The zero-expansion behavior for the new tridimensional $3D$ spherical warp drive vector compared with the Natario original warp drive

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Abstract

The Natario warp drive appeared for the first time in 2001.Although the idea of the warp dive as a spacetime distortion that allows a spaceship to travel faster than light predated the Natario work by 7 years Natario introduced in 2001 the new concept of a propulsion vector to define or to generate a warp drive spacetime.Natario defined a warp drive vector for constant speeds in Polar Coordinates and Polar Coordinates uses only two dimensions and we know that a real spaceship is a tridimensional $3D$ object inserted inside a tridimensional $3D$ warp bubble that must be defined in real $3D$ Spherical Coordinates.The "ex-libris" of the Natario 2001 original paper was the so-called "zero-expansion" behavior in the expansion of the normal volume elements that occurs only in $2D$ Polar Coordinates but not in tridimensional $3D$ Spherical Coordinates.We demonstrate this affirmation in the present work.

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1 Introduction:

The Natario warp drive appeared for the first time in 2001.([1]).Although the idea of the warp dive as a spacetime distortion that allows a spaceship to travel faster than light predated the Natario work by 7 years Natario introduced in 2001 the new concept of a propulsion vector to define or to generate a warp drive spacetime.

This propulsion vector nX uses the form $nX = X^i e_i$ where X^i are the shift vectors responsible for the spaceship propulsion or speed and e_i are the Canonical Basis of the Coordinates System where the shift vectors are based or placed.

Natario (See pg 5 in [1]) defined a warp drive vector $nX = vs * (dx)$ where vs is the constant speed of the warp bubble and $*(dx)$ is the Hodge Star taken over the x-axis of motion in Polar Coordinates(See pg 4 in [1]).(see Appendix D in [2] about Polar Coordinates).The final form of the original Natario warp drive vector is given by $nX = vs * d(r\cos\theta)$.However Polar Coordinates are not real tridimensional $3D$ coordinates since it uses only the two Canonical Basis $\mathbf{e}_r$ and $\mathbf{e}_\theta$.

Natario used Polar Coordinates(See pg 4 in [1]) but for a real $3D$ Spherical Coordinates another warp drive vector must be calculated.Remember that a real spaceship is a tridimensional $3D$ object inserted inside a tridimensional $3D$ warp bubble that must be defined in real $3D$ Spherical Coordinates.In this work we discuss a very interesting feature present in the new warp drive vector in tridimensional $3D$ Spherical Coordinates also for constant speeds(See section 4 in [2]).The final form of the Hodge Star for this warp drive vector is calculated no longer over $*d(r\cos\theta)$ but instead over $*d(r\sin\phi\cos\theta)$ since this form uses all the tridimensional $3D$ Canonical Basis $\mathbf{e}_r$,$\mathbf{e}_\theta$ and $\mathbf{e}_\phi$.(see Appendix E in [2] about tridimensional $3D$ Spherical Coordinates).

In order to fully understand the idea we wish to present in this work familiarity with the warp drive vector in tridimensional $3D$ Spherical Coordinates([2]) or familiarity with the Natario original warp drive paper([1]) are required.

The very interesting feature we intend to present in this work is the following one:

- 1)-The original Natario 2001 paper describes a warp drive vector in which the expansion of the normal volume elements is zero.A "warp drive with zero expansion"(see the title of [1]).

- 2)-This "ex-libris" feature of the Natario work was possible because Natario used Polar Coordinates in two dimensions and used only the Canonical Basis $\mathbf{e}_r$ and $\mathbf{e}_\theta$. (We discuss this in Section 2 of this work.)

This work will demonstrate that in real tridimensional $3D$ Spherical Coordinates with all the tridimensional $3D$ Canonical Basis $\mathbf{e}_r$,$\mathbf{e}_\theta$ and $\mathbf{e}_\phi$ already present in a real physical scenario(a real tridimensional $3D$ spaceship) the "zero expansion behavior" no longer exists.(We discuss this in Section 3 of this work.)

This work was designed to be a companion to our work in [2].

2 The zero expansion Natario original warp drive vector in polar coordinates with a constant speed vs

The equation of the Natario vector $nX = vs * dx$(see pg 5 in [1]) is given by:(see Appendix A in [2] for the mathematical demonstrations).The Natario warp drive vectors are not properly $nX = vs * dx$ or $nX = -vs * dx$ but instead $nX = vs * d\left[f(r)r^2\sin^2\theta d\varphi\right]$ and $nX = -vs * d\left[f(r)r^2\sin^2\theta d\varphi\right]$ being $*$ the Hodge Star operator.(see Appendix D in [2] about Polar Coordinates)

$$\mathbf{X} \sim -v_s(t) * d\left[f(r)r^2\sin^2\theta d\varphi\right] \sim -2v_s f\cos\theta\mathbf{e}_r + v_s(2f + rf')\sin\theta\mathbf{e}_\theta \tag{1}$$

or

$$\mathbf{X} \sim v_s(t) * d\left[f(r)r^2\sin^2\theta d\varphi\right] \sim 2v_s f\cos\theta\mathbf{e}_r - v_s(2f + rf')\sin\theta\mathbf{e}_\theta \tag{2}$$

$$nX = X^r e_r + X^\theta e_\theta \tag{3}$$

With the contravariant shift vector components X^r and X^θ given by:

$$X^r = -2v_s f(r)\cos\theta \tag{4}$$

$$X^\theta = v_s(2f(r) + (r)f'(r))\sin\theta \tag{5}$$

or

$$X^r = 2v_s f(r)\cos\theta \tag{6}$$

$$X^\theta = -v_s(2f(r) + (r)f'(r))\sin\theta \tag{7}$$

Natario in its warp drive uses the polar coordinates r and θ.(see pgs 4 and 5 in [1]).We must examine now the extrinsic curvatures and the rate-of-strain stress tensor as described in pgs 354 and 355 in [3],Natario in pg 5 in [1],pg 92 in [4](with $-p = 0$ and $\mu = \frac{1}{2}$),pg 141 eqs 5.130 to 5.135 in [5] (with $\mu = -(\frac{1}{2})$ and $(\frac{2}{3})(\nabla.U = 0)$),pg 52 eq 15.17 in [6](with $-p = 0$ and $\eta = \frac{1}{2}$).Here we use the equations:

$$\mathbf{X} \sim -v_s(t) * d\left[f(r)r^2\sin^2\theta d\varphi\right] \sim -2v_s f\cos\theta\mathbf{e}_r + v_s(2f + rf')\sin\theta\mathbf{e}_\theta \tag{8}$$

$$nX = X^r e_r + X^\theta e_\theta \tag{9}$$

$$X^r = -2v_s f(r)\cos\theta \tag{10}$$

$$X^\theta = v_s(2f(r) + (r)f'(r))\sin\theta \tag{11}$$

The extrinsic curvature equations are:(Natario pg 5 in [1])

$$K_{rr} = \frac{\partial X^r}{\partial r} = -2v_s f' \cos\theta \tag{12}$$

$$K_{\theta\theta} = \frac{1}{r}\frac{\partial X^\theta}{\partial\theta} + \frac{X^r}{r} = v_s f' \cos\theta \tag{13}$$

$$K_{\varphi\varphi} = \frac{1}{r\sin\theta}\frac{\partial X^\varphi}{\partial\varphi} + \frac{X^r}{r} + \frac{X^\theta \cot\theta}{r} = v_s f' \cos\theta \tag{14}$$

$$K_{r\theta} = \frac{1}{2}\left[r\frac{\partial}{\partial r}\left(\frac{X^\theta}{r}\right) + \frac{1}{r}\frac{\partial X^r}{\partial\theta}\right] = v_s \sin\theta \left(f' + \frac{r}{2}f''\right) \tag{15}$$

$$K_{r\varphi} = \frac{1}{2}\left[r\frac{\partial}{\partial r}\left(\frac{X^\varphi}{r}\right) + \frac{1}{r\sin\theta}\frac{\partial X^r}{\partial\varphi}\right] = 0 \tag{16}$$

$$K_{\theta\varphi} = \frac{1}{2}\left[\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{X^\varphi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial X^\theta}{\partial\varphi}\right] = 0 \tag{17}$$

Note that polar coordinates are not real tridimensional $3D$ spherical coordinates since it uses only the two canonical casis $\mathbf{e}_r$ and $\mathbf{e}_\theta$.So the canonical basis $\mathbf{e}_\varphi$ and the term X^φ from the rate-of-strain stress tensor $\mathbf{X} = X^r\mathbf{e}_r + X^\theta\mathbf{e}_\theta + X^\varphi\mathbf{e}_\varphi$ do not exists in polar coordinates and then we have only $\mathbf{X} = X^r\mathbf{e}_r + X^\theta\mathbf{e}_\theta$.

This is the reason why $K_{r\varphi}$ and $K_{\theta\varphi}$ are also zero.

Since the term X^φ do not exists we must rewrite the extrinsic curvature $K_{\varphi\varphi}$ as follows:

$$K_{\varphi\varphi} = \frac{X^r}{r} + \frac{X^\theta \cot\theta}{r} \tag{18}$$

The expansion of the normal volume elements (the trace or the sum of the diagonalized extrinsic curvatures components $Tr(K)$) is given by:

$$Tr(K) = K_{rr} + K_{\theta\theta} + K_{\varphi\varphi} \tag{19}$$

Now we must verify the original results of the Natario zero expansion warp drive but step by step:

Remember that the main goal of this work is to demonstrate that the zero expansion warp drive behavior or better the "ex-libris" of the Natario 2001 work occurs only in polar coordinates in two dimensions and not in spherical coordinates in three dimensions.

Since this is a very important and "hot" topic we choose to present the detailed calculations one by one.

$$X^r = -2v_s f(r)\cos\theta \tag{20}$$

$$X^\theta = v_s(2f(r) + (r)f'(r))\sin\theta \tag{21}$$

$$K_{rr} = \frac{\partial X^r}{\partial r} = \frac{\partial[-2v_s f(r)\cos\theta]}{\partial r} = -2v_s\cos\theta\frac{\partial[f(r)]}{\partial r} = -2v_s f'\cos\theta \tag{22}$$

$$K_{\theta\theta} = \frac{1}{r}\frac{\partial X^\theta}{\partial\theta} + \frac{X^r}{r} = \frac{1}{r}\frac{\partial[v_s(2f(r) + (r)f'(r))\sin\theta]}{\partial\theta} + \frac{[-2v_s f(r)\cos\theta]}{r} \tag{23}$$

$$\frac{1}{r}\frac{\partial[v_s(2f(r) + (r)f'(r))\sin\theta]}{\partial\theta} = \frac{1}{r}[v_s(2f(r) + (r)f'(r))]\frac{\partial\sin\theta}{\partial\theta} = \frac{1}{r}[v_s(2f(r) + (r)f'(r))]\cos\theta \tag{24}$$

$$\frac{1}{r}[v_s(2f(r) + (r)f'(r))]\cos\theta = \frac{1}{r}[v_s(2f(r)]\cos\theta + \frac{1}{r}v_s[(r)f'(r))]\cos\theta \tag{25}$$

$$\frac{X^r}{r} = \frac{[-2v_s f(r)\cos\theta]}{r} \tag{26}$$

$$K_{\theta\theta} = \frac{1}{r}\frac{\partial X^\theta}{\partial\theta} + \frac{X^r}{r} = \frac{1}{r}[v_s(2f(r)]\cos\theta + \frac{1}{r}v_s[(r)f'(r))]\cos\theta + \frac{[-2v_s f(r)\cos\theta]}{r} \tag{27}$$

$$K_{\theta\theta} = \frac{1}{r}\frac{\partial X^\theta}{\partial\theta} + \frac{X^r}{r} = \frac{1}{r}v_s[(r)f'(r))]\cos\theta = v_s f'(r)\cos\theta \tag{28}$$

$$K_{\varphi\varphi} = \frac{X^r}{r} + \frac{X^\theta\cot\theta}{r} \tag{29}$$

$$\frac{X^\theta\cot\theta}{r} = \frac{[v_s(2f(r) + (r)f'(r))\sin\theta]\cot\theta}{r} = \frac{[v_s(2f(r) + (r)f'(r))]\cos\theta}{r} \tag{30}$$

$$\frac{X^\theta\cot\theta}{r} = \frac{[v_s(2f(r) + (r)f'(r))]\cos\theta}{r} = \frac{[v_s(2f(r)]\cos\theta}{r} + \frac{v_s[(r)f'(r))]\cos\theta}{r} \tag{31}$$

$$K_{\varphi\varphi} = \frac{X^r}{r} + \frac{X^\theta\cot\theta}{r} = \frac{[v_s(2f(r)]\cos\theta}{r} + \frac{v_s[(r)f'(r))]\cos\theta}{r} + \frac{[-2v_s f(r)\cos\theta]}{r} \tag{32}$$

$$K_{\varphi\varphi} = \frac{X^r}{r} + \frac{X^\theta\cot\theta}{r} = \frac{v_s[(r)f'(r))]\cos\theta}{r} = v_s f'(r)\cos\theta \tag{33}$$

The expansion of the normal volume elements (the trace or the sum of the diagonalized extrinsic curvatures components $Tr(K)$) is given by:

$$Tr(K) = K_{rr} + K_{\theta\theta} + K_{\varphi\varphi} \tag{34}$$

With the diagonalized extrinsic curvatures components being:

$$K_{rr} = \frac{\partial X^r}{\partial r} = \frac{\partial[-2v_s f(r)\cos\theta]}{\partial r} = -2v_s\cos\theta\frac{\partial[f(r)]}{\partial r} = -2v_s f'\cos\theta \tag{35}$$

$$K_{\theta\theta} = \frac{1}{r}\frac{\partial X^\theta}{\partial\theta} + \frac{X^r}{r} = \frac{1}{r}v_s[(r)f'(r))]\cos\theta = v_s f'(r)\cos\theta \tag{36}$$

$$K_{\varphi\varphi} = \frac{X^r}{r} + \frac{X^\theta\cot\theta}{r} = \frac{v_s[(r)f'(r))]\cos\theta}{r} = v_s f'(r)\cos\theta \tag{37}$$

Then we have:

$$Tr(K) = K_{rr} + K_{\theta\theta} + K_{\varphi\varphi} = -2v_s f'\cos\theta + v_s f'(r)\cos\theta + v_s f'(r)\cos\theta = 0 \to ZERO!!! \tag{38}$$

Remember that Natario in its warp drive uses the polar coordinates r and θ.(see pgs 4 and 5 in [1]) and polar coordinates are not real tridimensional $3D$ spherical coordinates since it uses only the two canonical casis $\mathbf{e}_r$ and $\mathbf{e}_\theta$.So the canonical basis $\mathbf{e}_\varphi$ and the term X^φ from the rate-of-strain stress tensor $\mathbf{X} = X^r\mathbf{e}_r + X^\theta\mathbf{e}_\theta + X^\varphi\mathbf{e}_\varphi$ do not exists in polar coordinates and then we have only $\mathbf{X} = X^r\mathbf{e}_r + X^\theta\mathbf{e}_\theta$.

The zero expansion behavior in the original Natario warp drive vector occurs only in $2D$ polar coordinates because the canonical basis $\mathbf{e}_\varphi$ and the term X^φ from the rate-of-strain stress tensor $\mathbf{X} = X^r\mathbf{e}_r + X^\theta\mathbf{e}_\theta + X^\varphi\mathbf{e}_\varphi$ do not exists in polar coordinates and then we have only $\mathbf{X} = X^r\mathbf{e}_r + X^\theta\mathbf{e}_\theta$ implying in the cancellation of the term $\frac{1}{r\sin\theta}\frac{\partial X^\varphi}{\partial\varphi}$ in the diagonalized extrinsic curvature term $K_{\varphi\varphi} = \frac{1}{r\sin\theta}\frac{\partial X^\varphi}{\partial\varphi} + \frac{X^r}{r} + \frac{X^\theta\cot\theta}{r}$

The existence of the the canonical basis $\mathbf{e}_\varphi$ and the term X^φ from the rate-of-strain stress tensor $\mathbf{X} = X^r\mathbf{e}_r + X^\theta\mathbf{e}_\theta + X^\varphi\mathbf{e}_\varphi$ in real tridimensional $3D$ spherical coordinates implies in the existence of the extra term $\frac{1}{r\sin\theta}\frac{\partial X^\varphi}{\partial\varphi}$ that changes the whole picture in the expansion of the normal volume elements and the zero expansion behavior ceases to exists.We will demonstrate this in the next section.

Natario worked with this warp drive vector:(pg 5 in [1])

$$\mathbf{X} \sim -v_s(t) * d\left[f(r)r^2 \sin^2\theta d\varphi\right] \sim -2v_s f \cos\theta \mathbf{e}_r + v_s(2f + rf')\sin\theta \mathbf{e}_\theta \tag{39}$$

With these extrinsic curvatures:

$$K_{rr} = \frac{\partial X^r}{\partial r} = -2v_s f' \cos\theta \tag{40}$$

$$K_{\theta\theta} = \frac{1}{r}\frac{\partial X^\theta}{\partial\theta} + \frac{X^r}{r} = v_s f' \cos\theta \tag{41}$$

$$K_{\varphi\varphi} = \frac{1}{r\sin\theta}\frac{\partial X^\varphi}{\partial\varphi} + \frac{X^r}{r} + \frac{X^\theta \cot\theta}{r} = v_s f' \cos\theta \tag{42}$$

or better:

$$K_{\varphi\varphi} = \frac{X^r}{r} + \frac{X^\theta \cot\theta}{r} = v_s f' \cos\theta \tag{43}$$

Since the term $\frac{1}{r\sin\theta}\frac{\partial X^\varphi}{\partial\varphi}$ do not exists in polar coordinates.

All these terms cancels in the expansion of the normal volume elements
$K_{rr} + K_{\theta\theta} + K_{\varphi\varphi} = -2v_s f' \cos\theta + v_s f'(r)\cos\theta + v_s f'(r)\cos\theta = 0 \rightarrow ZERO$!!!

In front of the bubble $K_{rr} < 0$ indicating a compression in the radial direction; this is however exactly balanced by $K_{\theta\theta} + K_{\varphi\varphi} = -K_{rr}$, corresponding to an expansion in the perpendicular direction.

Using this warp drive vector:

$$\mathbf{X} \sim v_s(t) * d\left[f(r)r^2 \sin^2\theta d\varphi\right] \sim 2v_s f \cos\theta \mathbf{e}_r - v_s(2f + rf')\sin\theta \mathbf{e}_\theta \tag{44}$$

The respective extrinsic curvatures are:

$$K_{rr} = \frac{\partial X^r}{\partial r} = 2v_s f' \cos\theta \tag{45}$$

$$K_{\theta\theta} = \frac{1}{r}\frac{\partial X^\theta}{\partial\theta} + \frac{X^r}{r} = -v_s f' \cos\theta \tag{46}$$

$$K_{\varphi\varphi} = \frac{1}{r\sin\theta}\frac{\partial X^\varphi}{\partial\varphi} + \frac{X^r}{r} + \frac{X^\theta \cot\theta}{r} = -v_s f' \cos\theta \tag{47}$$

These terms also cancels each other in the expansion of the normal volume elements
$K_{rr} + K_{\theta\theta} + K_{\varphi\varphi} = 2v_s f' \cos\theta - v_s f'(r)\cos\theta - v_s f'(r)\cos\theta = 0 \rightarrow ZERO$!!!

However the results are inverted:Now in front of the bubble $K_{rr} > 0$ indicating an expansion in the radial direction; this is however exactly balanced by $K_{\theta\theta} + K_{\varphi\varphi} = -K_{rr}$, corresponding to a contraction in the perpendicular direction.

3 The zero expansion behavior of the new warp drive vector in tridimensional $3D$ spherical coordinates with a constant speed vs

The equation of the new warp drive vector in tridimensional $3D$ spherical coordinates with a constant speed vs nX is given by:

$$nX = X^r e_r + X^\theta e_\theta + X^\phi e_\phi \tag{48}$$

With the contravariant shift vector components X^{rs}, X^θ and X^ϕ given by:
(see Section 4 and Appendix J in [2] for details)(see Appendix E in [2] about tridimensional $3D$ Spherical Coordinates).

$$X^r = vs(t)[\sin\phi][2f(r)\ cos\theta] \tag{49}$$

$$X^\theta = -vs(t)[\sin\phi][2f(r) + rf'(r)]\sin\theta] \tag{50}$$

$$X^\phi = [vs(t)cos\phi][cot\theta[2(f(r)) + (rf'(r))] \tag{51}$$

Natario in its warp drive uses the polar coordinates r and θ.(see pgs 4 and 5 in [1]).Also the equatorial plane $x - y$ makes an angle of 90 degrees with the $z - axis$ so $\sin\phi = 1$ and $cos\phi = 0$.

Then the contravariant components reduces to:

$$X^r = vs(t)[2f(r)\ cos\theta] \tag{52}$$

$$X^\theta = -vs(t)[2f(r) + rf'(r)]\sin\theta] \tag{53}$$

And we recover again the Natario warp drive vector in polar coordinates with a constant speed vs.

We must examine now the extrinsic curvatures and the expansion of the normal volume elements using the rate-of-strain stress tensor as described in pgs 354 and 355 in [3],Natario in pg 5 in [1],pg 92 in [4](with $-p = 0$ and $\mu = \frac{1}{2}$),pg 141 eqs 5.130 to 5.135 in [5] (with $\mu = -(\frac{1}{2})$ and $(\frac{2}{3})(\nabla.U = 0)$),pg 52 eq 15.17 in [6](with $-p = 0$ and $\eta = \frac{1}{2}$).Here we use only the equations of the diagonalized extrinsic curvature components:

$$K_{rr} = \frac{\partial X^r}{\partial r} \tag{54}$$

$$K_{\theta\theta} = \frac{1}{r}\frac{\partial X^\theta}{\partial\theta} + \frac{X^r}{r} \tag{55}$$

$$K_{\phi\phi} = \frac{1}{r\sin\theta}\frac{\partial X^\phi}{\partial\phi} + \frac{X^r}{r} + \frac{X^\theta\cot\theta}{r} \tag{56}$$

Now we are in real tridimensional $3D$ spherical coordinates and beyond the two canonical basis $\mathbf{e}_r$ and $\mathbf{e}_\theta$ also used in polar coordinates we have a new canonical basis $\mathbf{e}_\phi$ and a new term X^ϕ for the rate-of-strain stress tensor $\mathbf{X} = X^r\mathbf{e}_r + X^\theta\mathbf{e}_\theta + X^\phi\mathbf{e}_\phi$.

Our focus is a detailed study of the expansion of the normal volume elements given by:

$$Tr(K) = K_{rr} + K_{\theta\theta} + K_{\phi\phi} \tag{57}$$

$$K_{rr} = \frac{\partial X^r}{\partial r} \tag{58}$$

$$K_{\theta\theta} = \frac{1}{r}\frac{\partial X^\theta}{\partial \theta} + \frac{X^r}{r} \tag{59}$$

$$K_{\phi\phi} = \frac{1}{r\sin\theta}\frac{\partial X^\phi}{\partial \phi} + \frac{X^r}{r} + \frac{X^\theta \cot\theta}{r} \tag{60}$$

$$X^r = vs(t)[\sin\phi][2f(r)\ cos\theta] \tag{61}$$

$$X^\theta = -vs(t)[\sin\phi][2f(r) + rf'(r)]\sin\theta] \tag{62}$$

$$X^\phi = [vs(t)cos\phi][cot\theta[2(f(r)) + (rf'(r))] \tag{63}$$

The existence of the the canonical basis $\mathbf{e}_\phi$ and the term X^ϕ from the rate-of-strain stress tensor $\mathbf{X} = X^r\mathbf{e}_r + X^\theta\mathbf{e}_\theta + X^\phi\mathbf{e}_\phi$ in real tridimensional $3D$ spherical coordinates implies in the existence of the extra term $\frac{1}{r\sin\theta}\frac{\partial X^\phi}{\partial \phi}$ that changes the whole picture in the expansion of the normal volume elements and in the diagonalized extrinsic curvature term $K_{\phi\phi} = \frac{1}{r\sin\theta}\frac{\partial X^\phi}{\partial \phi} + \frac{X^r}{r} + \frac{X^\theta \cot\theta}{r}$and the zero expansion behavior ceases to exists.We will demonstrate this in this section.

Remember that the main goal of this work is to demonstrate that the zero expansion warp drive behavior or better the "ex-libris" of the Natario 2001 work occurs only in polar coordinates in two dimensions and not in spherical coordinates in three dimensions.

Since this is a very important and "hot" topic we choose to present the detailed calculations one by one and step by step.

$$X^r = vs(t)[\sin\phi][2f(r)\ \ cos\theta] \tag{64}$$

$$X^\theta = -vs(t)[\sin\phi][2f(r) + rf'(r)]\sin\theta] \tag{65}$$

$$X^\phi = [vs(t)cos\phi][cot\theta[2(f(r)) + (rf'(r))] \tag{66}$$

$$K_{rr} = \frac{\partial X^r}{\partial r} = \frac{\partial[[vs(t)[\sin\phi][2f(r)\ \ cos\theta]]}{\partial r} = [vs(t)[\sin\phi][2f'(r)\ \ cos\theta]] \tag{67}$$

$$K_{\theta\theta} = \frac{1}{r}\frac{\partial X^\theta}{\partial\theta} + \frac{X^r}{r} \tag{68}$$

$$\frac{X^r}{r} = \frac{vs(t)[\sin\phi][2f(r)\ \ cos\theta]}{r} \tag{69}$$

$$\frac{1}{r}\frac{\partial X^\theta}{\partial\theta} = \frac{1}{r}\frac{\partial[-vs(t)[\sin\phi][2f(r) + rf'(r)]\sin\theta]]}{\partial\theta} = \frac{1}{r}[[-vs(t)[\sin\phi][2f(r) + rf'(r)]\cos\theta]]] \tag{70}$$

$$\frac{1}{r}\frac{\partial X^\theta}{\partial\theta} = \frac{1}{r}[[-vs(t)[\sin\phi][2f(r)+rf'(r)]\cos\theta]]] = \frac{1}{r}[[-vs(t)[\sin\phi][2f(r)]\cos\theta]]]+\frac{1}{r}[[-vs(t)[\sin\phi][rf'(r)]\cos\theta]]] \tag{71}$$

$$K_{\theta\theta} = \frac{1}{r}\frac{\partial X^\theta}{\partial\theta}+\frac{X^r}{r} = \frac{1}{r}[[-vs(t)[\sin\phi][2f(r)]\cos\theta]]]+\frac{1}{r}[[-vs(t)[\sin\phi][rf'(r)]\cos\theta]]]+\frac{vs(t)[\sin\phi][2f(r)\ \ cos\theta]}{r} \tag{72}$$

$$K_{\theta\theta} = \frac{1}{r}\frac{\partial X^\theta}{\partial\theta} + \frac{X^r}{r} = +\frac{1}{r}[[-vs(t)[\sin\phi][rf'(r)]\cos\theta]]] = [-vs(t)[\sin\phi][f'(r)]\cos\theta]] \tag{73}$$

$$K_{\phi\phi} = \frac{1}{r\sin\theta}\frac{\partial X^\phi}{\partial\phi} + \frac{X^r}{r} + \frac{X^\theta\cot\theta}{r} \tag{74}$$

$$\frac{X^\theta\cot\theta}{r} = \frac{[-vs(t)[\sin\phi][2f(r) + rf'(r)]\sin\theta]]\cot\theta}{r} = \frac{[-vs(t)[\sin\phi][2f(r) + rf'(r)]\cos\theta]]}{r} \tag{75}$$

$$\frac{X^\theta\cot\theta}{r} = \frac{[-vs(t)[\sin\phi][2f(r) + rf'(r)]\cos\theta]]}{r} = \frac{[-vs(t)[\sin\phi][2f(r)]\cos\theta]]}{r} + \frac{[-vs(t)[\sin\phi][rf'(r)]\cos\theta]]}{r} \tag{76}$$

$$\frac{X^r}{r} + \frac{X^\theta\cot\theta}{r} = \frac{vs(t)[\sin\phi][2f(r)\ \ cos\theta]}{r} + \frac{[-vs(t)[\sin\phi][2f(r)]\cos\theta]]}{r} + \frac{[-vs(t)[\sin\phi][rf'(r)]\cos\theta]]}{r} \tag{77}$$

$$\frac{X^r}{r} + \frac{X^\theta \cot\theta}{r} = \frac{[-vs(t)[\sin\phi][rf'(r)]\cos\theta]]}{r} = [-vs(t)[\sin\phi][f'(r)]\cos\theta]] \tag{78}$$

Back to the expansion of the normal volume elements we have:

$$Tr(K) = K_{rr} + K_{\theta\theta} + K_{\phi\phi} \tag{79}$$

$$K_{rr} = \frac{\partial X^r}{\partial r} = [vs(t)[\sin\phi][2f'(r)\ cos\theta]] \tag{80}$$

$$K_{\theta\theta} = \frac{1}{r}\frac{\partial X^\theta}{\partial\theta} + \frac{X^r}{r} = [-vs(t)[\sin\phi][f'(r)]\cos\theta]] \tag{81}$$

$$K_{\phi\phi} = \frac{1}{r\sin\theta}\frac{\partial X^\phi}{\partial\phi} + \frac{X^r}{r} + \frac{X^\theta\cot\theta}{r} \tag{82}$$

$$K_{\phi\phi} = \frac{1}{r\sin\theta}\frac{\partial X^\phi}{\partial\phi} + [-vs(t)[\sin\phi][f'(r)]\cos\theta]] \tag{83}$$

$$Tr(K) = [vs(t)[\sin\phi][2f'(r)\ cos\theta]] + [-vs(t)[\sin\phi][f'(r)]\cos\theta]] + \frac{1}{r\sin\theta}\frac{\partial X^\phi}{\partial\phi} + [-vs(t)[\sin\phi][f'(r)]\cos\theta]] \tag{84}$$

$$Tr(K) = \frac{1}{r\sin\theta}\frac{\partial X^\phi}{\partial\phi} \tag{85}$$

From the results above we can see that the zero expansion behavior that occurs in polar coordinates in two dimensions and was the "ex-libris" of the Natario 2001 work ceases to exists in three dimensions in real $3D$ spherical coordinates due to the term $\frac{1}{r\sin\theta}\frac{\partial X^\phi}{\partial\phi}$ that do not vanishes in the expansion of the normal volume elements.

$$Tr(K) = \frac{1}{r\sin\theta}\frac{\partial X^\phi}{\partial\phi} = \frac{1}{r\sin\theta}\frac{\partial[[vs(t)cos\phi][cot\theta[2(f(r)) + (rf'(r))]]}{\partial\phi} \tag{86}$$

$$Tr(K) = \frac{1}{r\sin\theta}[-[vs(t)sin\phi][cot\theta[2(f(r)) + (rf'(r))]] \tag{87}$$

A detailed study of the equation above will appear in a future work.

Remember that a real spaceship is a tridimensional $3D$ object inserted inside a tridimensional $3D$ warp bubble that must be defined in real $3D$ spherical coordinates.

4 Conclusion

In this work presented a very interesting feature of the expansion of the normal volume elements for the new tridimensional $3D$ spherical coordinates warp drive vector but only for constant speeds.

The "zero-expansion" behavior as the "ex-libris" of the Natario 2001 original warp drive paper occurs only in $2D$ polar coordinates but not in $3D$ tridimensional spherical coordinates and remember that a real spaceship is a tridimensional $3D$ object inserted inside a tridimensional $3D$ warp bubble that must be defined in real $3D$ spherical coordinates spacetime.

The Natario warp drive is probably the best candidate(known until now) for an interstellar space travel considering the fact that a spaceship in a real superluminal spaceflight will encounter(or collide against) hazardous objects(asteroids,comets,interstellar dust and debris etc) and the Natario spacetime offers an excellent protection to the crew members as depicted in the works [7],[8],[9] and [14].However since these works were based in the original Natario 2001 paper this line of reason must be extended to encompass the new tridimensional $3D$ spherical coordinates warp drive vector

The existence(or non-existence) of the "zero-expansion" behavior in the expansion of the normal volume elements is only a geometrical feature and a marginal consequence for the new tridimensional $3D$ spherical coordinates warp drive vector .

The application of the new tridimensional $3D$ spherical coordinates warp drive vector wether in constant or variable speeds to the ADM(Arnowitt-Dresner-Misner) formalism equations of General Relativity using the approach of MTW(Misner-Thorne-Wheeler)resembling the study already done in [10],[11][12] and [13] will appear in a future work.

A complete study of the expansion of the normal volume elements for the new tridimensional $3D$ spherical coordinates warp drive vector considering the case of variable speeds using the techniques of the rate-of-strain stress tensor as described in pgs 354 and 355 in [8],Natario in pg 5 in [1],pg 92 in [4](with $-p = 0$ and $\mu = \frac{1}{2}$),pg 141 eqs 5.130 to 5.135 in [5] (with $\mu = -(\frac{1}{2})$ and $(\frac{2}{3})(\nabla.U = 0)$),pg 52 eq 15.17 in [6](with $-p = 0$ and $\eta = \frac{1}{2}$) will also appear in a future work.

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