The problem of Newton's cannonball

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Abstract

Looking at the kinematics of Newton's thought experiment of the cannonball fired from a mountain, by considering a linear gravitational acceleration we face a non constant angular momentum, thus no Keplerian motion. Nonetheless Newton always referred to his gravitational force as centripetal, therefore the problem can be solved by using Hamilton's Keplerian velocity, which also forecasts a centripetal acceleration. We might then have misunderstood Newton by considering a linear instead of centripetal gravitational acceleration in some local experiments, like the body falling.

I. INTRODUCTION

One of the most famous Newton's thought experiments is the cannonball fired from a mountain, with increasing ejection velocity, until it is satellited. He does not relate it in the Principia, but in the "*Treatise of the System of the World*"[1]. The figure 1 is the illustration published in this document.

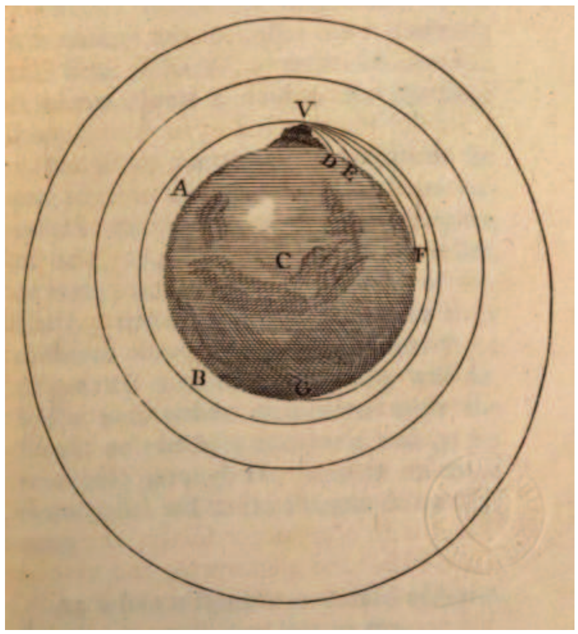

FIG. 1. Illustration by Newton, in "*Treatise of the System of the World*", of the cannonball fired from a mountain.

Newton explains that if the ejection velocity $\vec{v}_0$ of the cannonball is low, it will evolve on "*curve lines*" until falling back to Earth (paths VD, VE and VF in figure 1). If its ejection velocity is sufficient, "*it would reach at last quite beyond the circumference of the Earth, and return to the mountain from which it was projected*" (circular path in Figure 1). This last case of course describes the cannonball placed into orbit around the Earth, and $\vec{v}_0$ is the orbital velocity. It is then interesting to study the kinematics driving these "*curve lines*".

II. THE ANGULAR MOMENTUM IS NOT CONSTANT

Let us assume that the cannon is located at a distance $\vec{r}$ from the Earth center, which is the addition of the Earth radius R and the altitude h of the mountain. The consequent gravitational acceleration is therefore $\vec{g} = -\frac{GM}{r^3}\vec{r}$, where G is the gravitational constant, and M is the mass of the Earth. As far as h is very small with regard to R, we can make the approximation that the gravitational acceleration remains constant during the flight $g \approx GM/R^2$. In such conditions the gravitation will provide a gravitational velocity, or speed of fall, $\vec{v}_G = -gt\frac{\vec{r}}{r}$, where t is the flight duration, which is directed toward the Earth center because of the attraction. In addition, at fire the cannonball will be provided an ejection velocity $\vec{v}_0$, which added to the gravitational velocity gives the overall velocity : $\vec{v} = \vec{v}_G + \vec{v}_0$. The figure 2 presents this velocity scheme.

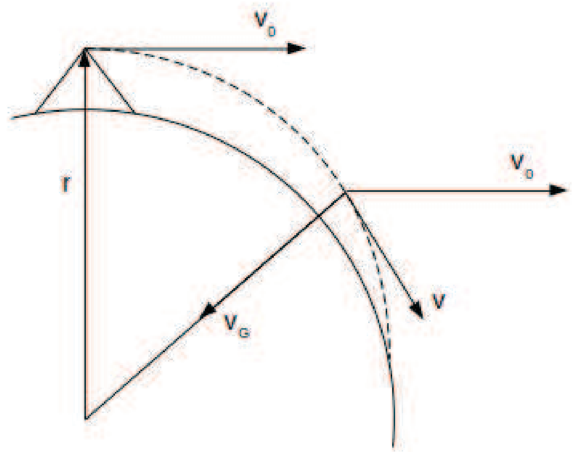

FIG. 2. Velocity $\vec{v}$ of the cannonball from Newton's experiment. $\vec{v}_G$ is the velocity due to the gravitational attraction, $\vec{v}_0$ is the ejection velocity.

However, as far as the gravitational velocity $\vec{v}_G$ is collinear to the vector radius $\vec{r}$, the geometric angular momentum $\vec{L} = \vec{r} \times \vec{v}$ is thus $\vec{L} = \vec{r} \times \vec{v}_0$. Therefore obviously $\vec{L}$ cannot be constant, and this is a problem. On this point the classical mechanics is strict : all bodies in a gravitational field, having or not an initial velocity, must have a motion respecting the 3 laws of Kepler, which second and third derive directly from the constancy of the angular momentum[2].

For the cannonball, the angular momentum not being a constant, we face a problem that has two solutions. One of them is to consider that the cannonball is not a body in a gravitational field at a Keplerian point of view, and thus it exists at least two different physical states for a body in a gravitational field. But Newton never considered such a possibility, and even at contrary, he proposed a universal, so unique, law of gravitation.

Furthermore this would also contradict the classical mechanics that does not make any place to another gravitational state than the Keplerian one[2]. The other solution is to consider that something is wrong in our interpretation of the attraction. Sure Newton's equations for the gravitation have been widely experienced, with an indisputable success ever, so we must believe their validity, but there might be a little tricky thing that we might have missed, because the cannonball's angular momentum must be constant in a gravitational field. On this W.R. Hamilton can give us a light.

III. HAMILTON TO THE RESCUE

In 1845 W.R. Hamilton demonstrated[3] that the velocity $\vec{v}$ of any Keplerian orbiter (body which motion respects Kepler's three laws) is always the addition of a constant rotation velocity $\vec{v}_R$ and a constant translation velocity $\vec{v}_T$:

$$\vec{v} = \vec{v}_R + \vec{v}_T \qquad \text{with } v_R = \textit{constant} \text{ and } v_T = \textit{constant} \tag{1}$$

Be careful, the index T stands here for translation, but not tangential, while the index R stands for rotation, but not radial. The figure 3 shows these velocities in action.

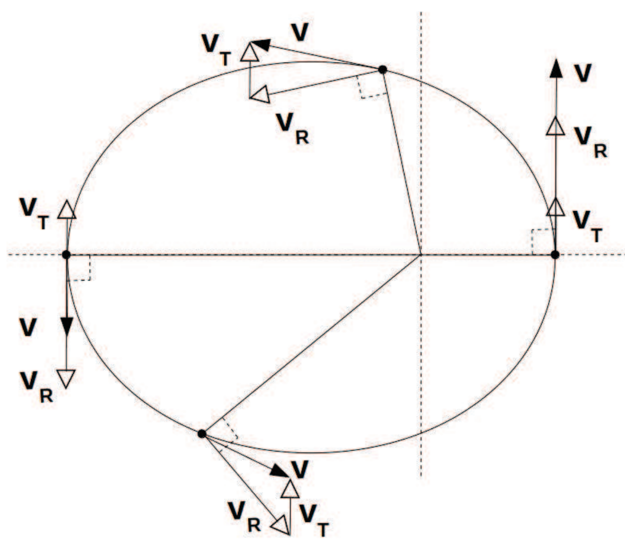

FIG. 3. Hamilton's velocity $\vec{v}$, with rotation velocity $\vec{v}_R$ and translation velocity $\vec{v}_T$ along a Keplerian orbit. Note that $\vec{v}_R$ is always perpendicular to the vector radius, and $\vec{v}_T$ is always perpendicular to the major axis.

Starting from this kinematic structure of the velocity, and defining the angular momentum

$\vec{L}$ such as $\vec{L} = \vec{r} \times \vec{v}$, then calculating the value of $\vec{v}_R \times \vec{L}$, it is trivial to get the equation of the conics :

$$\vec{v}_R \times \vec{L} = {v_R}^2 (1 + \frac{\vec{v}_R . \vec{v}_T}{{v_R}^2}) \vec{r} \tag{2}$$

or :

$$p = (1 + e \cos\theta) r \qquad \text{with} \quad p = \frac{L}{v_R} \qquad \text{and} \quad e = \frac{v_T}{v_R} \tag{3}$$

In this formula, p is the semi-latus rectum, e is the eccentricity and θ is the true anomaly.

Kepler's first law therefore derives from Hamilton's velocity 1, as logically expected. The second and third ones trivially derive from the same velocity, like Hamilton explained[3].

It is also easy to demonstrate that the acceleration $\vec{a} = \dot{\vec{v_R}} + \dot{\vec{v_T}}$ of the orbiter is centripetal, because it derives only from the rotation velocity $\vec{v}_R$, the derivative of $\vec{v}_T$ being zero. Deriving the velocity 1 with respect to time, by noting $\vec{v}_R = \vec{\omega} \times \vec{r}$, so $\omega r = constant$, $\vec{\omega}$ being the rotation frequency, and remembering that $\vec{r}.\vec{v} = \dot{r} r$, we get :

$$\vec{a} = \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times \vec{v} = -\frac{\dot{r}}{r} \vec{\omega} \times r + \vec{\omega} \times \vec{v} = -\frac{\vec{\omega}}{r^2} \times [\vec{r} \times (\vec{r} \times \vec{v})]$$

and thus :

$$\vec{a} = -\frac{L v_R}{r^3} \vec{r} \tag{4}$$

This formula for the acceleration is mathematically identical to the one of Newton, because since L and v_R are constants, the kinematic factor $L v_R$ at numerator is also constant, and then corresponds to Newton's physical factor GM, both factors having the same dimension $m^3 s^{-2}$. Thus to reconcile Newton and Hamilton we have to state :

$$L v_R = GM \tag{5}$$

As expected the gravitational acceleration predicted from Hamilton's velocity has the same mathematical structure as Newton's, although it is centripetal but not linearly attractive like the one we used in the cannonball experiment. Let us recall here that a centripetal acceleration causes a perpendicular trajectory, while an attractive one causes a collinear trajectory. These two accelerations are therefore geometrically, and physically, of absolute different natures. Here is the cause of the problem we faced in the cannonball experiment. We considered a linearly attractive acceleration instead of a centripetal one, although Newton wrote in his treatise[?] (5) " *We said , in a mathematical way, to avoid all questions* *about the nature or quality of this force* [of gravitation] *which we would not be understood to determine by any hypothesis ; and therefore call it by the general name of centripetal force as it is a force which is directed towards some center ;* [...]". Therefore Newton entitles us to use a centripetal acceleration in the cannonball experiment, we can then reinterpret it from Hamilton's kinematics point of view.

Let first explain what is $\vec{v}_T$ with regards to $\vec{v}_R$. If $\vec{v}_T$ is zero, only $\vec{v}_R$ exists, and the motion is a uniform rotation, the cannonball is satellited in a circular orbit, with a velocity $v = v_R = \sqrt{GM/r}$. Thus $\vec{v}_T$ is a velocity that prevents the cannonball from orbiting freely around the Earth, and the derivative of a velocity being an acceleration, we deduce that $\vec{v}_T$ is the integral of the accelerations (therefore the forces) of "friction" that prevent the cannonball from orbiting : the cannonball is slowed down by the cannon, which is slowed down by the ground, which is slowed down by the underground, etc. At start of the experiment, when the cannonball is at rest, it must nonetheless be a body in a gravitational field, and thus respect Hamilton's velocity 1. The only way to achieve this is to verify $\vec{v}_R = -\vec{v}_T$, so in this case both translation and rotation velocities have the same intensity, but opposite directions, therefore knowing 3 the cannonball is at rest on a parabola.

When adding the ejection velocity $\vec{v}_0$ to the cannonball, it is also a translation velocity, and then it is added to the existing one $\vec{v}_T$: $\vec{v}_T{}' = \vec{v}_T + \vec{v}_0$. The figure 4 shows all these velocities during the flight. Consequently, at all times during the flight the velocity is given by Hamilton's expression 1, and therefore the trajectory is a Keplerian conic, a circle, an ellipse, a parabola or an hyperbola depending on the intensity and direction of $\vec{v}_0$, and at all times the centripetal acceleration has the same value as Newton's. The "*curve lines*" of Newton are then all Keplerian conics, of course with a constant angular momentum.

A particular notice has to be made about the body falling. We all experienced that an apple falls from the tree on a straight line to the ground. But the straight line is not a particular case nor a limit of the conic equation 3, although the apple is a body in a gravitational field, thus a Keplerian orbiter. Such paradox is solved by considering an ellipse so flat that it can be approximated locally like a straight line. Indeed, when the apple is fixed to the tree, its velocity is null and therefore $\vec{v}_T = -\vec{v}_R$, but when it disconnects from the tree, it is freed of a small part of the frictions that disabled its orbiting, so the translation

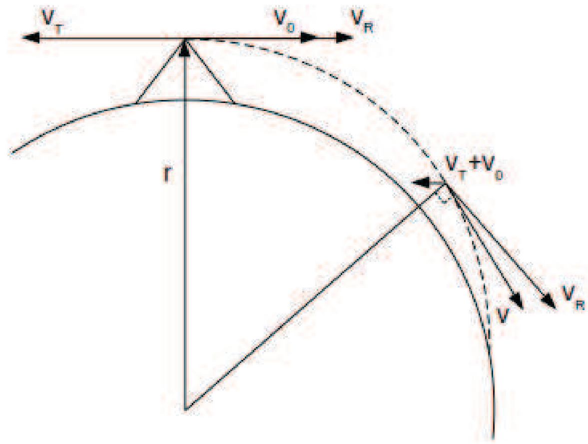

FIG. 4. Construction of the cannonball velocity $\vec{v}$ from Hamilton's kinematics point of view. $\vec{v}_R$ is the orbital velocity, $\vec{v}_T$ is the translation velocity which opposes to $\vec{v}_R$ at rest, $\vec{v}_0$ is the ejection velocity that must be added to $\vec{v}_T$. Note that $\vec{v}_R$ is always perpendicular to the vector radius.

velocity is slightly lowered : $v_T = 0.9999...v_R$. Knowing the definition 3 of the eccentricity, we have $e = 0.9999...$, and this describes a very flat ellipse which focus is at the Earth center, and minor axis is barely more than a few nanometers. Such a flat ellipse can be approximated locally to a straight line, because of its so small curvature. The Earth would be transparent and all its mass contained into a mathematical point at its center of mass, the apple will fall towards it, turn around it, and come back to its initial position, like any satellite does. A body falling is nothing else but a satellite on a Keplerian conic.

IV. CONCLUSION

We solved the non constancy of the angular momentum in Newton's cannonball experiment by replacing the linear attractive acceleration by Hamilton's centripetal one, consistently with Newton's own description[4] of the gravitational force being centripetal.

It is remarkable that Newton did not write nor say anything explicitly about this most simple effect of the gravitation that we all experience each day, the body falling. No record from Newton talks explicitly of this physical issue. One of the most emblematic thing that he wrote in his Principia about the attraction between bodies is the following theorem[5] : "*Two bodies attract each other according to the following laws. First, if two homogeneous spherical bodies attract each other by mutual forces, these forces will be mutually directed towards the* *centers of the spheres, and will be inversely proportional to the square of the distances between the centers of these spheres.*". From such a theorem, it is quite logical to think that a body will fall to the surface of the Earth on a straight line, and this is indeed what we inferred from Newton. But this might be a misinterpretation of his writings, because he was very clear from the above quotes : "*these forces will be mutually directed towards the centers of the spheres*" but "[I] *call it by the general name of centripetal force*". When looking at the apple falling on what appears to be a straight line, we choosed a linear attractive acceleration, infering implicitly that Newton's centripetal acceleration could transform into a linear one. But these two accelerations are geometrically incompatibles, one is not a particular case nor a limit of the other, one cannot transform into the other. Consequently, as we saw above, confusing both leads to break Kepler's laws. However, it did not harm our experimental processes as far as at the scale these experiments were done, approximating locally the real flat Keplerian ellipse with a straight line is acceptable. But for Newton's cannonball experiment we cannot stand this approximation any more, because the scale of the experiment is larger than local.

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[1] Isaac Newton, A Treatise of the System of the World, Londres, F. Fayram, 1728, p 6.

[2] L. Landau & E. Lifchitz, Mechanics, Ed. Mir, Moscow, 1966, §15.

[3] W. R. Hamilton, The hodograph, or a new method of expressing in symbolic language the Newtonian law of attraction, Proc. R. Ir. Acad. III , 344353 (1845).

[4] Isaac Newton, A Treatise of the System of the World, Londres, F. Fayram, 1728, p 5.

[5] Isaac Newton, Philosophiæ naturalis principia mathematica, Book I, Proposition 75, Theorem 35