Fundamental Algebraic Disproof of the Riemann Hypothesis in the logarithmic derivative

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Abstract

In this paper we prove the fundamental contradiction about the Riemann Hypothesis, expressing a function as a product and given the following summation, where R is the set of all solutions of $R(x) = 0$:

$$\frac{R'(x)}{R(x)} = \sum_{r \in R} \left(\frac{1}{x - r} \right) \tag{1}$$

And considering a regularization for hypertranscendental functions, then the expression applied in Riemann Zeta function of $\frac{1}{2}$, or the logarithmic derivative, where R_t is the set of tirivial zeros:

$$\frac{\zeta'(\frac{1}{2})}{\zeta(\frac{1}{2})} \neq \sum_{r \in R_t} \left(\frac{1}{\frac{1}{2} - r} \right) \tag{2}$$

1 Introduction

The Riemann Hypothesis, one of the most famous unsolved problems in mathematics, has profound implications for the distribution of prime numbers and connections to various areas of mathematics and physics [6]. Alongside this celebrated conjecture, the study of hypertranscendental functions represents another fascinating realm of mathematical inquiry, with deep connections to number theory and complex analysis [1].One of the most well-known examples of a hypertranscendental function is the Gamma function $\Gamma(z)$, which satisfies the functional equation:

$$\Gamma(z + 1) = z\Gamma(z) \tag{3}$$

The Riemann Hypothesis, proposed by Bernhard Riemann in 1859, states that all non-trivial zeros of the Riemann zeta function $\zeta(s)$ have real part equal to $\frac{1}{2}$ [4][p.9]. The zeta function is defined for complex s with $\Re(s) > 1$ as:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{4}$$

This function can be analytically continued to the entire complex plane, except for a simple pole at $s = 1$. The Riemann Hypothesis has far-reaching consequences, including insights into the distribution of prime numbers [5]. Hypertranscendental functions are a class of functions that cannot satisfy any algebraic differential equation with coefficients that are rational functions [2]. These functions extend beyond the realm of algebraic and even transcendental functions, exhibiting properties that make them both challenging and intriguing to study.

Theorem 1. *Given the transcendental or hypertranscendental continuous and differentiable function y, we can express in therms of a summation if it has a infinity multi valued inverse function for k, where $R(x)$ is the function minus a constant k, where R is the set of all the solutions.*

$$\frac{R'(x)}{R(x)} = \sum_{r\in R}\left(\frac{1}{x-r}\right) \tag{5}$$

Proof. We can express a polynomial function as the following factorization see [7] ,where R is the set of all roots of $f(x)$:

$$f(x) = \prod_{r\in R}(x-r) \tag{6}$$

We can express the derivative of this function:

$$f'(x) = \sum_{s\in R}\prod_{r\in(R-\{s\})}(x-r) \tag{7}$$

Then dividing both, because each therm of the summation hasn't a element s then the division cancelates all therms except $\frac{1}{x-s}$, but s can be substituted by r because is only an auxuiliary variable:

$$\frac{f'(x)}{f(x)} = \sum_{r\in R}\left(\frac{1}{x-r}\right) \tag{8}$$

The main problem in some transcendental functions like e^x is that there isn't any solution to $e^x = 0$ or functions like $\frac{1}{x}$ has not solution to $\frac{1}{x} = 0$. Then to make a disctintion to describe functions without zeros. By definition a trasncendental function is an analytic function that does not satisfy a polynomial finite equation see [9]. But the key part is that any of this can be calculated an expressed with basic operations. The second part is that the inverse function of a transcendental function if is multivalued we have the following:

$$y_i^{-1}(k) = r_i \implies y(r_i) - k = 0 \tag{9}$$

Then we can construct a function $R(x)$ as the product of all $x - r_i$, then the function satisfies:

$$0 = R(r_i) = y(r_i) - k \implies y_i^{-1}(k) - r_i = 0 \tag{10}$$

The function is factorized and returning to 6 and 7 and 8.

$$\frac{R'(x)}{R(x)} = \sum_{r\in R}\left(\frac{1}{x-r}\right) \tag{11}$$

It's valid for all $y(r) - k = 0\ \forall r \in R$, then if some a, and $R(a) \neq 0 \implies a \notin R$. The useless part is that we must know all the values, and the values are a infinity. Notice it is valid for hypertranscendental see [2], that is not a differential equation of this form in the case because the coefficients and solutions can be $R \subseteq \mathbb{C}$. ☐

Theorem 2. *Riemann Hypothesis is false an exists non trivial roots with Re$(r) \neq \frac{1}{2}$*

Proof. Then the special case for $R(x) = f(x) - k = \zeta(x) - 0$ by supposing true that all non trivial zeros has real part equal to $\frac{1}{2}$:

$$\frac{\zeta'(x)}{\zeta(x)} = \sum_{r\in R}\left(\frac{1}{x-r}\right) \tag{12}$$

$$\frac{\zeta'(\frac{1}{2})}{\zeta(\frac{1}{2})} = \sum_{r\in R}\left(\frac{1}{\frac{1}{2}-r}\right) \tag{13}$$

We need to separate trivial zeros and non trivial zeros, then R_0 is the set of non trivial zeros and R_t the set of trivial zeros.

$$\frac{\zeta'(\frac{1}{2})}{\zeta(\frac{1}{2})} = \sum_{r\in R_t}\left(\frac{1}{\frac{1}{2}-r}\right) + \sum_{r\in R_0}\left(\frac{1}{\frac{1}{2}-\mathrm{Re}(r)-\mathrm{Im}(r)i}\right) \tag{14}$$

$$\frac{\zeta'(\frac{1}{2})}{\zeta(\frac{1}{2})} = \sum_{r\in R_t}\left(\frac{1}{\frac{1}{2}-r}\right) + \sum_{r\in R_0}\left(\frac{1}{\frac{1}{2}-\frac{1}{2}-\mathrm{Im}(r)i}\right) \tag{15}$$

$$\frac{\zeta'(\frac{1}{2})}{\zeta(\frac{1}{2})} = \sum_{r\in R_t}\left(\frac{1}{\frac{1}{2}-r}\right) + \sum_{r\in R_0}\left(\frac{1}{-\mathrm{Im}(r)i}\right) \tag{16}$$

We simplify $\frac{1}{-i} = i$ for non trivial zero summation.

$$\frac{\zeta'(\frac{1}{2})}{\zeta(\frac{1}{2})} = \sum_{r\in R_t}\left(\frac{1}{\frac{1}{2}-r}\right) + i\sum_{r\in R_0}\left(\frac{1}{\mathrm{Im}(r)}\right) \tag{17}$$

Expanding the trivial zeros summation, are the negative even integers:

$$\frac{\zeta'(\frac{1}{2})}{\zeta(\frac{1}{2})} = \sum_{k=1}^{\infty}\left(\frac{1}{\frac{1}{2}-(-2k)}\right) + i\sum_{r\in R_0}\left(\frac{1}{\mathrm{Im}(r)}\right) \tag{18}$$

$$\frac{\zeta'(\frac{1}{2})}{\zeta(\frac{1}{2})} = \sum_{k=1}^{\infty}\left(\frac{1}{\frac{1}{2}+2k}\right) + i\sum_{r\in R_0}\left(\frac{1}{\mathrm{Im}(r)}\right) \tag{19}$$

Notice the summation of the imaginary part is 0, the cancelation is because if s is a non-trivial zero, the $1-s$ is a non trivial zero see [4][p.113-116]. If s is a non-trivial zero of the Riemann zeta function, then $1-s$ is also a non-trivial zero. This is known as the functional equation of the Riemann zeta function. According to the Riemann hypothesis, all non-trivial zeros have a real part equal to $\frac{1}{2}$. If this is true, then the zeros occur in conjugate pairs: $\frac{1}{2}+\sigma i$ and $\frac{1}{2}-\sigma i$, where σ is some positive real number. When we sum the reciprocals of these zeros, the imaginary parts cancel out:

$$\frac{\zeta'(\frac{1}{2})}{\zeta(\frac{1}{2})} = \sum_{k=1}^{\infty}\left(\frac{1}{\frac{1}{2}+2k}\right) \tag{20}$$

Even if the non trivial zero summation is 0 or actually any value, it does not have effect over the divergent summation because of is a special case of generalized harmonic series see [8].By evaluating the functions numerically:

$$\zeta(\frac{1}{2}) \approx -1.46035\ldots \tag{21}$$

$$\zeta'(\frac{1}{2}) \approx -3.92265\ldots \tag{22}$$

$$\frac{\zeta'(\frac{1}{2})}{\zeta(\frac{1}{2})} \approx 2.68609\ldots \tag{23}$$

The number does not diverges, then the contradiction is evident, hence Riemann Hypothesis is false. □

Theorem 3. *The redefined equation is, where γ is the Euler-Mascheroni constant. Divide the groups of non trivial zero's over $\frac{1}{2}$ and distinct real part denote the sets $R_{0,\frac{1}{2}}$, and $R^C_{0,\frac{1}{2}}$. The summation of the inverse of the new roots is:*

$$\sum_{r\in R^C_{0,\frac{1}{2}}}\left(\frac{1}{r}\right) = \frac{1}{2}\gamma + \lim_{n\to\infty}\left[\ln\left(\frac{n^{\frac{1}{2}}}{2\pi}\right)\right] - \sum_{r\in R_{0,\frac{1}{2}}}\left(\frac{2}{1+4t_r^2}\right) \tag{24}$$

Proof. We can evaluate in $x=0$ and the result is equivalent to the logarithmic derivative see [3][p.94], then returning to equation (12):

$$\ln(2\pi) = \frac{\zeta'(0)}{\zeta(0)} = \sum_{r\in R}\left(\frac{1}{0-r}\right) = -\sum_{r\in R}\left(\frac{1}{r}\right) \tag{25}$$

Expanding the summation in zero's sets R_t, R_0:

$$\ln(2\pi) = -\sum_{r\in R_t}\left(\frac{1}{r}\right) - \sum_{r\in R_0}\left(\frac{1}{\sigma_r + t_r i}\right) \tag{26}$$

$$\ln(2\pi) = -\sum_{k=1}^{\infty}\left(\frac{1}{-2k}\right) - \sum_{r\in R_0}\left(\frac{1}{\sigma_r + t_r i}\right) \tag{27}$$

$$\ln(2\pi) = -\sum_{k=1}^{\infty}\left(\frac{1}{-2k}\right) - \sum_{r\in R_0}\left(\frac{\sigma_r - t_r i}{\sigma_r^2 + t_r^2}\right) \tag{28}$$

$$\ln(2\pi) = \sum_{k=1}^{\infty}\left(\frac{1}{2k}\right) + \sum_{r\in R_0}\left(\frac{-\sigma_r + t_r i}{\sigma_r^2 + t_r^2}\right) \tag{29}$$

$$\ln(2\pi) = \frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{1}{k}\right) + \sum_{r\in R_0}\left(\frac{-\sigma_r + t_r i}{\sigma_r^2 + t_r^2}\right) \tag{30}$$

Using limit for harmonic series see [8]:

$$\ln(2\pi) = \frac{1}{2}\lim_{n\to\infty}\left[ln(n)\right] + \frac{1}{2}\gamma + \sum_{r\in R_0}\left(\frac{-\sigma_r + t_r i}{\sigma_r^2 + t_r^2}\right) \tag{31}$$

We can get the imaginary part:

$$\operatorname{Im}\left(\ln(2\pi) + 0i\right) = \operatorname{Im}\left(\frac{1}{2}\lim_{n\to\infty}\left[ln(n)\right] + \frac{1}{2}\gamma + \sum_{r\in R_0}\left(\frac{-\sigma_r + t_r i}{\sigma_r^2 + t_r^2}\right)\right) \tag{32}$$

$$0 = \sum_{r\in R_0}\left(\frac{t_r}{\sigma_r^2 + t_r^2}\right) \tag{33}$$

And the real part:

$$\operatorname{Re}\left(\ln(2\pi) + 0i\right) = \operatorname{Re}\left(\frac{1}{2}\lim_{n\to\infty}\left[ln(n)\right] + \frac{1}{2}\gamma + \sum_{r\in R_0}\left(\frac{-\sigma_r + t_r i}{\sigma_r^2 + t_r^2}\right)\right) \tag{34}$$

$$\ln(2\pi) = \frac{1}{2}\lim_{n\to\infty}\left[ln(n)\right] + \frac{1}{2}\gamma + \sum_{r\in R_0}\left(\frac{-\sigma_r}{\sigma_r^2 + t_r^2}\right) \tag{35}$$

$$\ln(2\pi) - \frac{1}{2}\lim_{n\to\infty}\left[ln(n)\right] - \frac{1}{2}\gamma = \sum_{r\in R_0}\left(\frac{-\sigma_r}{\sigma_r^2 + t_r^2}\right) \tag{36}$$

$$-\ln(2\pi) + \frac{1}{2}\lim_{n\to\infty}\left[ln(n)\right] + \frac{1}{2}\gamma = \sum_{r\in R_0}\left(\frac{\sigma_r}{\sigma_r^2 + t_r^2}\right) \tag{37}$$

$$\lim_{n\to\infty}\left[\ln\left(\frac{n^{\frac{1}{2}}}{2\pi}\right)\right] + \frac{1}{2}\gamma = \sum_{r\in R_0}\left(\frac{\sigma_r}{\sigma_r^2 + t_r^2}\right) \tag{38}$$

□

Divide the groups of non trivial zero's over $\frac{1}{2}$ and disctint real part denote sets $R_{0,\frac{1}{2}}$, and $R^C_{0,\frac{1}{2}}$.

$$\lim_{n\to\infty}\left[\ln\left(\frac{n^{\frac{1}{2}}}{2\pi}\right)\right] + \frac{1}{2}\gamma = \sum_{r\in R_{0,\frac{1}{2}}}\left(\frac{\frac{1}{2}}{\frac{1}{4} + t_r^2}\right) + \sum_{r\in R^C_{0,\frac{1}{2}}}\left(\frac{\sigma_r}{\sigma_r + t_r^2}\right) \tag{39}$$

$$\lim_{n\to\infty}\left[\ln\left(\frac{n^{\frac{1}{2}}}{2\pi}\right)\right]+\frac{1}{2}\gamma=\frac{4}{4}\sum_{r\in R_{0,\frac{1}{2}}}\left(\frac{\frac{1}{2}}{\frac{1}{4}+t_r^2}\right)+\sum_{r\in R^C_{0,\frac{1}{2}}}\left(\frac{\sigma_r}{\sigma_r+t_r^2}\right) \tag{40}$$

$$\lim_{n\to\infty}\left[\ln\left(\frac{n^{\frac{1}{2}}}{2\pi}\right)\right]+\frac{1}{2}\gamma=\sum_{r\in R_{0,\frac{1}{2}}}\left(\frac{2}{1+4t_r^2}\right)+\sum_{r\in R^C_{0,\frac{1}{2}}}\left(\frac{\sigma_r}{\sigma_r+t_r^2}\right) \tag{41}$$

$$\lim_{n\to\infty}\left[\ln\left(\frac{n^{\frac{1}{2}}}{2\pi}\right)\right]+\frac{1}{2}\gamma-\sum_{r\in R_{0,\frac{1}{2}}}\left(\frac{2}{1+4t_r^2}\right)=\sum_{r\in R^C_{0,\frac{1}{2}}}\left(\frac{\sigma_r}{\sigma_r+t_r^2}\right) \tag{42}$$

From the functional equation see [4][p.113-116] we can get that if a non trivial zero r with real part disctint to $\frac{1}{2}$ then $1-r$ is a non-trivial zero, and the result of the summation for every peer is 0, the case is the same in the non trivial zeros of $R_{0,\frac{1}{2}}$.

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