with a *maximal* proper force (which is also compatible with a *maximal* and *minimal* length duality) [5].

We explored in [5] some novel consequences of Born's reciprocal Relativity theory in flat phase-space and generalized the theory to the curved spacetime scenario. We provided, in particular, some specific results resulting from Born's reciprocal Relativity and which are *not* present in Special Relativity. These are : momentum-dependent time delay in the emission and detection of photons; relativity of chronology; energy-dependent notion of locality; superluminal behavior; relative rotation of photon trajectories due to the aberration of light; invariance of areas-cells in phase-space and modified dispersion relations.

The generalized velocity and force (acceleration) boosts (rotations) transformations of the $flat$ $8D$ Phase space coordinates , where $X^i, T, E, P^i; i = 1, 2, 3$ are **c** -valued (classical) variables which are *all* boosted (rotated) into each-other, were given by [3] based on the group $U(1,3)$ and which is the Born version of the Lorentz group $SO(1,3)$. The $U(1,3) = SU(1,3) \times U(1)$ group transformations leave invariant the symplectic 2-form $\Omega = - \, dT \wedge dE + \delta_{ij} dX^i \wedge dP^j; i, j = 1, 2, 3$ and also the following Born-Green line interval in the $flat$ $8D$ phase-space

$$(d\omega)^2 \ = \ c^2 (dT)^2 \ - \ (dX)^2 \ - \ (dY)^2 \ - \ (dZ)^2 \ +$$

$$\frac{1}{b^2} \ \left((dE)^2 \ - \ c^2 (dP_x)^2 \ - \ c^2 (dP_y)^2 \ - \ c^2 (dP_z)^2 \right) \qquad (1.1)$$

The maximal proper force is set to be given by b. The symplectic group is relevant because $U(1,3) = Sp(8,R) \cap O(2,6)$; $U(3,1) = Sp(8,R) \cap O(6,2)$, and $U(2,2) = Sp(8,R) \cap O(4,4)$.

The generators Z_{ab} of the $U(1,3)$ algebra can be decomposed into the Lorentz sub-algebra generators $L_{[ab]}$ and the "shear"-like generators $M_{(ab)}$ as

$$Z_{ab} \equiv \frac{1}{2} \left(M_{(ab)} + L_{[ab]} \right) \ \Rightarrow \ L_{ab} \equiv L_{[ab]} \ = \ (Z_{ab} - Z_{ba}); \ \ M_{ab} \equiv M_{(ab)} \ = \ (Z_{ab} + Z_{ba}), \qquad (1.2)$$

the "shear"-like generators $M_{(ab)}$ and the Lorentz generators $L_{[ab]}$ are Hermitian. The explicit commutation relations of the M_{ab}, L_{ab} generators is given by

$$[L_{ab}, \ L_{cd}] \ = \ i \ (\eta_{bc} L_{ad} - \eta_{ac} L_{bd} - \eta_{bd} L_{ac} + \eta_{ad} L_{bc}). \qquad (1.3a)$$

$$[M_{ab}, \ M_{cd}] \ = \ - \, i \ (\eta_{bc} L_{ad} + \eta_{ac} L_{bd} + \eta_{bd} L_{ac} + \eta_{ad} L_{bc}). \qquad (1.3b)$$

$$[L_{ab}, \ M_{cd}] \ = \ i \ (\eta_{bc} M_{ad} - \eta_{ac} M_{bd} + \eta_{bd} M_{ac} - \eta_{ad} M_{bc}). \qquad (1.3c)$$

Therefore, given $Z_{ab} = \frac{1}{2}(M_{ab} + L_{ab}), \ Z_{cd} = \frac{1}{2}(M_{cd} + L_{cd})$ after straightforward algebra it leads to the $U(1,3)$ commutators

$$[\ Z_{ab}, \ Z_{cd} \ ] \ = \ - \, i \ (\ \eta_{bc} \ Z_{ad} \ - \ \eta_{ad} \ Z_{cb} \ ). \qquad (1.3d)$$

as expected.

The commutators of the Lorentz boosts generators L_{ab} and X_c, P_c are of the form

$$[L_{ab},\ X_c]\ =\ i\ (\ \eta_{bc}\ X_a\ -\ \eta_{ac}\ X_b\ );\quad [L_{ab},\ P_c]\ =\ i\ (\ \eta_{bc}\ P_a\ -\ \eta_{ac}\ P_b\ )\quad (1.4)$$

The Hermitian M_{ab} generators are the "reciprocal" boosts/rotation transformations which *exchange* X for P, in addition to boosting (rotating) those variables, and one ends up with the commutators of M_{ab} and X_c, P_c given by

$$[M_{ab},\ \frac{X_c}{\lambda_l}]\ =\ -\ \frac{i}{\lambda_p}\ (\ \eta_{bc}\ P_a + \eta_{ac}\ P_b\ );\quad [M_{ab},\ \frac{P_c}{\lambda_p}]\ =\ -\frac{i}{\lambda_l}\ (\ \eta_{bc}\ X_a + \eta_{ac}\ X_b\ )\quad (1.5)$$

where λ_l, λ_p are suitable length and momentum scales, for instance the Compton wavelength and momentum associated to a particle of proper mass m.

The rotations, velocity and force (acceleration) boosts leaving invariant the symplectic 2-form and the line interval in the $8D$ phase-space are rather elaborate. In four spacetime dimensions the velocity-boosts generators along the x_i spatial directions $(i = 1, 2, 3)$ are given by $K_i = L_{0i}$. The force-boots (acceleration boosts) generators along the x_i spatial directions are given by $N_i = M_{0i}$. The rotation generators are $J_i = \epsilon_i^{jk} L_{jk}$. The shear generators are M_{ij}, M_{00}. In general, given the $U(1,3)$ generator $Z = \frac{1}{2}\theta^{AB} Z_{AB}$ the transformations of $\mathbf{X} = (T, X_i); \mathbf{P} = (E, P_i)$ are given by

$$\mathbf{X}'\ =\ e^{\frac{1}{2}\theta^{AB} Z_{AB}}\ \mathbf{X}\ e^{-\frac{1}{2}\theta^{AB} Z_{AB}},\quad \mathbf{P}'\ =\ e^{\frac{1}{2}\theta^{AB} Z_{AB}}\ \mathbf{P}\ e^{-\frac{1}{2}\theta^{AB} Z_{AB}}\qquad (1.6)$$

leading to

$$\mathbf{X}'\ =\ \mathbf{X}\ +\ [Z,\ \mathbf{X}]\ +\frac{1}{2!}\ [Z,\ [Z,\ \mathbf{X}]]\ +\frac{1}{3!}\ [Z,\ [Z,\ [Z,\ \mathbf{X}]]]\ +\ \ldots\qquad (1.7)$$

and a similar relation for $\mathbf{P}'$ in terms of the nested commutators.

By recurring to the commutation relations (1.5) and the nested commutators in eq-(1.7), one finds that the group transformations of the 8-dim phase space coordinates involving both velocity and force boosts are given by [3] (page 18)

$$T'\ =\ T\ cosh\xi\ +\ (\frac{\xi_v^i\ X_i}{c}\ +\ \frac{\xi_a^i\ P_i}{b})\ \frac{sinh\xi}{\xi}\qquad (1.8a)$$

$$E'\ =\ E\ cosh\xi\ +\ (-b\ \xi_a^i\ X_i\ +\ c\ \xi_v^i\ P_i)\ \frac{sinh\xi}{\xi}\qquad (1.8b)$$

$$X'^i\ =\ X^i + (cosh\xi\ -1)\ \frac{(\xi_v^i\ \xi_v^j\ +\ \xi_a^i\ \xi_a^j)\ X_j}{\xi^2} + (c\ \xi_v^i\ T - \frac{\xi_a^i\ E}{b})\ \frac{sinh\xi}{\xi}\quad (1.8c)$$

$$P'^i \;=\; P^i + (cosh\xi\;-1)\;\frac{(\xi^i_v\;\xi^j_v\;+\;\xi^i_a\;\xi^j_a)\;P_j}{\xi^2} + (b\;\xi^i_a\;T + \frac{\xi^i_v\;E}{c})\;\frac{sinh\xi}{\xi} \quad (1.8d)$$

where ξ^i_v are the velocity-boost rapidity parameters along the e_i directions; ξ^i_a are the force (acceleration) boost rapidity parameters along the e_i directions, $i = 1, 2, 3$, and ξ is the *net* effective rapidity parameter of the primed-reference frame given by

$$\xi = \sqrt{(\xi^i_v)^2 + (\xi^i_a)^2}, \quad i = 1, 2, 3 \quad (1.9)$$

A straightforward way of understanding how one obtains the above transformations of eqs-(1.8) can be found by simply recalling the most general (Lorentz) velocity boosts transformations of the spacetime coordinates after splitting the vectors $\vec{X}, \vec{P}$ into the parallel $\vec{X}_\parallel$ and transverse $\vec{X}_\perp$ components with respect to the velocity boost parameter $\vec{\xi} = (\xi_1, \xi_2, \xi_3); \xi = \sqrt{(\xi_1)^2 + (\xi_2)^2 + (\xi_3)^2}$. Such decomposition is of the form

$$\vec{X}_\parallel \;=\; (\vec{X}\cdot\vec{\xi})\;\frac{\vec{\xi}}{\xi^2},\;\; \vec{X}_\perp \;=\; \vec{X} \;-\; \vec{X}_\parallel \;=\; \vec{X} \;-\; (\vec{X}\cdot\vec{\xi})\;\frac{\vec{\xi}}{\xi^2} \quad (1.10)$$

$$\vec{P}_\parallel \;=\; (\vec{P}\cdot\vec{\xi})\;\frac{\vec{\xi}}{\xi^2},\;\; \vec{P}_\perp \;=\; \vec{P} \;-\; \vec{P}_\parallel \;=\; \vec{P} \;-\; (\vec{P}\cdot\vec{\xi})\;\frac{\vec{\xi}}{\xi^2} \quad (1.11)$$

so that the Lorentz transformations of $\vec{X}, \vec{P}$ can be written in vector form as

$$\vec{X}' \;=\; \left(\vec{X} \;-\; (\vec{X}\cdot\vec{\xi})\;\frac{\vec{\xi}}{\xi^2}\right) \;+\; (\vec{X}\cdot\vec{\xi})\;\frac{\vec{\xi}}{\xi^2}\;cosh\xi \;+\; \frac{c\;T\;sinh\xi}{\xi}\;\vec{\xi} \quad (1.12)$$

$$\vec{P}' \;=\; \left(\vec{P} \;-\; (\vec{P}\cdot\vec{\xi})\;\frac{\vec{\xi}}{\xi^2}\right) \;+\; (\vec{P}\cdot\vec{\xi})\;\frac{\vec{\xi}}{\xi^2}\;cosh\xi \;+\; \frac{E\;sinh\xi}{c\;\xi}\;\vec{\xi} \quad (1.13)$$

where the modulus $\xi = |\vec{\xi}|$ of the velocity boost parameters, and the modulus $|\vec{v}|$ of the velocity $\vec{v}$ of the moving frame of reference are related by $tanh(\xi) = \beta = \frac{\sqrt{v_1^2+v_2^2+v_3^2}}{c}$. One then finds that the transverse directions to the velocity remain unaffected by the Lorentz transformations, while the parallel directions are. One can see by simple inspection that by setting the force-boost parameters to zero $\xi^i_a = 0$ in eqs-(1.8), one recovers the standard Lorentz transformations.

These transformations can be *simplified* drastically when the velocity and force (acceleration) boosts are both parallel to the x-direction and leave the transverse directions Y, Z, P_y, P_z intact. There is now a subgroup $U(1,1) = SU(1,1) \times U(1) \subset U(1,3)$ which leaves invariant the following line interval

$$(d\omega)^2 \;=\; c^2(dT)^2 \;-\; (dX)^2 \;+\; \frac{(dE)^2 \;-\; c^2(dP)^2}{b^2} =$$

$$(d\tau)^2 \left(1 + \frac{(dE/d\tau)^2 - c^2(dP/d\tau)^2}{b^2} \right) = (d\tau)^2 \left(1 - \frac{F^2}{F^2_{max}} \right), \quad P = P_x \tag{1.14}$$

where one has factored out the proper time infinitesimal $(d\tau)^2 = c^2 dT^2 - dX^2$ in (1.2). The proper force interval $(dE/d\tau)^2 - c^2(dP/d\tau)^2 = -F^2 < 0$ is "spacelike" when the proper velocity interval $c^2(dT/d\tau)^2 - (dX/d\tau)^2 > 0$ is timelike. The analog of the Lorentz relativistic factor in eq-(1.14) involves the ratios of two proper *forces*.

One may set the maximal proper-force acting on a fundamental particle of Planck mass to be given by $F_{max} = b \equiv m_P c^2/L_P$, where m_P is the Planck mass and L_P is the postulated minimal Planck length. Invoking a minimal/maximal length duality one can also set $b = M_U c^2/R_H$, where R_H is the Hubble scale and M_U is the observable mass of the universe. Equating both expressions for b leads to $M_U/m_P = R_H/L_P \sim 10^{60}$. The value of b may also be interpreted as the maximal string tension.

The $U(1,1)$ group transformations involving the velocity and force boosts along the X direction of the phase-space coordinates X, T, P, E which leave the interval (1.14) invariant are obtained directly from eqs-(1.8) in this special case as follows

$$T' = T\ cosh\xi + (\frac{\xi_v\ X}{c} + \frac{\xi_a\ P}{b})\ \frac{sinh\xi}{\xi} \tag{1.15a}$$

$$E' = E\ cosh\xi + (-b\ \xi_a\ X + c\ \xi_v P)\ \frac{sinh\xi}{\xi} \tag{1.15b}$$

$$X' = X\ cosh\xi + (c\ \xi_v\ T - \frac{\xi_a\ E}{b})\ \frac{sinh\xi}{\xi} \tag{1.15c}$$

$$P' = P\ cosh\xi + (\frac{\xi_v\ E}{c} + b\ \xi_a\ T)\ \frac{sinh\xi}{\xi} \tag{1.15db}$$

ξ_v is the velocity-boost rapidity parameter; ξ_a is the force (acceleration) boost rapidity parameter, and ξ is the net effective rapidity parameter of the primed-reference frame. These parameters ξ_a, ξ_v, ξ are defined respectively in terms of the velocity $v = dX/dT$ and force $f = dP/dT$ (related to acceleration) as

$$tanh(\xi_v) = \frac{v}{c}; \quad tanh(\xi_a) = \frac{F}{F_{max}}, \quad \xi = \sqrt{(\xi_v)^2 + (\xi_a)^2} \tag{1.16}$$

It is straight-forward to verify that the transformations (1.15) leave invariant the phase space interval $c^2(dT)^2 - (dX)^2 + ((dE)^2 - c^2(dP)^2)/b^2$ but *do not* leave separately invariant the proper time interval $(d\tau)^2 = c^2 dT^2 - dX^2$, nor the interval in energy-momentum space $\frac{1}{b^2}[(dE)^2 - (dP)^2]$. Only the *combination*

$$(d\omega)^2 = (d\tau)^2 \left(1 - \frac{F^2}{F^2_{max}} \right) \tag{1.17}$$

is truly left invariant under force (acceleration) boosts. They also leave invariant the symplectic 2-form (phase space areas) $\Omega = -\ dT \wedge dE + dX \wedge dP$.

A physical picture of how velocity-boosts and force-boosts transformations operate on reference frames is best captured if one focus on the full phase-space configuration. Since $d\tau \neq d\omega$, one has $F^\mu = \frac{dp^\mu}{d\tau}; F = \sqrt{|F_\mu F^\mu|}$, and $\mathcal{F}^\mu = \frac{dp^\mu}{d\omega}; \mathcal{F} = \sqrt{|\mathcal{F}_\mu \mathcal{F}^\mu|}$, such that

$$\mathcal{F} = \frac{F}{\sqrt{1 - F^2/b^2}} = \frac{m\ a}{\sqrt{1 - F^2/b^2}} = m(F)\ a, \quad m(F) = \frac{m}{\sqrt{1 - F^2/b^2}} \tag{1.18}$$

which is the BRRT version of the Special Relativistic relation

$$P = \frac{m\ v}{\sqrt{1 - v^2/c^2}} = m(v)\ v, \quad m(v) = \frac{m}{\sqrt{1 - v^2/c^2}} \tag{1.19}$$

depicting the velocity-dependence of the mass.

From eq-(1.18) one learns that when $F = b \Rightarrow m(F = b) = \infty$, when the proper (rest) mass $m \neq 0$. Hence, a natural UV cutoff at $F = b$ appears. In case of a photon whose proper mass is zero, one may recall the (right) Rindler wedge corresponding to a family of accelerated observers describing hyperbolic trajectories in $D = 1 + 1$ spacetime given by $t = \frac{1}{a} sinh(a\tau); x = \frac{1}{a} cosh(a\tau)$ in $c = 1$ units. a is the proper acceleration and τ is the proper time. In the limit $a \to \infty$, the hyperbolas degenerate into the light-cone lines (Rindler horizon) corresponding to the null trajectories of a massless particle (like a photon). Consequently, one may choose to have the following double scaling limit

$$m \to 0, \quad a \to \infty, \quad m\ a \to F_0 \leq b \tag{1.19}$$

and which corresponds to reaching the null-lines of the Rindler horizon. One then learns that $m(F_0 < b) = 0$ and the effective mass remains zero. However, if $F_0 = b$, then $m(F_0) = \frac{m}{\sqrt{1 - F_0^2/b^2}} = \frac{0}{0}$ is undetermined when $m = 0$ and $F_0 = b$. In this limiting case, when $ma \to F_0 = b$, the effective (photon) mass is no longer zero and resembles the introduction of an infrared cutoff. This may have far reaching consequences in the resolution of the dark matter problem. See [5] for further applications of BRRT in Cosmology.

The Fulling-Davies-Unruh effect [12] states that a uniformly accelerating observer experiences the vacuum state of a quantum field in Minkowski spacetime as a mixed state in thermodynamic equilibrium. Such mixed state is comprised of a thermal bath (warm gas) of Rindler particles whose temperature is proportional to the acceleration. By invoking the expression of Unruh's temperature in terms of the acceleration $T = \frac{a}{2\pi}$ one finds that the Planck temperature $T_P = m_P$ corresponds to an acceleration $a_P = 2\pi m_P$ [1] so that the ratio $\frac{T}{T_P} = \frac{a}{a_P} = \frac{a}{2\pi m_P}$. Hence, if one sets the maximal proper force to be given by $F_{max} = b = 2\pi m_P^2$ (instead of m_P^2), then one arrives at

[1]Reinstating the units one has $2\pi m_P c^3/\hbar = a_P = 2\pi c^2/L_P$ furnishing a *huge* acceleration associated with the Planck temperature

$$\frac{F}{b} = \frac{m\ a}{2\pi m_P^2} = \frac{m}{m_P}\ \frac{a}{2\pi m_P} = \frac{m}{m_P}\ \frac{a}{a_P} = \frac{m}{m_P}\ \frac{T}{T_P} \tag{1.20}$$

and in doing so, one may rewrite $m(F)$ in terms of the temperature $m(T)$ as follows

$$m(F) = \frac{m}{\sqrt{1 - F^2/b^2}} \Rightarrow m(T) = \frac{m}{\sqrt{1\ - \frac{m^2}{m_P^2}\ T^2/T_P^2}} \tag{1.21}$$

and one arrives at an interesting relationship between mass and temperature. The reader might object invoking the Fulling-Davies-Unruh effect (which requires the use of Quantum Field Theory) in obtaining eq-(1.21) and which involves a classical theory. However one may notice that the $\hbar$ factors cancel out (decouple) in the ratio a/a_P. This can be verified by simply reinstating the dimensionful constants $\hbar = c = k_B = 1$ that were set to unity in the ratio

$$\frac{a}{a_P} = \frac{(\hbar a/2\pi k_B c)}{(\hbar a_P/2\pi k_B c)} = \frac{T}{T_P} \tag{1.22}$$

The Hawking radiation of black holes was derived based on treating the gravitational field as classical field while the matter was treated quantum mechanically. The temperature-dependence of the mass $m(T)$ bears some resemblance to the energy-scale dependence of the mass and other physical parameters in the Renormalization program of QFT, like the difference between the bare and renormalized mass.

2 The Relativistic Oscillator and the Fulling-Davies-Unruh effect

Bars [8] has rigorously shown that the familiar Fock space commonly used to describe the relativistic harmonic oscillator, for example as part of string theory, is insufficient to describe all the states of the relativistic oscillator. He found that there are three different vacua leading to three disconnected Fock sectors, all constructed with the same creation-annihilation operators. These have different spacetime geometric properties as well as different algebraic symmetry properties or different quantum numbers. Two of these Fock spaces include negative norm ghosts (as in string theory) while the third one is completely free of ghosts. He discussed a gauge symmetry in a worldline theory approach that supplies appropriate constraints to remove all the ghosts from all Fock sectors of the single oscillator. The resulting ghost free quantum spectrum in $D = d+1$ dimensions is then classified in unitary representations of the Lorentz group SO(d,1). Moreover all states of the single oscillator put together make up a single infinite dimensional unitary representation of a hidden global symmetry

$SU(d,1)$, whose Casimir eigenvalues are computed. One of the purpose of this section is to exploit the $U(d,1)$ symmetry.

As it is customary, Bars [8] began his results by *absorbing* all the dimensionful parameters, as well as the frequency of the oscillator, by rescaling the phase space coordinates x^μ, p^μ, such that the relativistic oscillator eigenvalue equation (in units $\hbar = c = 1$) turns out to be

$$\frac{1}{2} \, (-\partial^\mu \partial_\mu \ + \ x_\mu x^\mu) \Psi_\lambda(x^\mu) \ = \ \lambda \ \Psi_\lambda(x^\mu) \tag{2.1}$$

One of the key findings in [8] was in understanding that the symmetry properties of the solutions of eq-(2.1) were based precisely on the $U(1,3)$ group ($U(3,1)$ group depending on the signature). As usual, eq-(2.1) can be recast as an operator equation $Q\Psi_\lambda = \lambda \Psi_\lambda$ in terms of Lorentz covariant oscillators

$$a_\mu \ = \ \frac{1}{\sqrt{2}} \, (x_\mu + i p_\mu), \ \ \bar{a}_\mu \ = \ \frac{1}{\sqrt{2}} \, (x_\mu - i p_\mu) \tag{2.2}$$

where the operator Q is

$$Q \ = \ \frac{1}{2} \, (p_\mu p^\mu \ + \ x_\mu x^\mu) \ = \ \eta^{\mu\nu} \ \bar{a}_\mu \ a_\nu \ + \ \frac{d+1}{2} \tag{2.3}$$

The hidden symmetry is $U(d,1)$ with generators : $\bar{a}_\mu a_\nu$, and all of these $(d+1)^2$ generators commute with Q.

The author [8] remarked that in a unitary Hilbert space the operators x_μ, p_μ are Hermitian; in that case $\bar{a}_\mu$ is the Hermitian conjugate of a_μ i.e. $\bar{a}_\mu = a^\dagger_\mu$. A unitary Hilbert space without ghosts (negative norm states) is possible only if and only if x_μ, p_μ are hermitian or equivalently if $\bar{a}_\mu = a^\dagger_\mu$. The canonical commutation relations are

$$[x_\mu, \ p_\nu] \ = \ i\eta_{\mu\nu}, \ \ [a_\mu, \ \bar{a}_\nu] \ = \ \eta_{\mu\nu} \ = \ diag \ (-1,1,1,1) \tag{2.4}$$

Lorentz covariant solutions based on a vacuum state $\Psi_{vac} \sim e^{-\frac{1}{2}(x^\mu x_\mu)}$ (that is a Lorentz invariant Gaussian) have a number of problems, including issues of infinite norm and negative norm states. Bars [8] explored the spacelike $x_\mu x^\mu > 0$, and the timelike $x_\mu x^\mu < 0$ cases, and assigned the vacuum states $|0\rangle \leftrightarrow e^{-\frac{1}{2}(x^\mu x_\mu)}$; $|0'\rangle \leftrightarrow e^{\frac{1}{2}(x^\mu x_\mu)}$, respectively, to the spacelike and timelike cases.

The relevant vacuum state we shall explore in this work is the unitary Fock space based on the non-symmetric vacuum (Lorentz non-invariant vacuum) in $D = d+1$ spacetime, $\Psi_{vac} \sim e^{-\frac{1}{2}(x^i x_i + t^2)}, i = 1,2,\ldots,d$. Note that $x^\mu x_\mu \neq x^i x_i + t^2$. This vacuum state was labeled as $|\tilde{0}\rangle$ by [8].

The solutions we are interested in are found by separating the spatial variables from the temporal one as follows

$$\frac{1}{2} \, [\, (-\partial^i \partial_i + x_i x^i) - (-\partial_t^2 + t^2) \,] \, \Psi_\lambda(x^i, t) \ = \ \lambda \, \Psi_\lambda(x^i, t), \ \ i = 1,2,\ldots,d \tag{2.5}$$

A factorization

$$\Psi_\lambda(x^i,t) \ = \ \Psi_{\lambda_x}(x^i)\ \Psi_{\lambda_t}(t), \quad \lambda \ = \ \lambda_x \ - \ \lambda_t \tag{2.6}$$

yields the equations for the Euclidean harmonic oscillator in d dimensions and 1 dimension, respectively

$$\frac{1}{2}\ (-\partial^i\partial_i \ + \ x_i x^i)\ \Psi_{\lambda_x}(x^i) \ = \ \lambda_x\ \Psi_{\lambda_x}(x^i), \ \ \frac{1}{2}\ (-\partial_t^2 \ + \ t^2)\ \Psi_{\lambda_t}(t) \ = \ \lambda_t\ \Psi_{\lambda_t}(t) \tag{2.7}$$

and whose solutions are well known. The possible eigenvalues for Euclidean harmonic oscillator in d spatial dimensions and 1 temporal dimension are respectively

$$\lambda_x \ = \ n_1 + n_2 + n_3 \ + \ \ldots \ + n_d \ + \ \frac{d}{2}, \ \ \lambda_t \ = \ n_0 \ + \ \frac{1}{2}, \ \ \Rightarrow$$

$$\lambda \ = \ \lambda_x - \lambda_t \ = \ n_1+n_2+n_3 \ +\ldots \ +n_d - n_0 + \frac{d-1}{2} \ = \ N \ + \ \frac{d-1}{2} \tag{2.8}$$

For any given eigenvalue $\lambda = N + \frac{d-1}{2}$, with $N = 0, \pm 1, \pm 2, \pm 3, \ldots$ there is an infinite degeneracy of values $n_0, n_1, n_2, n_3, \ldots, n_d$

All solutions have the form $\Psi_\lambda(x^\mu) \sim e^{-\frac{1}{2}(t^2+x_ix^i)}\times$ Hermite polynomials in the variables x_i, t. The wavefunction of an arbitrary excited state of the d-dimensional Euclidean (isotropic) harmonic oscillator with eigenvalue $n+\frac{d}{2}$ and $SO(d)$ orbital angular momentum quantum number l, has the form [8]

$$\Psi^{nl}_{i_1 i_2 \ldots i_l}(\vec{x}) \ = \ e^{-\vec{x}^2/2}\ |\vec{x}|^l\ L_n^{l-1+d/2}\ (\vec{x}^2)\ T_{i_1 i_2 \ldots i_l}(\frac{x_i}{|\vec{x}|}) \tag{2.9}$$

where $T_{i_1 i_2 \ldots i_l}(\frac{x_i}{|\vec{x}|})$ is the symmetric traceless tensor of rank l constructed from the unit vector $\hat{x}_i = \frac{x_i}{|\vec{x}|}$ and which can also be recast in terms of the hyperspherical harmonic functions based on the $d-1$ angles associated with the hypersphere S^{d-1}. The function $L_n^{l-1+d/2}\ (\vec{x}^2)$ is a generalized Laguerre polynomial. The excitation level n is any positive integer $n = 0, 1, 2, 3, \ldots$ while at fixed n the allowed values of l are $l = n, (n-2), (n-4), \ldots$... (1 or 0) .

The solutions associated with the Lorentz symmetric invariant vacuum $|0\rangle \leftrightarrow e^{-\frac{1}{2}x_\mu x^\mu}$ are of the form

$$\Psi_k \ \sim \ e^{-\frac{1}{2}x^2}\ L_k^{\frac{d-1}{2}}(x^2), \quad x^2 = x_\mu x^\mu > 0 \tag{2.10}$$

and whose eigenvalue λ is

$$\lambda \ = \ n_1 + n_2 + n_3 + \ldots + n_d \ - \ n_0 \ + \ \frac{d+1}{2} \ = \ 2k \ + \ \frac{d+1}{2} \tag{2.11}$$

The solutions for the other Lorentz invariant vacuum $|0'\rangle$ are obtained by replacing $x^2 = x_\mu x^\mu \rightarrow -x^2 = -x_\mu x^\mu$ and $\lambda \rightarrow -\lambda$ [8].

The $U(1,1)$ algebra generators associated to a 4-dim phase corresponding to a $1+1$-dim spacetime, can be realized in terms of the creation and annihilation operators as follows

$$J_{\mu\nu} \;=\; \bar{a}_\mu \; a_\nu, \quad \mu, \nu = 0, 1 \tag{2.12}$$

However there is a *subtlety* in assigning the creation and annihilation operator for the temporal components : $a_0 = x_0 - ip_0 = -x^0 - ip_0 = -x^0 + \frac{\partial}{\partial x^0}$ is an annihilation operator for the Lorentz invariant symmetric vacuum $e^{-\frac{1}{2}[-(x^0)^2+(x^i)^2]}$, but it is a creation operator for the Lorentz non-invariant vacuum $e^{-\frac{1}{2}[(x^0)^2+(x^i)^2]}$. Consequently, the double creation and double annihilation operators for the Lorentz non-invariant vacuum in $D = 1+1$ are respectively given by [8]

$$J_{10} \;=\; \bar{a}_1 a_0, \;\; J_{01} \;=\; \bar{a}_0 a_1 \tag{2.13}$$

and such that $J_{01} \neq J_{10}$.

In a $D = d+1 = 1+1$ spacetime one has $\lambda = n_1 - n_0 + \frac{d-1}{2} = n_1 - n_0$. In the particular case when $n_1 = n_0$ one has $\lambda = 0$ and the infinite tower of states originating from $|\tilde{0}\rangle_{\lambda=0}$ is obtained by successive applications of the double creation operator $J_{10} = \bar{a}_1 a_0$ as follows

$$(Tower)_{\lambda=0} \;=\; \bigoplus_{k=0}^{\infty} \; (\bar{a}_1^k \; a_0^k) \; |\tilde{0}\rangle_{\lambda=0} \tag{2.14}$$

There are an infinite number of towers, parametrized by the eigenvalue of $\lambda = n_1 - n_0$, and with an infinite amount of degeneracy (an infinite number of states within each tower). This result is consistent with the fact that unitary irreducible representations of non-compact groups are infinite-dimensional.

Defining the state in the Fock space with $n_0 = n_1 = k$ as

$$|k, \; k\rangle \;\equiv\; |n_1 = k, \; n_0 = k\rangle \;=\; \frac{\bar{a}_1^k}{\sqrt{k!}} \; \frac{a_0^k}{\sqrt{k!}} \; |\tilde{0}\rangle_{\lambda=0} \tag{2.15}$$

one has the following infinite superposition of states belonging to the infinite tower

$$|\Psi\rangle_{\lambda=0} \;=\; e^{\theta_{10} J_{10}} \; |\tilde{0}\rangle_{\lambda=0} \;=\; e^{\theta_{10} \bar{a}_1 a_0} \; |\tilde{0}\rangle_{\lambda=0} \;=$$

$$\sum_{k=0}^{\infty} \; \theta_{10}^k \; \frac{\bar{a}_1^k}{\sqrt{k!}} \; \frac{a_0^k}{\sqrt{k!}} \; |\tilde{0}\rangle_{\lambda=0} \;=\; \sum_{k=0}^{\infty} \; \theta_{10}^k \; |k, k\rangle \tag{2.16}$$

and which bears some analogy to the construction of coherent states by applying the displacement operator to the ground state

$$|z\rangle \;=\; D(z)|0\rangle \;=\; e^{za^\dagger - \bar{z}a}|0\rangle \;=\; e^{-|z|^2/2} \; \sum_{n=0}^{\infty} \; \frac{z^n \; (a^\dagger)^n}{n!}|0\rangle \tag{2.17}$$

with $z = x + ip$ complex; $|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle$, and $\hat{a}|z\rangle = z|z\rangle$. Consequently, one could have rewritten in the above eq-(2.16) : $|\Psi\rangle_{\lambda=0} = |\theta_{10}\rangle$. Multimode minimal uncertainty squeezed states based on the Quaplectic group, given by the semi-direct product of $U(1,3)$ with the Weyl-Heisenberg group $H(1,3)$, were constructed by [6] from the action of the exponential of the Quaplectic algebra generators on the ground state, in the same vein as displayed in eq-(2.16), and involving the judicious group parameters multiplying the generators.

The symmetric part $M_{01} = \frac{1}{2}(J_{01} + J_{10})$ is the generator of force-boosts transformations along the x_1 direction, and the antisymmetric part $L_{10} = \frac{1}{2}(J_{01} - J_{10})$ is the generator of Lorentz boost along x_1. When θ_{10} is chosen to be complex-valued (as it occurs in the construction of coherent states $|z\rangle$) then $\theta_{10}J_{10} = \theta_{(10)}M_{10} + i\theta_{[10]}L_{10}$, where $\theta_{(10)} = \xi_a$ is the force-boost parameter, and $\theta_{[10]} = \xi_v$ is the velocity boost parameter. This decomposition results from breaking the complex-valued $\theta_{10} \equiv \theta_{(10)} + i\theta_{[10]}$ into a real-symmetric and imaginary-antisymmetric piece. [2]

If the relativistic oscillator is studied from two different frames of reference (two different observers) given by a fixed frame of reference, and another frame with a linear uniform acceleration with respect to the first one, the result found in eq-(1.20)

$$tanh(\xi_a = \theta_{(10)}) \ = \ \frac{F}{b} \ = \ \frac{m}{m_p}\ \frac{T}{T_P} \ \Rightarrow \ T \ = \ T_P\ \frac{m_P}{m}\ tanh(\xi_a = \theta_{(10)}) \tag{2.18}$$

combined with the fact that one had *absorbed* all the dimensionful parameters, as well as the frequency of the oscillator, by rescaling the phase space coordinates x^μ, p^μ,

$$p_\mu\ p^\mu \leftrightarrow \frac{p_\mu\ p^\mu}{m}, \quad x_\mu\ x^\mu \leftrightarrow (m\omega^2) x_\mu\ x^\mu, \quad \lambda \leftrightarrow \lambda\ \omega \tag{2.19}$$

allows us to identify the mass m appearing in eq-(2.18) with the relativistic oscillator proper mass appearing in eq-(2.19)[3]. One should note once more that the action of $e^{\theta_{10}J_{10}}$ (with θ_{10} complex-valued) on $|\tilde{0}\rangle$ is tantamount of a *combined* velocity and force-boost transformation on the vacuum $|\tilde{0}\rangle_{\lambda=0}$ generating a superposition of an $infinite$ number of states $|n_1 = k, n_0 = k\rangle$ in a Fock space, with complex-valued coefficients.

This picture can be contrasted with a warm gas (thermal bath) of Rindler scalar particles[4] of mass m at an equilibrium temperature of $T = T_P(\frac{m_P}{m})tanh(\xi_a = \xi)$ and experienced by a uniformly accelerated observer in Minkowski space whose acceleration is $a = 2\pi T$. A Planck relativistic oscillator of mass $m = m_P$ yields a temperature of $T = T_P\ tanh(\xi_a = \xi) \leq T_P$. When the force-boost parameter is $\xi_a = \xi = \infty$, one ends up with $T = T_P$ making contact with the Thermal Relativity proposal in [14] where the Planck temperature is postulated as the maximal temperature.

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[2] Velocity boosts with imaginary parameters are equivalent to ordinary rotations, and vice versa, rotations with imaginary angles are equivalent to velocity boosts

[3] One may note that $m \neq \lambda\omega$

[4] Because in $D = 1 + 1$ the little group is trivial, all the particles are scalars

More recently, Popov [9] has studied the relativistic oscillator from a Geometric Quantization point of view. He showed that turning on the interaction of relativistic spinless particles with the vacuum of relativistic quantum mechanics leads to the replacement of the Klein-Gordon equation with the Klein-Gordon oscillator equation. In this case, coordinate time becomes an operator and free relativistic particles go into a virtual state. He also discussed the geometry associated with classical and quantum Klein-Gordon oscillators, and its relation to the geometry underlying the description of free particles.

3 Concluding Remarks

A continuation of the Born Reciprocal Relativity Theory (BRRT) program in phase space showed that a natural temperature-dependence of mass occurs after recurring to the Fulling-Davies-Unruh effect [12]. The temperature dependence of the mass $m(T)$ resemblances the energy-scale dependence of mass and other physical parameters in the renormalization (group) program of QFT. It was found in a special case that the effective photon mass is no longer zero, which may have far reaching consequences in the resolution of the dark matter problem.

The Fulling-Davies-Unruh effect in a $D = 1+1$-dim spacetime was analyzed entirely from the perspective of BRRT, and we explained how it may be interpreted in terms of a linear superposition of an infinite number of states resulting from the action of the group $U(1,1)$ on the Lorentz non-invariant vacuum $|\tilde{0}\rangle$ of the relativistic oscillator studied by [8].

It is worth mentioning that a granularity of spacetime within the context of Born reciprocity and the Schrödinger-Robertson inequality for relativistic position and momentum operators $X^\mu, P^\nu, \mu, \nu = 0,1,2,3$ has been proposed by [6] and leading to the more generalized uncertainty relation

$$\Delta X_0 \; \Delta X_1 \; \Delta X_2 \; \Delta X_3 \; \Delta P_0 \; \Delta P_1 \; \Delta P_2 \; \Delta P_3 \; \geq \; (\frac{\hbar}{2})^4 \tag{3.1}$$

The authors [6], [7] studied the constraint quantization of a worldline system invariant under Born Reciprocal Relativity. The reciprocal transformations are *not* spin-conserving in general. The physical state space is vastly enriched as compared with the covariant approach, and contains towers of integer spin massive states, as well as unconventional massless representations, with continuous Euclidean momentum and arbitrary integer helicity.

The Dirac-Born oscillator as the "square-root" of the relativistic oscillator for a spinless particle was studied by [10]. For some other aspects based on maximal proper acceleration rather than maximal proper force see [2], [11]. The role of quantum groups, non-commutative Lorentzian spacetimes and curved momentum spaces see [13]. Curved phase space within the context of Finsler Geometry was studied in [15] and allowed us to find novel avenues to tackle the cosmological constant problem [15]. A study of Born's deformed reciprocal complex gravitational theory and noncommutative gravity can be found in [17]. The role of maximal acceleration in strings with dynamical Tension and Rindler worldsheets was analyzed in [16].

To finalize, the study of observers moving in $uniform$ circular motion (besides linear uniform acceleration) deserves careful consideration within the realm of this work. The problem is more subtle since it requires instantaneous reference frames where the velocity-boost and force-boosts *directions* change at every instant.

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