A Dynamical Systems Model for Population and Depleting Resources

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Abstract

We investigate a coupled, non-linear dynamical systems model for the relationship between population and depleting resources inspired by limits to growth. The model is determined by logistic growth in population with carrying capacity determined by resources. The rate of decline of resources is determined linearly by the population. The model produces an initial exponential increase in population followed by a decrease to the fixed point while congruently resources decrease in a sigmoidal fashion to the fixed point. We fit the model to world population over the period 10000 BC to 2021 in different time intervals corresponding to different growth rates. We show a number of projections to 2500 based on fitting to the time period of 1950 to 2021 with various parameter constraints.

1 Introduction

Since the dawn of civilisation human world population has been increasing at roughly an exponential rate with the rate changing over different time periods, see Figure 2. This surge in population has been maintained mostly by non-renewable resource depletion of biomass, fossil fuels and other raw materials such as aluminium, steel, and rare-earth metals [5, 1]. The recent rapid decrease in birth rates across increasing proportions of the world is suggestive that population could soon reach it's peak. There are many proposed reasons for the decrease in birth rates but perhaps the underlying factor is a critical point in the decrease of resources.

We analyse a coupled dynamical system where population is limited by resources in a logistic fashion and resources decline proportionally with increasing population. This model gives similar results for population and resources to the famous limits to growth model (LtG) [3]. Indeed the main aim of this paper is to simplify the LtG by focusing on the two primary variables of population and resources. Although the LtG is based on a dynamical system of differential equations it is not immediately obvious what these equations are as they are represented either in causal diagrams or algorithms [4]. In addition the number of variables and complexity of the relations in the LtG hinders an easy understanding of the equations [4].

We fit our model to world population data over different time intervals from 10000 BC to 2021 corresponding to different exponential growth rates. We do not investigate any resource data but this would be useful future work to assess how well this fits to the model. We project the model into the year 2500 and show four of many possible projections for future population. How fast population declines is dependent on all four of the parameters in the model. The main takeaway of this model is that population rises to a peak and then falls back to a stable point, the only difference being the rate at which this happens. All code for simulation and fitting is found on the author's GitHub [2].

2 Model

Let P_t and R_t be the population and resources respectively at time t. Then with positive parameters, $\alpha, \beta, \gamma, \delta > 0$, the coupled, non-linear dynamical systems model is defined (when convenient we drop the t subscript)

$$\dot{P} = P(\alpha - \beta P/R)\,, \tag{1}$$

$$\dot{R} = \delta - \gamma P\,. \tag{2}$$

where the dot above the letters is the usual representation of the derivative with respect to t, i.e. $\dot{P} := \frac{\mathrm{d}}{\mathrm{d}t}P$. We note that (1) is the logistic growth equation but with carrying capacity R dependent on P. We refer to the model represented by (1) and (2) as the **population resource model**.

2.1 Fixed Point

The following theory to analyse fixed points is found, for example, in Sections 5.2 and 6.3 of [7]. Fixed points occur when $\dot{P} = 0$ and $\dot{R} = 0$ simultaneously. We find there is a **single fixed point** at

$$P^* = \delta/\gamma\,, \quad R^* = \frac{\beta\delta}{\alpha\gamma} = \frac{\beta}{\alpha}P^*\,. \tag{3}$$

To analyse the stability of the fixed point (P^*, R^*) we find the Jacobian

$$J = \begin{pmatrix} \frac{\partial \dot{P}}{\partial P} & \frac{\partial \dot{P}}{\partial R} \\ \frac{\partial \dot{R}}{\partial P} & \frac{\partial \dot{R}}{\partial R} \end{pmatrix} = \begin{pmatrix} \alpha - \frac{2\beta P}{R} & \beta\frac{P^2}{R^2} \\ -\gamma & 0 \end{pmatrix}.$$

Noting $P^*/R^* = \alpha/\beta$ then evaluating J at the fixed point gives

$$J(P^*, R^*) = \begin{pmatrix} -\alpha & \frac{\alpha^2}{\beta} \\ -\gamma & 0 \end{pmatrix}.$$

Let τ and Δ be the trace and determinant of $J(P^*, R^*)$ respectively. As $\tau = -\alpha < 0$ the fixed point (3) is **stable**. Although not really of interest to this paper we shall briefly categorise the type of stable fixed point. The type of stable fixed point is determined by the sign of

$$D = \tau^2 - 4\Delta = \alpha^2(1 - \gamma/\beta)\,.$$

When $D > 0 \Leftrightarrow \gamma < \beta$ the fixed point is a stable node, when $D < 0 \Leftrightarrow \gamma > \beta$ the fixed point is a stable spiral and when $D = 0 \Leftrightarrow \gamma = \beta$ the fixed point is an edge case such as a star or degenerate node.

2.2 General Comments and Simulations

We shall choose the initial population and resources greater than their respective fixed points $R_0 > R^*, P_0 > P^*$. Also we choose initial resources far greater than initial population $R_0 >> P_0$ giving initial population dynamics

$$\dot{P} \approx \alpha P$$

so that $P \approx P_0 e^{\alpha t}$ initially grows roughly exponentially with rate α. We have three cases for the sign for $\dot{P}$ corresponding to whether the population is increasing, reaching a critical point or decreasing:

$$\dot{P} \begin{cases} > 0\,, & P/R < \alpha/\beta\,, \\ = 0\,, & P/R = \alpha/\beta\,, \\ < 0\,, & P/R > \alpha/\beta\,. \end{cases}$$

These cases, the initial conditions and the stability of the fixed point indicate that after the initial population increase the population hits a maximum before descending down to the fixed point. When $P > P^*$ then $\dot{R} < 0$ and so resources will decrease towards the fixed point R^*. From (2) we see that resources decrease at a proportional rate to population.

We can simulate the population resource model numerically using the numerical ODE solver odeint in the Python SciPy library. We evaluate the numerical solution at each time step $\Delta t = 10^{-2}$. The results for changing one of the parameters while keeping the others fixed is shown in Figure 1. Higher α, γ and smaller β give a faster decline in population and resources. Higher α and smaller β, γ give a higher population peak. The δ parameter controls primarily only the size of the final fixed points. From the simulations we can see that in general resources decrease in a sigmoid fashion.

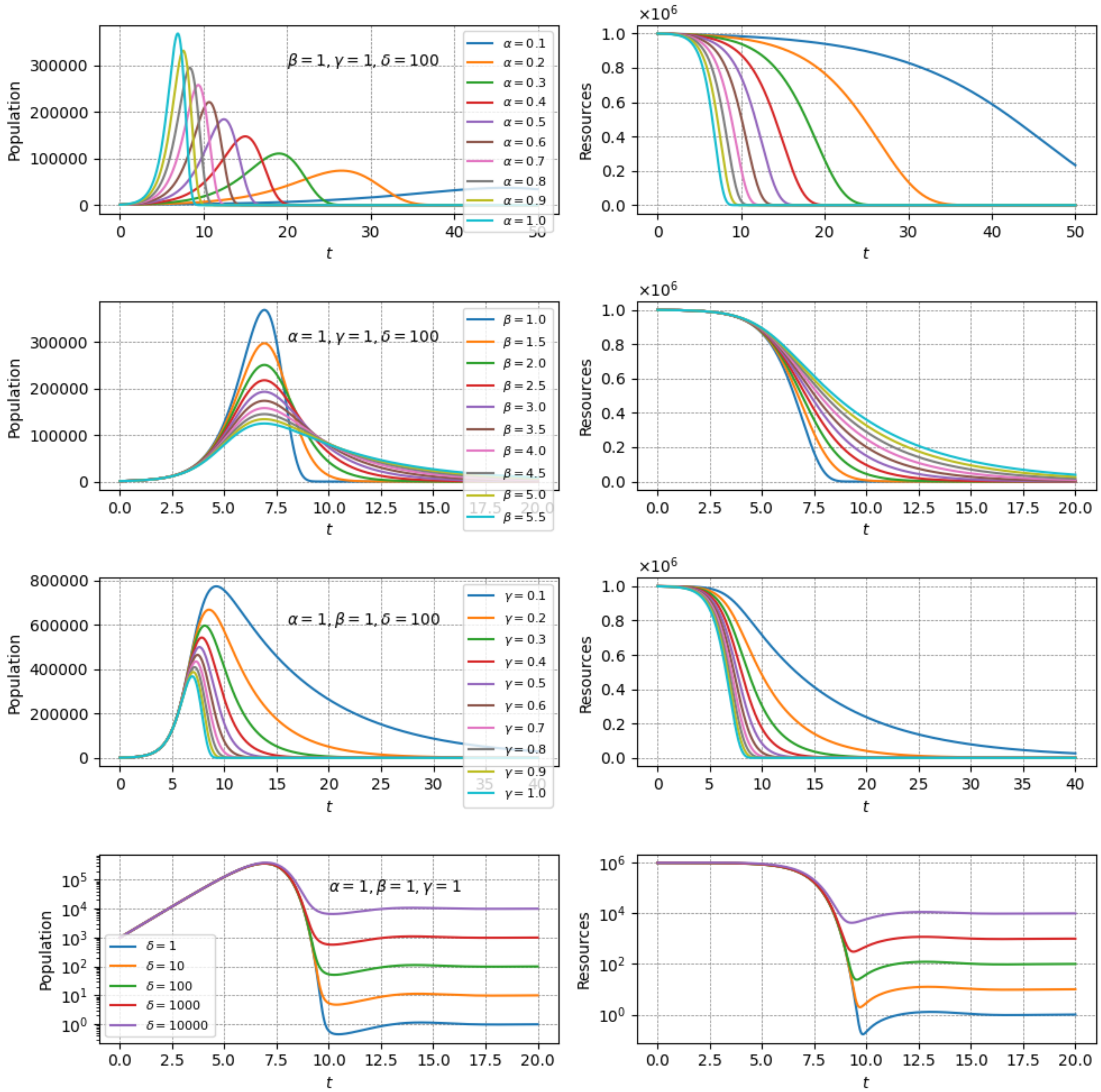

Figure 1: Numerical simulation of the population resource model with $P_0 = 10^3$, $R_0 = 10^6$ for different times with step size $\Delta t = 10^{-2}$. Left plots show the population and right plots shows the concurrent resources. The plots show the four combinations of changing one parameter while keeping the other three parameters fixed. The final row for the δ parameter changing is plotted on a log-axis to see the difference between the simulations more clearly. We note from the final row the possibility of population and resources to go under the fixed point.

3 Application to World Population

World population of humans has been increasing roughly exponentially though at different rates for potentially at least the last 12000 years coinciding with the beginning of civilisation, see Figure 2. We fit the population resource model to different growth intervals of world population data from 10000 BC to 2021. Each time step of $\Delta t = 10^{-2}$ in the numerical solution of the model is counted as a year. We start the fit at $t = 0$ equivalent to year -10000 or 10000 BC and choose $\beta = \gamma = 1$, $\delta = P_0 = 4432266$, $R_0 = 2.5 \cdot 10^{10}$. The units of resources are not specified and as mentioned in the introduction we do not fit to any resource data. We fit the α over the following time intervals where the growth rate appears to change

$$\begin{gathered}[-10000,\ -6001]\,,\ [-6000,\ -1]\,,\ [0,\ 999]\,,\ [1000,\ 1699]\,,\\ [1700,\ 1799]\,,\ [1800,\ 1869]\,,\ [1870,\ 1949]\,,\ [1950,\ 2021]\end{gathered} \tag{4}$$

as shown in Figure 2. We note for each of the intervals beyond the first, $[-10000,\ -6001]$, we choose P_0 and R_0 to be the population and resources at end of the previous interval. The fits use the Python Scipy minimize function to minimise the sum of squares between the data and the numerical solution of the model [2]. The initial resources $R_0 = 2.5 \cdot 10^{10}$ was chosen by trial and error so that there is a slight drop off in the rate of growth in the final time interval $[1950, 2021]$ as is evident in the data. The δ and initial resources P_0 were chosen to be the estimated population at the first time of 10000 BC that we have data. $\gamma = 1$ is chosen so $P^* = P_0$ (of the first interval above) and so the eventual population will drop back to what it was at at 10000 BC. Of course potentially more realistic values could be chosen for δ so that the eventual population is higher. The final parameter $\beta = 1$ is chosen through trial and error to allow for convergence when fitting α through all the time periods whilst also keeping resources declining in a smooth fashion.

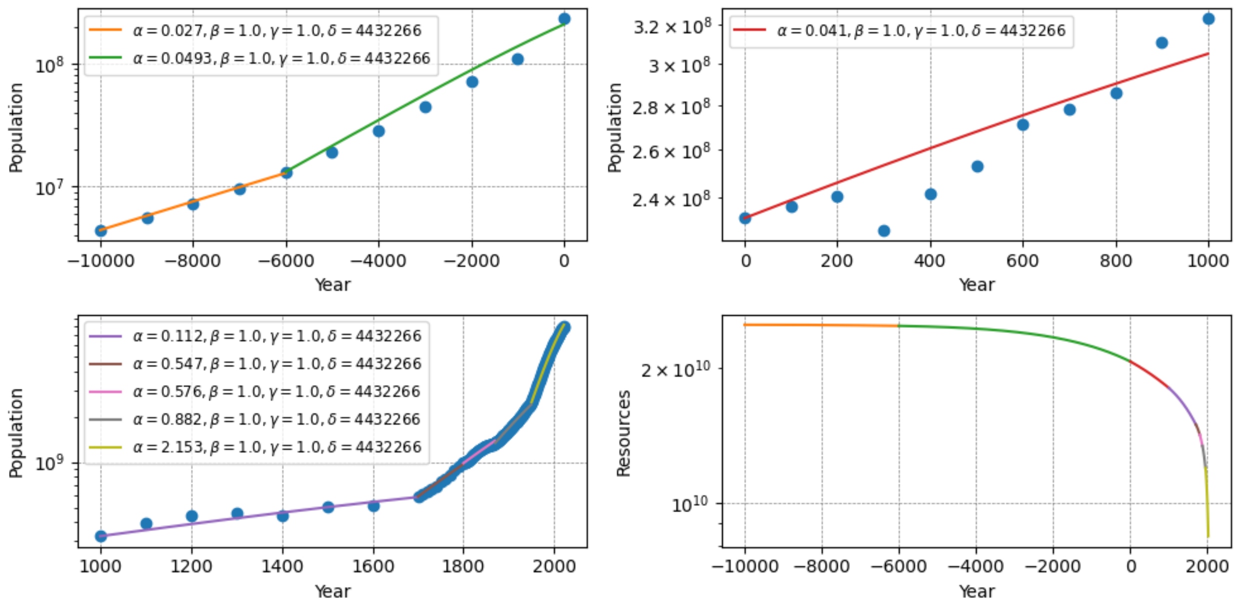

Figure 2: Numerical simulation of the population resource model with step size $\Delta t = 10^{-2}$ scaled to a year with α fits to the population data at the different time periods (4) exhibiting different growth rates with remaining parameters chosen as above. Population data from our world in data [6].

In Figure 3 we project the population to the year 2500 based on fitting the final time interval $[1950, 2021]$ with different parameter bounds. We see that there are multiple possibilities that fit the last time period reasonably but lead to different projections. We note that the parameters are fixed through the projections whereas in reality as when fitting the historic data the parameters are likely to change across time.

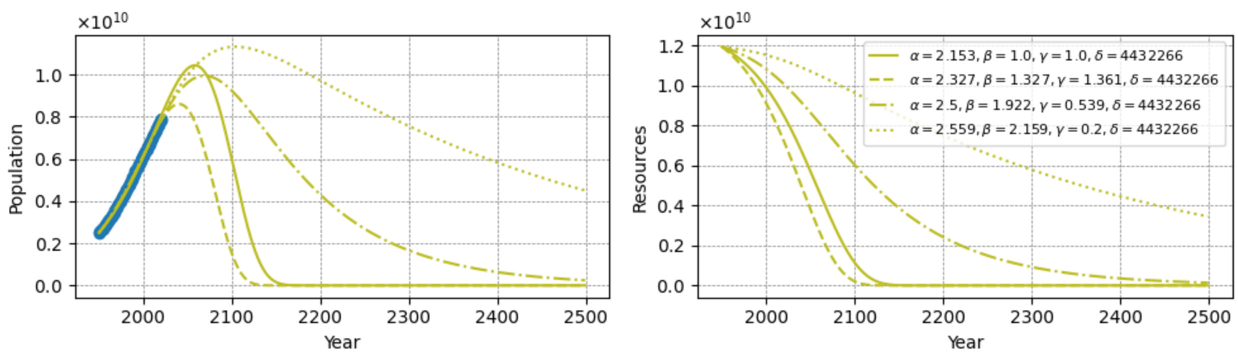

Figure 3: Numerical simulation of the population resource model with step size $\Delta t = 10^{-2}$ scaled to a year with α, β, γ fits to the population data over [1950, 2021] and then projecting this fit to the year 2500. Full line, same fit as Figure 2, only α fitted with remaining parameters chosen as above. Dashed lines α, β, γ fitted, dashed dotted line chosen $\alpha = 2.5$ with β, γ fitted and dotted line chosen $\gamma = 0.2$ with α, β fitted.

4 Conclusion

Inspired by simplifying the limits to growth model, we presented a coupled, non-linear dynamical system of population and depleting resources based on population logistic growth with carrying capacity dependent on resources. We then applied this model to world population and found due to the rough exponential increase of population we could apply a reasonable fit to several growth intervals from 10000 BC to 2021. Finally we project the model into the future to the year 2500 based on fitting the final time period of [1950, 2021]. There is a wide array of possibilities with a common feature of population hitting a maximum before falling to a fixed point. We chose this fixed point to be the population at 10000 BC which may be unrealistic and represents an extreme case of the complete fall of civilisation. We also note that in reality populations are usually in flux so an exact fixed point (apart form zero) is unlikely. Time will tell if this model will still be able to fit to the population particularly if/when the population hits a maximum and then decreases.

References

[1] Catton, W. R. (1982). *Overshoot: The ecological basis of revolutionary change*. University of Illinois Press. (Cited on page 1.)

[2] Forbes, S. (2024). Python code. `https://github.com/saf92/population-resources`. (Cited on pages 2 and 5.)

[3] Meadows, D. H., Meadows, D. L., Randers, J., and Behrens, W. W. (1972). *The limits to growth*. Universe Books. (Cited on page 1.)

[4] Meadows, D. L., Behrens, W. W., Meadows, D. H., Naill, R. F., Randers, J., and Zahn, E. (1974). *Dynamics of growth in a finite world*. Wright-Allen Press Cambridge, MA. (Cited on page 1.)

[5] Murphy Jr, T. W. (2022). Limits to economic growth. *Nature Physics*, 18(8):844–847. (Cited on page 1.)

[6] Our world in data (2024). World population estimate data 10000 BCE to 2021 CE. `https://ourworldindata.org/grapher/population`. (Cited on page 6.)

[7] Strogatz, S. H. (2018). *Nonlinear dynamics and chaos: with applications to physics, biology, chemistry, and engineering*. CRC press. (Cited on page 2.)