An interpretation of Ramanujan Mock θ-functions

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0-Abstract:

In this paper I show a several generalizations of the Ramanujan's Mock θ-functions. Using sigma and pi operators and defining the series.

1- Functions and their generalization:

In a letter of 1920 S. Ramanujan send from India to G. H. Hardy the following formulas which I had generalized, I used [1] to copy from original sequences and I used [2] to help me in some numerical series, but mostly is a creative and deductive thinking. I first enunciate the original writing of Ramanujan and then adapt it by my way.

$$f(q)=1+\frac{q}{(1+q)^2}+\frac{q^4}{(1+q)^2(1+q^2)^2}+\ldots$$

$$f(q)=1+\sum_{i=1}^{\infty}\left(\frac{q^{\sum_{n=1}^{\infty}(2n-1)}}{\prod_{k=1;\,j=1}^{\infty}(1+q^k)^2}\right)$$

$$\varphi(q)=1+\frac{q}{1+q^2}+\frac{q^4}{(1+q^2)(1+q^4)}+\ldots$$

$$\varphi(q)=1+\sum_{i=1}^{\infty}\left(\frac{q^{\sum_{n=1}^{\infty}(2n-1)}}{\prod_{k=1;\,j=1}^{\infty}(1+q^{2k})}\right)$$

$$\psi(q)=\frac{q}{1-q}+\frac{q^4}{(1-q)(1-q^3)}+\frac{q^9}{(1-q)(1-q^3)(1-q^5)}+\ldots$$

$$\psi(q)=\sum_{i=1}^{\infty}\left(\frac{q^{\sum_{n=1}^{\infty}(2n-1)}}{\prod_{k=1;\,j=1}^{\infty}(1-q^{(2k-1)})}\right)$$

$$\chi(q)=1+\frac{q}{1-q+q^2}+\frac{q^4}{(1-q+q^2)(1-q^2+q^4)}+\ldots$$

$$\chi(q)=1+\sum_{i=1}^{\infty}\left(\frac{q^{\sum_{n=1}^{\infty}(2n-1)}}{\prod_{k=1;j=1}^{\infty}(1-q^k+q^{2k})}\right)$$

$$2\varphi(-q)-f(q)=f(q)+4\psi(-q)=\frac{1-2q+2q^4-2q^9+\ldots}{(1+q)(1+q^2)(1+q^3)\ldots}=$$

$$=\frac{1+\sum_{i=1}^{\infty}(-1)^i 2q^{\sum_{n=1}^{\infty}(2n-1)}}{\prod_{k=1;j=1}^{\infty}(1+q^k)}$$

$$4\chi(q)-f(q)=3\frac{(1-2q^3+2q^{12}-\ldots)^2}{(1-q)(1-q^2)(1-q^3)\ldots}=$$

$$=3\frac{1+(\sum_{i=1;n=1}^{\infty}(-1)^i 2q^{(3n^2)})^2}{\prod_{k=1;j=1}^{\infty}(1-q^k)}$$

Mock θ-functions (of 5th order)

$$f(q)=1+\frac{q}{1+q}+\frac{q^4}{(1+q)(1+q^2)}+\ldots$$

$$f(q)=1+\sum_{i=1}^{\infty}\left(\frac{q^{\sum_{n=1}^{\infty}(2n-1)}}{\prod_{j=1;k=1}^{\infty}(1+q^k)}\right)$$

$$\psi(q)=q+q^3(1+q)+q^6(1+q)(1+q^2)+q^{10}(1+q)(1+q^2)(1+q^3)+\ldots$$

$$\psi(q)=q+\sum_{i=1}^{\infty}\left(q^{(1+\sum_{n=2}^{\infty}n)}\right)\prod_{j=1;k=1}^{\infty}(1+q^k)$$

$$\chi(q)=1+\frac{q}{1-q^2}+\frac{q^2}{(1-q^3)(1-q^4)}+\frac{q^3}{(1-q^4)(1-q^5)(1-q^6)}+\ldots$$

$$\chi(q)=1+\sum_{i=1;n=1}^{\infty}\left(\frac{q^n}{\prod_{j=1;k=0}^{\infty}(1-q^{(n+1+k)})}\right)$$

$$\chi(q)=1+\frac{q}{1-q}+\frac{q^3}{(1-q^2)(1-q^3)}+\frac{q^5}{(1-q^3)(1-q^4)(1-q^5)}+\ldots$$

$$\chi(q)=1+\sum_{i=1;n=1}^{\infty}\left(\frac{q^{(2n-1)}}{\prod_{j=1;k=[2n-i,2n-1]}^{\infty}(1-q^k)}\right)$$

$$f(-q)+2F(q^2)-2=\varphi(-q^2)+\psi(-q)=2\varphi(-q^2)-f(q)=\frac{1-2q+2q^4-2q^9+\ldots}{(1-q)(1-q^4)(1-q^6)(1-q^9)\ldots}=$$

$$=\frac{1\sum_{i=1}^{\infty}\left((-1)^i 2q^{(\sum_{n=1}^{\infty}2n-1)}\right)}{\prod_{j=1;k=\sum_{n\geq 2}^{\infty}\left(\frac{1}{a(n)^s}=\frac{\zeta(s)^2+\zeta(2s)}{2\zeta(s)}\right)}^{\infty}(1-q^k)}$$

$$\psi(q)-F(q^2)+1=q\frac{1+q^2+q^6+q^{12}+\ldots}{(1-q^8)(1-q^{12})(1-q^{28})\ldots}=$$

$$=q\frac{1+\sum_{i=1}^{\infty}q^{(\sum_{n=1}^{\infty}2n)}}{\prod_{j=1;k=F_{(2n-1)}+F_{(2n+1)}}^{\infty}(1-q^k)}$$

Mock θ-functions (of 5th order)

$$f(q)=1+\frac{q^2}{(1+q)}+\frac{q^6}{(1+q)(1+q^2)}+\frac{q^{12}}{(1+q)(1+q^2)(1+q^3)}+\ldots$$

$$f(q)=1+\sum_{i=1}^{\infty}\left(\frac{q^{\sum_{n=1}^{\infty}(2n)}}{\prod_{k=1;j=1}^{\infty}(1+q^k)}\right)$$

$$\varphi(q)=q+q^4(1+q)+q^9(1+q)(1+q^2)+\ldots$$

$$\varphi(q)=\sum_{i=1}^{\infty}q^{\sum_{n=1}^{\infty}(2n-1)}\prod_{j=1;i=2,k=1}^{\infty}(1+q^k)$$

Notice that Product starts act in i=2.

$$\psi(q)=1+q(1+q)+q^3(1+q)(1+q^2)+q^6(1+q)(1+q^2)(1+q^3)+\ldots$$

$$\psi(q)=1+\sum_{i=1}^{\infty}q^{\sum_{n=1}^{\infty}n}\prod_{k=1,j=1}^{\infty}(1+q^k)$$

$$\chi(q)=\frac{1}{1-q}+\frac{q}{(1-q^2)(1-q^3)}+\frac{q^2}{(1-q^3)(1-q^4)(1-q^5)}+\ldots$$

$$\chi(q)=\sum_{i=1;n=0}^{\infty}\left(\frac{q^n}{\prod_{j=1;k=[n+1,2n+1]}^{\infty}(1+q^k)}\right)$$

$$F(q)=\frac{1}{1-q}+\frac{q^4}{(1-q)(1-q^3)}+\frac{q^{12}}{(1-q)(1-q^3)(1-q^5)}+\ldots$$

$$F(q)=\sum_{i=1}^{\infty}\left(\frac{q^{\sum_{n=0}^{\infty}(4n)}}{\prod_{j=1;k=[1,2i-1]}^{\infty}(1-q^k)}\right)$$

Mock θ-functions (of 7th order)

$$(i)=1+\frac{q}{1-q^2}+\frac{q^4}{(1-q^3)(1-q^4)}+\frac{q^9}{(1-q^4)(1-q^5)(1-q^6)}+\ldots$$

$$(i)=1+\sum_{i=1}^{\infty}\left(\frac{q^{\sum_{n=1}^{\infty}(2n-1)}}{\prod_{j=1;k=[i+1,i+m];m\in\mathbb{Z}^+}^{\infty}(1-q^k)}\right)$$

$$(ii)=\frac{q}{1-q}+\frac{q^4}{(1-q^2)(1-q^3)}+\frac{q^9}{(1-q^3)(1-q^4)(1-q^5)}+\ldots$$

$$(ii)=\sum_{i=1}^{\infty}\left(\frac{q^{\sum_{n=1}^{\infty}(2n-1)}}{\prod_{j=1;k=[i,i+m];m\in\mathbb{N}_0}^{\infty}(1-q^k)}\right)$$

$$(iii)=\frac{1}{1-q}+\frac{q^2}{(1-q^2)(1-q^3)}+\frac{q^6}{(1-q^3)(1-q^4)(1-q^5)}+\ldots$$

$$(iii)=\sum_{i=1}^{\infty}\left(\frac{q^{\sum_{n=0}^{\infty}(2n)}}{\prod_{j=1;k=[1,i+m];m\in\mathbb{N}_0}^{\infty}(1-q^k)}\right)$$

Ramanujan said that these are not related each other.

REFERENCES

[1] Ramanujan, Srinivasa, Hardy, G. H. , Aiyar, P. V. Seshu, Wilson, B. M. - Collected Papers of Srinivasa Ramanujan - Cambridge University Press, London [1927]
[2] https://oeis.org/