Any rational number N is the sum and difference of four rational fourth powers in an infinite number of ways

Seiji Tomita

Abstract

It is known that the equation $X^4+Y^4-(Z^4+W^4)=N$ has infinitely many rational solutions for any rational number N. We present three new solutions for the equation $X^4+Y^4-(Z^4+W^4)=N$.

1. Introduction

S. Ryley[1] proved that "Any non-zero rational number a is the sum of three rational cubes in an infinite number of non-trivial ways."

$$\Big(\frac{9d^6-30a^2d^3+a^4)(a^2+3d^3)+72a^4d^3}{6ad(a^2+3d^3)^2}\Big)^3+\Big(\frac{30a^2d^3-9d^6-a^4}{6ad(a^2+3d^3)}\Big)^3+\Big(\frac{6ad^2(a^2+3d^3)-12a^3d^2}{(a^2+3d^3)^2}\Big)^3=a.$$

H. W. Richmond[2, 3] gave a simpler solution:

$$\Big(\frac{s^9+27n^3}{d}\Big)^3+\Big(\frac{(9ns^6-s^9-27n^3}{d}\Big)^3+\Big(\frac{9s^3n(s^3-3n)}{d}\Big)^3=n \text{ where } d=3(s^6-3ns^3+9n^2)s^2.$$

R. Norrie[4] proved that "Any rational number a is the sum and difference of four rational fourth powers in an infinite number of ways."

$$\Big(\frac{(2a+b)c^3d}{A}\Big)^4+\Big(\frac{2ac^4-bd^4}{A}\Big)^4-\Big(\frac{2ac^4+bd^4}{A}\Big)^4-\Big(\frac{(2a-b)c^3d}{A}\Big)^4=a \text{ where } A=2bcd \text{ and } b=c^8-d^8.$$

As far as I am aware, no new results on the fourth powers have been reported since Norrie.
We will give the three new solutions for fourth powers.

2. Solving the diophantine equation $X^4+Y^4-(Z^4+W^4)=N$

$$X^4+Y^4-(Z^4+W^4)=N \tag{1}$$

We will give three methods to prove that there are infinitely many rational solutions of $X^4+Y^4-(Z^4+W^4)=N$ where $N\in\mathbb{Q}$.

2.1. Method-1

Substitute $X=ax+b,\ Y=cx+d,\ Z=ax-b,\ W=cx-d$ into equation (1) where $a,b,c,d\in\mathbb{Z}$. We obtain

$$(8dc^3+8ba^3)x^3+(8d^3c+8b^3a)x=N. \tag{2}$$

To vanish the coefficient of x^3 of equation (2), we take $b=c^3$ and $d=-a^3$.
Hence, we get $x=-N/(8ac(a^8-c^8))$.
Finally, we obtain a parametric solution as follows:

$$\begin{aligned}X&=a(-N+8c^4a^8-8c^{12})/f,\\Y&=-c(N+8a^{12}-8a^4c^8)/f,\\Z&=-a(N+8c^4a^8-8c^{12})/f,\\W&=c(-N+8a^{12}-8a^4c^8)/f,\\f&=8ac(a^8-c^8).\end{aligned}$$

Thus, any rational number N is the sum and difference of four rational fourth powers in an infinite number of ways.
Numerical examples:

Table 1

$(a,c),\ N=1\cdots 10$
$(2,1),\ 1=(2039/2040)^4+(32641/4080)^4-(2041/2040)^4-(32639/4080)^4$
$(2,1),\ 2=(1019/1020)^4+(16321/2040)^4-(1021/1020)^4-(16319/2040)^4$
$(2,1),\ 3=(679/680)^4+(10881/1360)^4-(681/680)^4-(10879/1360)^4$
$(2,1),\ 4=(509/510)^4+(8161/1020)^4-(511/510)^4-(8159/1020)^4$
$(2,1),\ 5=(407/408)^4+(6529/816)^4-(409/408)^4-(6527/816)^4$
$(2,1),\ 6=(339/340)^4+(5441/680)^4-(341/340)^4-(5439/680)^4$
$(2,1),\ 7=(2033/2040)^4+(32647/4080)^4-(2047/2040)^4-(32633/4080)^4$
$(2,1),\ 8=(254/255)^4+(4081/510)^4-(256/255)^4-(4079/510)^4$
$(2,1),\ 9=(677/680)^4+(10883/1360)^4-(683/680)^4-(10877/1360)^4$
$(2,1),\ 10=(203/204)^4+(3265/408)^4-(205/204)^4-(3263/408)^4$

Table 2

$(a,c),\ N=7$
$(2,1),7=(2033/2040)^4+(32647/4080)^4-(2047/2040)^4-(32633/4080)^4$
$(3,1),7=(52473/52480)^4+(4250887/157440)^4-(52487/52480)^4-(4250873/157440)^4$
$(4,1),7=(524273/524280)^4+(134215687/2097120)^4-(524287/524280)^4-(134215673/2097120)^4$
$(5,1),7=(3124985/3124992)^4+(1953120007/15624960)^4-(3124999/3124992)^4-(1953119993/15624960)^4$
$(6,1),7=(1919559/1919560)^4+(2487749761/11517360)^4-(1919561/1919560)^4-(2487749759/11517360)^4$
$(7,1),7=(46118393/46118400)^4+(15818611201/46118400)^4-(46118407/46118400)^4-(15818611199/46118400)^4$
$(8,1),7=(19173959/19173960)^4+(78536540161/153391680)^4-(19173961/19173960)^4-(78536540159/153391680)^4$
$(9,1),7=(344373753/344373760)^4+(2259436239367/3099363840)^4-(344373767/344373760)^4-(2259436239353/3099363840)^4$
$(10,1),7=(799999985/799999992)^4+(7999999920007/7999999920)^4-(799999999/799999992)^4-(7999999919993/7999999920)^4$

2.2. Method-2

Substitute $X=ax+c,\ Y=bx+d,\ Z=ax+e,\ W=bx+f$ into equation (1) where $a,b,c,d\in\mathbb{Z}$. We obtain

$$(-4ea^3+4db^3-4fb^3+4a^3c)x^3+(-6f^2b^2-6e^2a^2+6d^2b^2+6a^2c^2)x^2+(-4f^3b+4c^3a-4e^3a+4d^3b)x+c^4+d^4-f^4-e^4=N. \quad (3)$$

To vanish the coefficients of x^3 and x^2 of equation (3), we take

$$e=(2db^3a-b^4c+a^4c)/(a^4+b^4),$$
$$f=-(-2ba^3c-db^4+da^4)/(a^4+b^4).$$

Hence, we get

$$\begin{aligned}x=&\big(Na^{12}-8bcd^3a^{11}+24b^2c^2d^2a^{10}-24db^3c^3a^9-8a^8d^4b^4\\&+3a^8Nb^4+8a^8b^4c^4+32b^5cd^3a^7-48d^2b^6c^2a^6+32db^7c^3a^5\\&+3a^4Nb^8+8a^4b^8d^4-8a^4b^8c^4-24d^3b^9ca^3+24d^2b^{10}c^2a^2-8db^{11}c^3a+Nb^{12}\big)\\&/\big(8(a^4+b^4)ba(d^3a^7-3bcd^2a^6+3b^2c^2da^5-b^3c^3a^4-d^3b^4a^3+3b^5cd^2a^2-3b^6c^2da+b^7c^3)\big).\end{aligned}$$

Finally, we obtain a parametric solution as follows:

$$\begin{aligned}
X =& a(-8a^4b^8c^4 - 8a^8d^4b^4 + 3a^8Nb^4 + 8a^4b^8d^4 + 3a^4Nb^8 + 32b^5cd^3a^7 \\
&- 48d^2b^6c^2a^6 + 32db^7c^3a^5 - 32d^3b^9ca^3 + 48d^2b^{10}c^2a^2 - 32db^{11}c^3a + 8b^{12}c^4 + Nb^{12} + Na^{12})/g, \\
Y =& b(-8a^4b^8c^4 - 8a^8d^4b^4 + 8a^8b^4c^4 + 3a^8Nb^4 + 3a^4Nb^8 - 32bcd^3a^{11} \\
&+ 48b^2c^2d^2a^{10} - 32db^3c^3a^9 + 32b^5cd^3a^7 - 48d^2b^6c^2a^6 + 32db^7c^3a^5 + Nb^{12} + Na^{12} + 8d^4a^{12})/g, \\
Z =& a(8a^4b^8c^4 + 8a^8d^4b^4 + 3a^8Nb^4 - 8a^4b^8d^4 + 3a^4Nb^8 - 32b^5cd^3a^7 \\
&+ 48d^2b^6c^2a^6 - 32db^7c^3a^5 + 32d^3b^9ca^3 - 48d^2b^{10}c^2a^2 + 32db^{11}c^3a - 8b^{12}c^4 + Nb^{12} + Na^{12})/g, \\
W =& -b(-8a^4b^8c^4 - 8a^8d^4b^4 + 8a^8b^4c^4 - 3a^8Nb^4 - 3a^4Nb^8 - 32bcd^3a^{11} \\
&+ 48b^2c^2d^2a^{10} - 32db^3c^3a^9 + 32b^5cd^3a^7 - 48d^2b^6c^2a^6 + 32db^7c^3a^5 - Nb^{12} - Na^{12} + 8d^4a^{12})/g, \\
g =& 8ab(a^8 - b^8)(da - bc)^3.
\end{aligned}$$

Thus, any rational number N is the sum and difference of four rational fourth powers in an infinite number of ways.

Table 3

$(a,b,c,d),\ N = 1 \cdots 10$
$(3,1,1,1),\ 1 = (67641/52480)^4 + (172601/157440)^4 - (70201/52480)^4 - (34759/157440)^4$
$(2,1,1,1),\ 2 = (4853/1020)^4 + (5873/2040)^4 - (4973/1020)^4 - (3953/2040)^4$
$(2,1,1,1),\ 3 = (4873/680)^4 + (5553/1360)^4 - (4953/680)^4 - (4273/1360)^4$
$(2,1,1,1),\ 4 = (4883/510)^4 + (5393/1020)^4 - (4943/510)^4 - (4433/1020)^4$
$(2,1,1,1),\ 5 = (4889/408)^4 + (5297/816)^4 - (4937/408)^4 - (4529/816)^4$
$(2,1,1,1),\ 6 = (4893/340)^4 + (5233/680)^4 - (4933/340)^4 - (4593/680)^4$
$(2,1,1,1),\ 7 = (34271/2040)^4 + (36311/4080)^4 - (34511/2040)^4 - (32471/4080)^4$
$(2,1,1,1),\ 8 = (4898/255)^4 + (5153/510)^4 - (4928/255)^4 - (4673/510)^4$
$(2,1,1,1),\ 9 = (14699/680)^4 + (15379/1360)^4 - (14779/680)^4 - (14099/1360)^4$
$(2,1,1,1),\ 10 = (4901/204)^4 + (5105/408)^4 - (4925/204)^4 - (4721/408)^4$

Table 4

$(a,b,c,d),\ N = 7$
$(1,2,1,1),\ 7 = (36311/4080)^4 + (34271/2040)^4 - (32471/4080)^4 - (34511/2040)^4$
$(1,2,2,1),\ 7 = (189911/110160)^4 + (24671/55080)^4 - (121129/110160)^4 - (44111/55080)^4$
$(1,2,2,2),\ 7 = (65111/32640)^4 + (32471/16320)^4 - (3671/32640)^4 - (36311/16320)^4$
$(2,1,1,1),\ 7 = (34271/2040)^4 + (36311/4080)^4 - (34511/2040)^4 - (32471/4080)^4$
$(2,1,1,2),\ 7 = (24671/55080)^4 + (189911/110160)^4 - (44111/55080)^4 - (121129/110160)^4$
$(2,1,2,2),\ 7 = (32471/16320)^4 + (65111/32640)^4 - (36311/16320)^4 - (3671/32640)^4$
$(3,1,1,1),\ 7 = (481167/52480)^4 + (586127/157440)^4 - (483727/52480)^4 - (378767/157440)^4$
$(3,1,1,2),\ 7 = (432447/820000)^4 + (4532447/2460000)^4 - (532447/820000)^4 - (3567553/2460000)^4$
$(3,1,2,1),\ 7 = (482367/6560)^4 + (488927/19680)^4 - (482527/6560)^4 - (475967/19680)^4$
$(3,1,2,2),\ 7 = (461967/419840)^4 + (2141327/1259520)^4 - (502927/419840)^4 - (1176433/1259520)^4$

2.3. Method-3

Substitute $X = ax + p,\ Y = bx + q\ Z = cx + r,\ W = dx + s$ into equation (1) where $a, b, c, d, p, q \in \mathbb{Z}$. We obtain

$$\begin{aligned}
&(b^4 + a^4 - c^4 - d^4)x^4 \\
&+ (-4rc^3 - 4sd^3 + 4qb^3 + 4pa^3)x^3 \\
&+ (-6s^2d^2 - 6r^2c^2 + 6q^2b^2 + 6p^2a^2)x^2 \\
&+ (4p^3a - 4s^3d + 4q^3b - 4r^3c)x + p^4 + q^4 - s^4 - N - r^4 = 0.
\end{aligned} \tag{4}$$

Taking (a, b, c, d) such as the solution of $A^4 + B^4 = C^4 + D^4$.
To vanish the coefficients of x^3 and x^2 of equation (4), we take

$$\begin{aligned}
s &= (-rc^3 + qb^3 + pa^3)/d^3, \\
r &= \big(2c^2qb^3 + 2c^2pa^3 + 2abd^2(-pb + aq)\big)/\big(2(c^4 + d^4)c\big).
\end{aligned}$$

Hence, we get

$$x = \big((4c^6a^6b^6d^2 - 4c^2a^6b^6d^6 + 6c^4a^8b^4d^4 - 3c^{12}d^8 - 3c^8d^{12} + c^8a^4b^8 - c^4d^4a^4b^8 + a^4b^8d^8 - c^{16}d^4 + c^4d^4a^{12} - c^4d^{16})p^4$$
$$+ (-8c^4a^9b^3d^4 - 4c^2b^9a^3d^6 + 16c^4a^5b^7d^4 + 12c^2a^7b^5d^6 + 4c^6a^3b^9d^2 - 4c^8a^5b^7 - 4a^5b^7d^8 - 12c^6a^7b^5d^2)qp^3$$
$$+ (6c^4a^{10}b^2d^4 + 12c^6a^8b^4d^2 - 12c^2a^8b^4d^6 - 24c^4a^6b^6d^4 + 12c^2a^4b^8d^6 + 6c^8a^6b^6 + 6b^6d^8a^6 - 12c^6a^4b^8d^2 + 6c^4a^2b^{10}d^4)q^2p^2$$
$$+ (12c^6a^5b^7d^2 - 4c^6a^9b^3d^2 + 16c^4a^7b^5d^4 - 4a^7b^5d^8 - 8c^4a^3b^9d^4 + 4c^2a^9b^3d^6 - 12c^2a^5b^7d^6 - 4c^8a^7b^5)q^3p$$
$$+ (-3c^{12}d^8 + c^4d^4b^{12} + 4c^2a^6b^6d^6 - c^4d^{16} + 6c^4d^4a^4b^8 + a^8b^4d^8 - c^4a^8b^4d^4 - 3c^8d^{12} + c^8a^8b^4 - c^{16}d^4 - 4c^6a^6b^6d^2)q^4$$
$$+ c^4Nd^{16} + 3c^8Nd^{12} + c^{16}Nd^4 + 3c^{12}Nd^8\big)$$
$$/ \big(d^2c^2(c^4+d^4)(c^{10}q^3bd^2 + c^{10}p^3ad^2 + 2c^6q^3bd^6 + 2c^6p^3ad^6 + 3c^4a^4b^5p^2q - c^4a^3b^6p^3 - 3c^4a^5b^4pq^2 + c^4a^6b^3q^3 - 3c^2a^2b^7p^2qd^2$$
$$- c^2d^2p^3a^9 + 3c^2a^3b^6pq^2d^2 + c^2q^3bd^{10} - 3c^2a^5b^4p^3d^2 - c^2d^2q^3b^9 + 3c^2a^6b^3p^2qd^2 - 3c^2a^7b^2q^2pd^2 - 3c^2a^4b^5q^3d^2 + c^2p^3ad^{10}$$
$$- 3d^4a^4b^5p^2q + d^4a^3b^6p^3 + 3d^4a^5b^4pq^2 - d^4a^6b^3q^3)\big).$$

Finally, we obtain a parametric solution as follows:

$$X = \big((-c^8a^5b^8 - 3ac^{16}d^4 + 4d^2c^{10}a^3b^6 - 9ac^{12}d^8 - c^4d^4a^{13} - 4d^{10}c^2a^3b^6 + 12d^4c^8a^5b^4 - 9ac^8d^{12} - 4c^6a^7b^6d^2 - 3ac^4d^{16}$$
$$+ 4c^2a^7b^6d^6 - a^5b^8d^8 + c^4d^4a^5b^8 + 12d^8c^4a^5b^4 + 4d^4c^8a^9 - 6c^4a^9b^4d^4 + 4d^8c^4a^9)p^4$$
$$+ (4a^6b^7d^8 - 12d^4c^8a^6b^3 + 12d^{10}c^2a^4b^5 + 4c^2b^9a^4d^6 - 12d^8c^4a^6b^3 + 12c^6a^8b^5d^2 + 12d^8c^4a^2b^7 - 12c^2a^8b^5d^6$$
$$+ 12d^4c^8a^2b^7 + 4c^8a^6b^7 - 12d^2c^{10}a^4b^5 + 8c^4a^{10}b^3d^4 - 4c^6a^4b^9d^2 - 16c^4a^6b^7d^4)qp^3$$
$$+ (-6d^8b^6a^7 + 24c^4a^7b^6d^4 + 12d^8c^4a^7b^2 - 12d^8c^4a^3b^6 - 6c^4a^3b^{10}d^4 - 12c^6a^9b^4d^2 + 12d^4c^8a^7b^2 + 12d^2c^{10}a^5b^4$$
$$- 12d^{10}c^2a^5b^4 - 6c^4a^{11}b^2d^4 + 12c^6a^5b^8d^2 - 12c^2a^5b^8d^6 - 12d^4c^8a^3b^6 + 12c^2a^9b^4d^6 - 6c^8a^7b^6)q^2p^2$$
$$+ (12d^4c^8a^4b^5 - 12c^6a^6b^7d^2 + 4d^{10}c^2a^6b^3 + 4a^8b^5d^8 - 12d^8c^{12}b - 4c^2a^{10}b^3d^6 + 8c^4a^4b^9d^4 - 12d^{12}c^8b + 4d^4c^8b^9$$
$$- 4d^{16}c^4b - 4d^4c^{16}b + 4c^6a^{10}b^3d^2 + 12c^2a^6b^7d^6 - 16c^4a^8b^5d^4 + 4c^8a^8b^5 - 4d^2c^{10}a^6b^3 + 12d^8c^4a^4b^5 + 4d^8c^4b^9)q^3p$$
$$+ (-ac^4d^4b^{12} + c^4a^9b^4d^4 + 3ac^8d^{12} - 6c^4d^4a^5b^8 + 4c^6a^7b^6d^2 - 4c^2a^7b^6d^6 + 3ac^{12}d^8 + ac^{16}d^4 - c^8a^9b^4 - a^9b^4d^8 + ac^4d^{16})q^4$$
$$- ac^{16}Nd^4 - ac^4Nd^{16} - 3ac^8Nd^{12} - 3ac^{12}Nd^8\big)/g,$$

$$Y = \big((-bc^4d^4a^{12} + d^4c^{16}b + 4c^2a^6b^7d^6 - 4c^6a^6b^7d^2 + 3d^{12}c^8b - 6c^4a^8b^5d^4 + c^4a^4b^9d^4 + 3d^8c^{12}b - c^8a^4b^9 + d^{16}c^4b - a^4b^9d^8)p^4$$
$$+ (-4d^{10}c^2a^3b^6 + 4a^5b^8d^8 - 4ac^{16}d^4 + 4d^2c^{10}a^3b^6 + 4d^8c^4a^9 + 12d^4c^8a^5b^4 - 16c^4d^4a^5b^8 + 4d^4c^8a^9 + 4c^8a^5b^8 + 4c^2b^{10}a^3d^6$$
$$+ 12c^6a^7b^6d^2 - 4ac^4d^{16} - 12ac^{12}d^8 - 12ac^8d^{12} - 12c^2a^7b^6d^6 - 4c^6a^3b^{10}d^2 + 8c^4a^9b^4d^4 + 12d^8c^4a^5b^4)qp^3$$
$$+ (12d^8c^4a^2b^7 + 12c^6a^4b^9d^2 - 12d^2c^{10}a^4b^5 - 6c^4a^2b^{11}d^4 - 12d^8c^4a^6b^3 + 12d^4c^8a^2b^7 - 6a^6b^7d^8 - 12c^2b^9a^4d^6$$
$$+ 12d^{10}c^2a^4b^5 + 24c^4a^6b^7d^4 + 12c^2a^8b^5d^6 - 6c^8a^6b^7 - 12d^4c^8a^6b^3 - 6c^4a^{10}b^3d^4 - 12c^6a^8b^5d^2)q^2p^2$$
$$+ (12c^2a^5b^8d^6 + 12d^8c^4a^7b^2 - 12d^{10}c^2a^5b^4 - 16c^4a^7b^6d^4 - 12c^6a^5b^8d^2 + 12d^4c^8a^7b^2 - 12d^8c^4a^3b^6$$
$$+ 4c^8a^7b^6 + 4d^8b^6a^7 - 12d^4c^8a^3b^6 + 12d^2c^{10}a^5b^4 - 4c^2a^9b^4d^6 + 4c^6a^9b^4d^2 + 8c^4a^3b^{10}d^4)q^3p$$
$$+ (4d^4c^8b^9 + 12d^8c^4a^4b^5 - 4c^2a^6b^7d^6 + 4d^{10}c^2a^6b^3 - 4d^2c^{10}a^6b^3 + c^4a^8b^5d^4 - 9d^{12}c^8b - c^8a^8b^5$$
$$- 6c^4a^4b^9d^4 - a^8b^5d^8 + 4d^8c^4b^9 - 3d^4c^{16}b + 4c^6a^6b^7d^2 + 12d^4c^8a^4b^5 - c^4d^4b^{13} - 3d^{16}c^4b - 9d^8c^{12}b)q^4,$$
$$- bc^4Nd^{16} - 3bc^8Nd^{12} - 3bc^{12}Nd^8 - bc^{16}Nd^4\big)/g,$$

$$Z = \big((4d^6a^2b^2c^{10} + c^{16}d^4 - c^8a^4b^8 - 4d^6a^{10}b^2c^2 + 8d^{10}a^2b^2c^6 - 8d^8c^8a^4 + 3c^8d^{12} + 6c^4a^8b^4d^4 - 4d^4c^{12}a^4$$
$$+ 3c^{12}d^8 - 4d^{12}c^4a^4 + 4d^{14}a^2b^2c^2 - 12c^2a^6b^6d^6 + 3a^4b^8d^8 + c^4d^{16} - 3c^4d^4a^4b^8 + 3c^4d^4a^{12})p^4$$
$$+ (-4d^{12}c^4b^3a - 8d^{10}a^3bc^6 + 4c^8a^5b^7 + 4d^6a^{11}bc^2 + 24c^2a^7b^5d^6 - 4d^6a^3bc^{10} - 12c^2b^9a^3d^6$$
$$+ 24c^4a^5b^7d^4 - 4d^4c^{12}b^3a - 4d^{14}a^3bc^2 - 8d^8c^8b^3a - 12a^5b^7d^8)qp^3$$
$$+ (6c^4a^2b^{10}d^4 + 18d^8b^6a^6 - 24c^4a^6b^6d^4 - 24c^2a^8b^4d^6 + 6c^4a^{10}b^2d^4 - 6c^8a^6b^6 + 24c^2a^4b^8d^6)q^2p^2$$
$$+ (-4d^6ab^{11}c^2 - 12a^7b^5d^8 + 4d^{14}ab^3c^2 + 4d^6ab^3c^{10} + 8d^{10}ab^3c^6 + 4c^8a^7b^5 + 24c^4a^7b^5d^4 - 4d^4c^{12}a^3b$$
$$- 8d^8c^8a^3b + 12c^2a^9b^3d^6 - 24c^2a^5b^7d^6 - 4d^{12}c^4a^3b)q^3p$$
$$+ (-4d^{14}a^2b^2c^2 + 6c^4d^4a^4b^8 - 4d^6a^2b^2c^{10} + 3c^{12}d^8 + 4d^6a^2b^{10}c^2 - c^8a^8b^4 + 12c^2a^6b^6d^6 - 3c^4a^8b^4d^4$$
$$+ 3c^4d^4b^{12} + 3c^8d^{12} + c^{16}d^4 - 4d^{12}c^4b^4 - 4d^4c^{12}b^4 - 8d^{10}a^2b^2c^6 - 8d^8c^8b^4 + c^4d^{16} + 3a^8b^4d^8)q^4$$
$$- c^{16}Nd^4 - c^4Nd^{16} - 3c^8Nd^{12} - 3c^{12}Nd^8\big)c/g,$$

$$
\begin{aligned}
W = \big(&d^2(3c^8a^4b^8 - 8d^8c^8a^4 - 8d^6a^2b^2c^{10} + 6c^4a^8b^4d^4 - 4d^{10}a^2b^2c^6 + c^{16}d^4 + 3c^8d^{12} + 4c^6a^{10}b^2d^2 \\
&- 4d^{12}c^4a^4 + 3c^{12}d^8 + 12c^6a^6b^6d^2 - 3c^4d^4a^4b^8 - 4d^4c^{12}a^4 - 4c^{14}a^2b^2d^2 + c^4d^{16} - a^4b^8d^8 + 3c^4d^4a^{12})p^4 \\
&+ d^2(12c^6a^3b^9d^2 - 24c^6a^7b^5d^2 - 8d^8c^8b^3a - 4d^4c^{12}b^3a + 4c^{14}a^3bd^2 + 4a^5b^7d^8 - 12c^8a^5b^7 + 24c^4a^5b^7d^4 \\
&- 4d^{12}c^4b^3a + 4d^{10}a^3bc^6 - 4c^6a^{11}bd^2 + 8d^6a^3bc^{10})qp^3 \\
&+ d^2(-24c^4a^6b^6d^4 + 6c^4a^{10}b^2d^4 - 24c^6a^4b^8d^2 + 24c^6a^8b^4d^2 - 6d^8b^6a^6 + 6c^4a^2b^{10}d^4 + 18c^8a^6b^6)q^2p^2 \\
&+ d^2(24c^6a^5b^7d^2 - 12c^8a^7b^5 - 4d^{12}c^4a^3b - 8d^6ab^3c^{10} + 24c^4a^7b^5d^4 + 4c^6ab^{11}d^2 - 8d^8c^8a^3b - 4d^4c^{12}a^3b \\
&- 4d^{10}ab^3c^6 + 4a^7b^5d^8 - 4c^{14}ab^3d^2 - 12c^6a^9b^3d^2)q^3p \\
&+ d^2(-8d^8c^8b^4 - 4c^6a^2b^{10}d^2 + 3c^{12}d^8 - 12c^6a^6b^6d^2 + 6c^4d^4a^4b^8 + 3c^8d^{12} + c^4d^{16} - 3c^4a^8b^4d^4 + 3c^8a^8b^4 \\
&+ 4c^{14}a^2b^2d^2 + c^{16}d^4 - 4d^{12}c^4b^4 + 3c^4d^4b^{12} - 4d^4c^{12}b^4 + 4d^{10}a^2b^2c^6 + 8d^6a^2b^2c^{10} - a^8b^4d^8)q^4 \\
&+ d^2(-c^{16}Nd^4 - c^4Nd^{16} - 3c^8Nd^{12} - 3c^{12}Nd^8)\big)/(dg),
\end{aligned}
$$

$$
\begin{aligned}
g = &4d^2c^2(c^4 + d^4)(c^{10}q^3bd^2 + c^{10}p^3ad^2 + 2c^6q^3bd^6 + 2c^6p^3ad^6 + 3c^4a^4b^5p^2q - c^4a^3b^6p^3 - 3c^4a^5b^4pq^2 \\
&+ c^4a^6b^3q^3 - 3c^2a^2b^7p^2qd^2 - c^2d^2p^3a^9 + 3c^2a^3b^6pq^2d^2 + c^2q^3bd^{10} - 3c^2a^5b^4p^3d^2 - c^2d^2q^3b^9 \\
&+ 3c^2a^6b^3p^2qd^2 - 3c^2a^7b^2q^2pd^2 - 3c^2a^4b^5q^3d^2 + c^2p^3ad^{10} - 3d^4a^4b^5p^2q + d^4a^3b^6p^3 + 3d^4a^5b^4pq^2 - d^4a^6b^3q^3).
\end{aligned}
$$

Thus, any rational number N is the sum and difference of four rational fourth powers in an infinite number of ways.

Numerical examples: $(a,b,c,d,p,q) = (158, 59, 133, 134, 1, 1), N = 1 \cdots 10.$

$1 = (\frac{1152088938475343058370283651}{468406456552475819404377120})^4 + (\frac{1447411222389118310947848991}{936812913104951638808754240})^4 - (\frac{16154536110020269283069549}{7043706113571064953449280})^4 - (\frac{13432560164358510764707469}{6991141142574265961259360})^4$

$2 = (\frac{74014647394230755323350303529}{13817990468298036672429125040})^4 + (\frac{97200766993578308383866970969}{37004110067645589732945792480})^4 - (\frac{77676170113610372665620519769}{16415357097677366874013547040})^4 - (\frac{71332605172245644438577692329}{16292854432769326822714938480})^4$

$3 = (\frac{76028447402292593616518159569}{9211993645532024448286083360})^4 + (\frac{91485860468524295656862604529}{24669406711763726488630528320})^4 - (\frac{78469462548545671844698303729}{10943571398451577916009031360})^4 - (\frac{74240419254302519693336418769}{10861902955179551215143292320})^4$

$4 = (\frac{77035347406323512763102087589}{6908995234149018336214562520})^4 + (\frac{88628407205997289293360421309}{18502055033822794866472896240})^4 - (\frac{78866108766013321434237195709}{8207678548838683437006773520})^4 - (\frac{75694326295330957320715781989}{8146427216384663411357469240})^4$

$5 = (\frac{77639487408742064251052444401}{5527196187319214668971650016})^4 + (\frac{86913935248481085475259111377}{14801644027058235893178316992})^4 - (\frac{79104096496493911187960530897}{6566142839070946749605418816})^4 - (\frac{76566670519948019897143399921}{6517141773107730729085975392})^4$

$6 = (\frac{78042247410354431909686015609}{4605996822766012224143041680})^4 + (\frac{85770953943470282929858238089}{12334703355881863244315264160})^4 - (\frac{79262754983480971023776087689}{5471785699225788958004515680})^4 - (\frac{77148233336359394948095145209}{5430951477589775607571646160})^4$

$7 = (\frac{548309531880542861660969965303}{27635980936596073344858250080})^4 + (\frac{594681771079237967782003300183}{74008220135291179465891584960})^4 - (\frac{555632577319302096345510397783}{32830714195354733748027094080})^4 - (\frac{542945447436572639891424742903}{32585708865538653645429876960})^4$

$8 = (\frac{78545697412369891482977979619}{3454497617074509168107281260})^4 + (\frac{84342227312206779748107146479}{9251027516911397433236448120})^4 - (\frac{79461078092214795818545533679}{4103839274419341718503386760})^4 - (\frac{77875186856873613761784826819}{4073213608192331705678734620})^4$

$9 = (\frac{78713514079708378007408634289}{3070664548510674816095361120})^4 + (\frac{83865985101785612020856782609}{8223135570587908829543509440})^4 - (\frac{79527185795126070750135349009}{3647857132817192638669677120})^4 - (\frac{78117504697045020033014720689}{3620634318393183738381097440})^4$

$10 = (\frac{78847767413579167226953158025}{2763598093659607334485825008})^4 + (\frac{83484991333448677839056491513}{7400822013529117946589158496})^4 - (\frac{79580071957455090695407201273}{3283071419535473374802709408})^4 - (\frac{78311358969182145049998635785}{3258570886553865364542987696})^4$

References

[1] L.E. Dickson: History of the theory of numbers,vol.2, p.726

[2] L. J. Mordell: Diophantine equations, Academic Press, London, 1969, p.84

[3] H. W. Richmond: On Rational Solutions of $x^3 + y^3 + z^3 = R$. Proceedings of the Edinburgh Mathematical Society, 2(2), 92-100.

[4] Tito Piezas, A Collection of Algebraic Identities, https://web.archive.org/web/20230326022140/https://sites.google.com/site/tpiez