ON THE DISTRIBUTION OF THE ℓ FUNCTION

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Abstract. In this short note, we study the distribution of the ℓ function, particularly on the primes. We study various elementary properties of the ℓ function. We relate the ℓ function to prime gaps, offering a motivation for a further studies of this function.

1. **Introduction**

The factorial function, $n!$, is of fundamental importance in numerous fields of both pure and applied mathematics, serving as a cornerstone in combinatorics, number theory, and analysis. Its significance arises not only from its combinatorial interpretation, as it counts permutations of a set of size n, but also from its deep connections to various branches of mathematics such as complex analysis, asymptotic analysis, and even probability theory. The factorial function is one among many transcendental functions that has been rigorously studied for centuries, providing insights into both theoretical and practical problems. Historically, the factorial function has evolved through the contributions of many mathematicians, including Euler, Stirling, and Gauss, who explored its properties in connection with infinite products and the gamma function. For a detailed historical account concerning the evolution of the factorial function, the reader is referred to [2], which offers a comprehensive survey of its development from antiquity to modern mathematics. The function itself is formally defined as

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$$

for any natural number $n \geq 1$, with the convention $0! = 1$. For large values of n, the factorial function exhibits rapid growth, and its asymptotic behavior is well-captured by Stirling's approximation, an essential result in asymptotic analysis. Stirlings formula is given by

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

which provides an efficient approximation of $n!$ for sufficiently large values of n, and has been extensively used in various branches of mathematical analysis and statistical mechanics (see [1] for refinements and improvements of this approximation).

In this note, we introduce a new function, termed the ℓ function, which, while distinct from the factorial, exhibits certain recursive properties that parallel the behavior of $n!$. The ℓ function is recursively defined based on the parity of its

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input and is motivated by a novel approach to studying integer sequences and their relationships with fundamental operations. Unlike the factorial, which grows super-exponentially, the ℓ function grows at a more tempered rate, yet retaining a significant mathematical structure. In some sense, it can be viewed as a variant of the factorial function, possessing deep recursive characteristics that warrant further exploration in the context of number theory and combinatorial analysis. By exploring this function, we aim to uncover new insights, particularly in relation to its interactions with prime numbers and the intriguing behavior of prime gaps.

2. The ℓ function

Definition 2.1. Let n be a natural number, then we set

$$\ell(n) := \begin{cases} \frac{n}{2}\ell(\frac{n}{2}) \quad \text{if} \quad n \quad \text{is} \quad \text{even} \\ \frac{n-1}{2}\ell(\frac{n-1}{2}) \quad \text{if} \quad n > 1 \quad \text{is} \quad \text{odd} \\ 1 \quad \text{if} \quad n = 1. \end{cases}$$

The little ℓ function, though completely different from the factorial function, can be seen to be performing a role similar to the factorial function. The factorial function on an integer n iterates downwards on n till it's value gets to 1, so does the little ℓ function. That is, $n! = n(n-1)!$ where as $\ell(n) = \frac{n}{2}\ell(\frac{n}{2})$ or $\ell(n) = \frac{n-1}{2}\ell(\frac{n-1}{2})$, according as n is odd or even.

Example 2.2. (a) $\ell(24) = 12 \cdot 6 \cdot 3 \cdot 1 = 216$, $\ell(100) = 50 \cdot 25 \cdot 12 \cdot 6 \cdot 3 \cdot 1 = 270000$.

(b) $\ell(55) = 27 \cdot 13 \cdot 6 \cdot 3 \cdot 1 = 6318$, $\ell(127) = 63 \cdot 31 \cdot 15 \cdot 7 \cdot 3 \cdot 1 = 615195$.

Remark 2.3. The little ℓ function is somewhat akin to the factorial function. However the growth rate of the factorial function supersedes that of the ℓ function. Below is a table that gives the distribution of the little ℓ function for the first thirty values of the integers.

3. Distribution of the little ℓ function

Values of n	The little ℓ function on n	Values of the little $\ell(n)$ function
1	$\ell(1)$	1
2	$\ell(2)$	1
3	$\ell(3)$	1
4	$\ell(4)$	2
5	$\ell(5)$	2
6	$\ell(6)$	3
7	$\ell(7)$	3
8	$\ell(8)$	8
9	$\ell(9)$	8
10	$\ell(10)$	10
11	$\ell(11)$	10
12	$\ell(12)$	18
13	$\ell(13)$	18
14	$\ell(14)$	21
15	$\ell(15)$	21
16	$\ell(16)$	64
17	$\ell(17)$	64
18	$\ell(18)$	72
19	$\ell(19)$	72
20	$\ell(20)$	100
21	$\ell(21)$	100
22	$\ell(22)$	110
23	$\ell(23)$	110
24	$\ell(24)$	216
25	$\ell(25)$	216
26	$\ell(26)$	234
27	$\ell(27)$	234
28	$\ell(28)$	294
29	$\ell(29)$	294
30	$\ell(30)$	315

Proposition 1 (Parity behaviour)**.** *For any even number n, the ℓ function satisfies $\ell(n) = \ell(n+1)$. That is, the little ℓ function returns the same value for consecutive even and odd numbers.*

Proof. The proof is trivial. Observe that from the definition of the ℓ function, we have $\ell(n) = \frac{n}{2}\ell(\frac{n}{2})$ if n is even and for $n+1$ (which is odd), we have $\ell(n+1) = \frac{n}{2}\ell(\frac{n}{2})$. Both recursive cases evaluate to the same value, proving that for all even n, the function is equal for consecutive even and odd numbers. □

Indeed, we can observe from the above table that $\ell(n) = \ell(n+1)$ for any even number n.

Proposition 2 (monotonicity of $\ell(n)$)**.** *The function ℓ function is non-decreasing for all $n \geq 1$. That is, if $n_1 < n_2$, then $\ell(n_1) \leq \ell(n_2)$.*

Proof. The proof is trivial. We can proceed by induction on n. In the case $n = 1$, we have by definition, $\ell(1) = 1$. For $n = 2$, we have $\ell(2) = \frac{2}{2} \cdot \ell(1) = 1$. Assume that for all $k < n$, $\ell(k) < \ell(k+1)$. We consider two cases: when n is even and when n is odd. In the case n is even, $n = 2m$, then $\ell(n) = \frac{n}{2}\ell\left(\frac{n}{2}\right) = m\ell(m)$. By the inductive hypothesis $\ell(m) \geq \ell(m-1)$ and $m > 1$, it follows that $\ell(n) \geq \ell(n-1)$. If n is odd $(n = 2m+1)$ then $\ell(n) = \frac{n-1}{2}\ell\left(\frac{n-1}{2}\right) = m\ell(m)$. Similarly, by the inductive hypothesis, $\ell(n) \geq \ell(n-1)$ holds, as $m > 1$. Thus, by induction, the function $\ell(n)$ is non-decreasing for all $n \geq 1$. □

3.1. **Some Heuristics.** Heuristically, choose any integer n at random and apply the little ℓ function on n. We remark that n can either be odd or even. Without loss of generality let us assume n is even, then the little ℓ on n is given by $\ell(n) = \frac{n}{2}\ell(\frac{n}{2})$. Again $\frac{n}{2}$ can either be odd or even. Without loss of generality, let us suppose $\frac{n}{2}$ is odd, then $\ell(n) = \frac{n}{2}(\frac{n/2-1}{2})\ell(\frac{n/2-1}{2}) \approx (\frac{n}{2})(\frac{n}{2})\frac{1}{2}\ell(\frac{n/2-1}{2})$. Again $\frac{n/2-1}{2}$ can either be odd or even. Without loss of generality, let us assume $\frac{n/2-1}{2}$ is even, then we find that $\ell(n) \approx (\frac{n}{2})^2\frac{1}{2}(\frac{n/2-1}{4})\ell(\frac{n/2-1}{4}) \approx (\frac{n}{2})^3(\frac{1}{2})^3\ell(\frac{n/2-1}{4})$. Again without loss of generality, let us assume $\frac{n/2-1}{4}$ is even, then we find that

$$\ell(n) \approx \left(\frac{n}{2}\right)^3\left(\frac{1}{2}\right)^3\ell\left(\frac{n/2-1}{4}\right) \approx \left(\frac{n}{2}\right)^3\left(\frac{1}{2}\right)^3\left(\frac{n}{16}\right)\ell\left(\frac{n/2-1}{8}\right)$$
$$\approx \left(\frac{n}{2}\right)^4\left(\frac{1}{2}\right)^6\ell\left(\frac{n/2-1}{8}\right).$$

Let $P(\ell(n))$ denotes the number of factors of the $\ell(n)$ function, or the frequency at which the $\ell(n)$ converges to 1, then it can be seen that $P(\ell(n)) = (1+o(1))\log_2 n$. Then by iterating the little $\ell(n)$ function, we obtain approximately

$$\ell(n) \sim C\left(\frac{n}{2}\right)^{\log_2 n}\left(\frac{1}{2}\right)^{\log_2 n}$$

where $\log_2 n := \lfloor\frac{\log n}{\log 2}\rfloor$. The asymptotic and inequality are therefore obvious from the ensuing heuristics:

Theorem 3.1. *For any natural number n, we have that*

$$\ell(n) \sim C\left(\frac{n}{2}\right)^{\log_2 n}\left(\frac{1}{2}\right)^{\log_2 n}$$

for some constant $C > 0$ that depends on the initial condition and the recursive structure of n.

We pose the following problem:

Problem 1. Determine the constant C in Theorem 3.1.

Theorem 3.2. *For all $n > 1$, we have*

$$\ell(n) < \left(\frac{n}{2}\right)^{\log_2 n}.$$

4. The distribution of the ℓ function on the primes

This section is devoted to the of the elementary properties of the ℓ function on the primes.

Theorem 4.1. *Let $p > 2$ be a prime number. Then the inequality*

$$\frac{p-1}{2} \leq \ell(p) \leq \left(\frac{p}{2}\right)^{\log_2 p}$$

holds.

Proof. Suppose p is an odd prime, then by definition of the ℓ function we have $\ell(p) = (\frac{p-1}{2})\ell(\frac{p-1}{2})$. Since $\ell(\frac{p-1}{2}) \geq 1$, the lower bound follows immediately. Using the upper bound $\ell(n) < (\frac{n}{2})^{\log_2 n}$, the upper bound follows immediately. □

Theorem 4.2 (Bound on gaps between primes)**.** *Let p and q be odd primes with $q > p$, then the inequality*

$$(q-p)\ell(\frac{p-1}{2}) < 2(\ell(q) - \ell(p))$$

holds.

Proof. Using the definition of the ℓ function, we have $\ell(p) = (\frac{p-1}{2})\ell(\frac{p-1}{2})$ and $\ell(q) = (\frac{q-1}{2})\ell(\frac{q-1}{2})$ so that by taking their difference and using the fact that the ℓ function is non-decreasing, we deduce that $\ell(q) - \ell(p) = (\frac{q-1}{2})\ell(\frac{q-1}{2}) - (\frac{p-1}{2})\ell(\frac{p-1}{2}) \geq (\frac{q-1}{2})\ell(\frac{p-1}{2}) - (\frac{p-1}{2})\ell(\frac{p-1}{2}) = (\frac{q-p}{2})\ell(\frac{p-1}{2})$ and the inequality follows immediately. □

The theorem of prime gap is an interesting and yet a challenging area of research. It is often considered a hub for most prime number theory problems and conjectures. The next result relates the gap between consecutive primes to the ℓ function on the primes. This does suggest the importance and the profound relation of the ℓ function in aiding our understanding of the gap between consecutive primes.

Theorem 4.3 (Gap between consecutive primes)**.** *Let p_n and p_{n+1} with $g_n = p_{n+1} - p_n$ denoting the prime gap. Then the following inequality*

$$\frac{2\ell(p_{n+1}) - p_n\ell(p_n)}{\ell(p_n)} \leq g_n \leq \frac{2\ell(p_{n+1}) - (p_n - 1)\ell(\frac{p_n-1}{2})}{\ell(\frac{p_n-1}{2})}$$

holds for all $n \geq 2$.

Proof. We note that $\ell(p_{n+1}) = \left(\frac{p_{n+1}-1}{2}\right)\ell\left(\frac{p_{n+1}-1}{2}\right)$ and $\ell(p_n) = \left(\frac{p_n-1}{2}\right)\ell\left(\frac{p_n-1}{2}\right)$. Using the relation $g_n = p_{n+1} - p_n$, we can write $\ell(p_{n+1}) = \left(\frac{p_n+g_n-1}{2}\right)\ell\left(\frac{p_n+g_n-1}{2}\right)$. Since the primes p_n and p_{n+1} are consecutive, it is clear that $g_n < p_n$. It is known that the function $\ell(n)$ is non-decreasing, so that we deduce

$$\ell\left(\frac{p_n + g_n - 1}{2}\right) \leq \ell(p_n).$$

Putting everything together, we obtain

$$\ell(p_{n+1})-\frac{p_n}{2}\ell(p_n)\leq \frac{g_n}{2}\ell(p_n)$$

and the claimed lower bound for the gap g_n is a consequence. We now analyze the upper bound. Since the function ℓ is non-decreasing, we have the inequality

$$\ell(p_{n+1})=\left(\frac{p_n+g_n-1}{2}\right)\ell\left(\frac{p_n+g_n-1}{2}\right)\geq \left(\frac{p_n+g_n-1}{2}\right)\ell\left(\frac{p_n-1}{2}\right)$$

and the upper bound follows immediately from this relation. □

We now use the prime number theorem to obtain an asymptotic for the ℓ function on the primes. We have the following growth rate on the sequence of primes.

Theorem 4.4 (Asymptotic on the primes)**.** *Let p_n denotes the n^{th} prime number, then*

$$\ell(p_n)\sim C\frac{(n\log n)^{\log_2 n}}{(2^{\log_2 n})^2}$$

for some constant $C>0$ depending on the recursive structure of p_n.

Proof. Using the asymptotic $\ell(p_n)\sim C\frac{(p_n)^{\log_2 p_n}}{(2^{\log_2 p_n})^2}$ and the prime number theorem of the form $p_n\sim n\log n$, the asymptotic is an immediate consequence. □

5. Some additive properties of the ℓ function

Some additive properties of the ℓ function on consecutive integers is explored in this section. We make the following observation, which allows us to control the sum of ℓ function.

Proposition 3 (Additive property)**.** *The following identity and inequality*

$$\ell(n)+\ell(n+1)=n\ell(\frac{n}{2})$$

if n is even and

$$(n-1)\ell\left(\frac{n-1}{2}\right)\leq \ell(n)+\ell(n+1)\leq (n+1)\ell\left(\frac{n+1}{2}\right)$$

if n is odd hold.

Proof. Suppose n is even, then $\ell(n)+\ell(n+1)=\frac{n}{2}\ell(\frac{n}{2})+\frac{n}{2}\ell(\frac{n}{2})=n\ell(\frac{n}{2})$. For the case n is odd, we find that $\ell(n)+\ell(n+1)=\left(\frac{n-1}{2}\right)\ell\left(\frac{n-1}{2}\right)+\left(\frac{n+1}{2}\right)\ell\left(\frac{n+1}{2}\right)\leq (n+1)\ell\left(\frac{n+1}{2}\right)$ using the property that ℓ is non-decreasing. On the other hand, we obtain $\ell(n)+\ell(n+1)=\left(\frac{n-1}{2}\right)\ell\left(\frac{n-1}{2}\right)+\left(\frac{n+1}{2}\right)\ell\left(\frac{n+1}{2}\right)\geq (n-1)\ell\left(\frac{n-1}{2}\right)$ where we have used the property that the function ℓ is non-decreasing. □

6. Relationship of the ℓ function to other number theoretic functions

In this section we study the relationship between the ℓ function with other number theoretic function. This relationship could offer us a better understanding the distribution of this function on the integers.

Theorem 6.1. *The following asymptotic and identities hold:*

(i)
$$\sum_{d|n} \ell(d) \sim C \sum_{d|n} d^{\log_2 n}$$
for all n sufficiently large and for some constant depending on the recursive structure ℓ. In particular, we have
$$\sum_{d|n} \ell(d) \sim C\sigma_{\log_2 n}(n)$$
and additionally that
$$\sum_{d|n} \ell(d) \le K(\sigma(n))^{\log_2 n}$$
where $\sigma_k := \sum_{d|n} d^k$.

(ii) *For all n sufficiently large positive integers, there exists a constant $C > 0$ such that $\phi(n) \le C\ell(n)$, where ϕ denotes the Euler totient function.*

Proof. The proof follows trivially by using the asymptotic of the $\ell(d)$. □

7. Conclusion and further discussions

A study of the $\ell(n)$ function presents an intriguing avenue for further research within number theory. Some elementary properties of the $\ell(n)$ function have now been established in this study. Investigating the distribution of values taken by the $\ell(n)$ function in relation to prime numbers could yield insights into patterns akin to those seen in the distribution of primes. This could involve statistical analyses or heuristic evaluations that compare the growth rates of $\ell(n)$ with known prime distributions.

A further studies of how the $\ell(n)$ function interacts with other number theoretic functions, such as the divisor function or the sum of divisors function, could unveil new relationships. For instance, conjecturing whether the ratio $\frac{\ell(n)}{d(n)}$, where $d(n)$ is the number of divisors of n exhibits specific asymptotic behavior might provide fertile ground for conjectures and proofs.

The multiplicative structure of $\ell(n)$ invites conjectures is also a potentially fruitful area to study. In particular, investigating whether $\ell(mn) = \ell(m) \cdot \ell(n)$ holds for certain classes of integers m and n could lead to new findings.

Identities and inequalities relating the $\ell(n)$ function to other known number theoretic functions, such as the sum of divisors function, could reveal further properties of the $\ell(n)$ function. A conjecture could posit that there exists a constant $C > 0$ such that $\ell(n) \le C \cdot \sigma(n)$ for all integers n. We make the following conjecture

Conjecture 7.1 (Divisor conjecture)**.** For any prime p the function $\ell(n)$ satisfies the congruence $\ell(n) \equiv 0 \pmod{p}$ for infinitely many n.

Conjecture 7.2 (Prime factorization conjecture)**.** The prime factorization of $\ell(n)$ for $n \geq 7$ contains at least two distinct prime factors. Furthermore, the set of distinct prime factors of $\ell(n)$ has positive density.

Conjecture 7.3 (Bounded growth prime factor conjecture)**.** For any integer n, the number of distinct prime factors of $\ell(n)$ is bounded above by $\log(\ell(n))$.

The potential connections between the $\ell(n)$ function and broader themes in number theory offer a rich landscape for conjecture and exploration. A further investigation of these areas may not only illuminate the properties of $\ell(n)$ but also enhance our overall understanding of distribution of the primes.

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