Figure 1 The simplest model universe's scaled radius

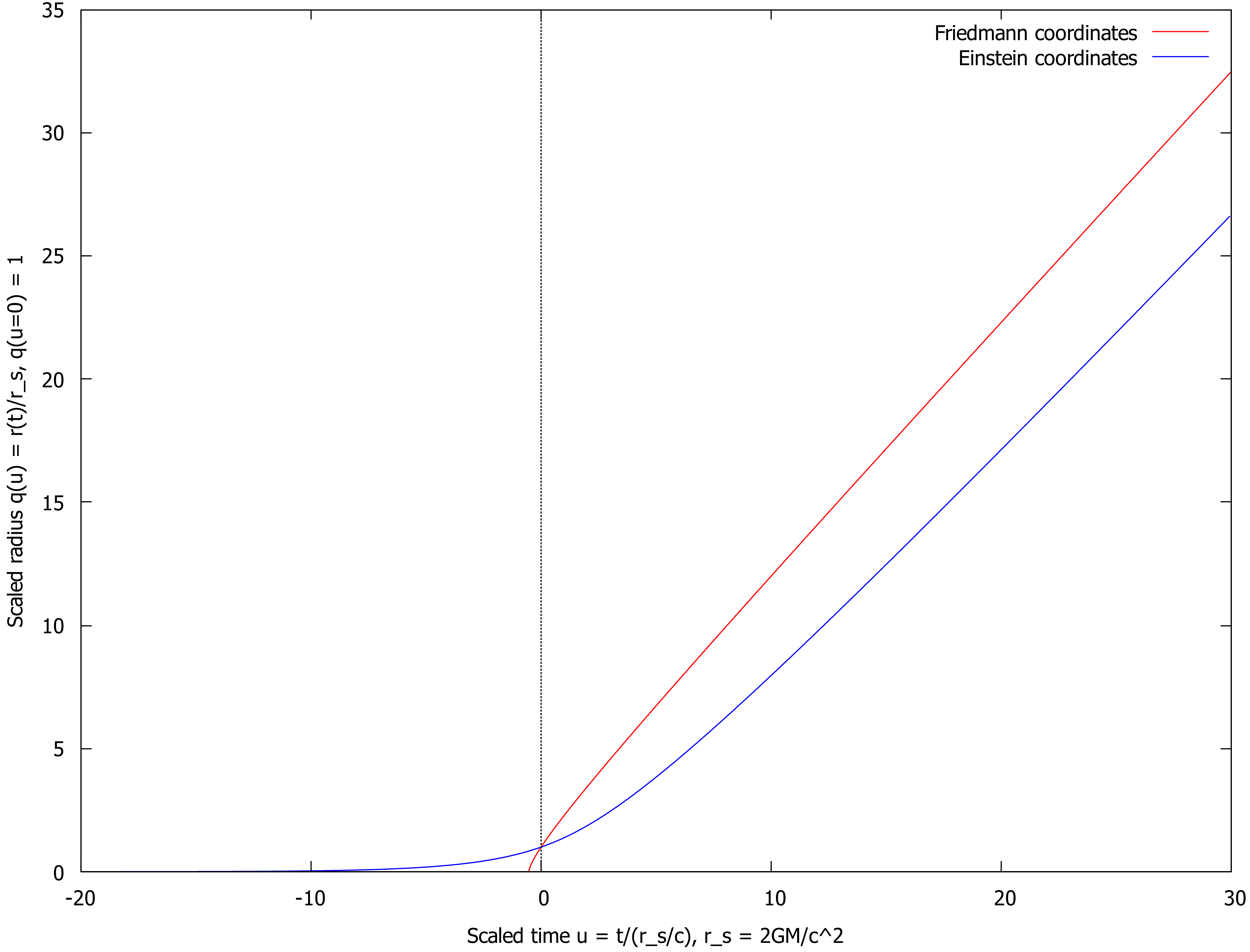


Figure 2 The simplest model universe's scaled radial velocity

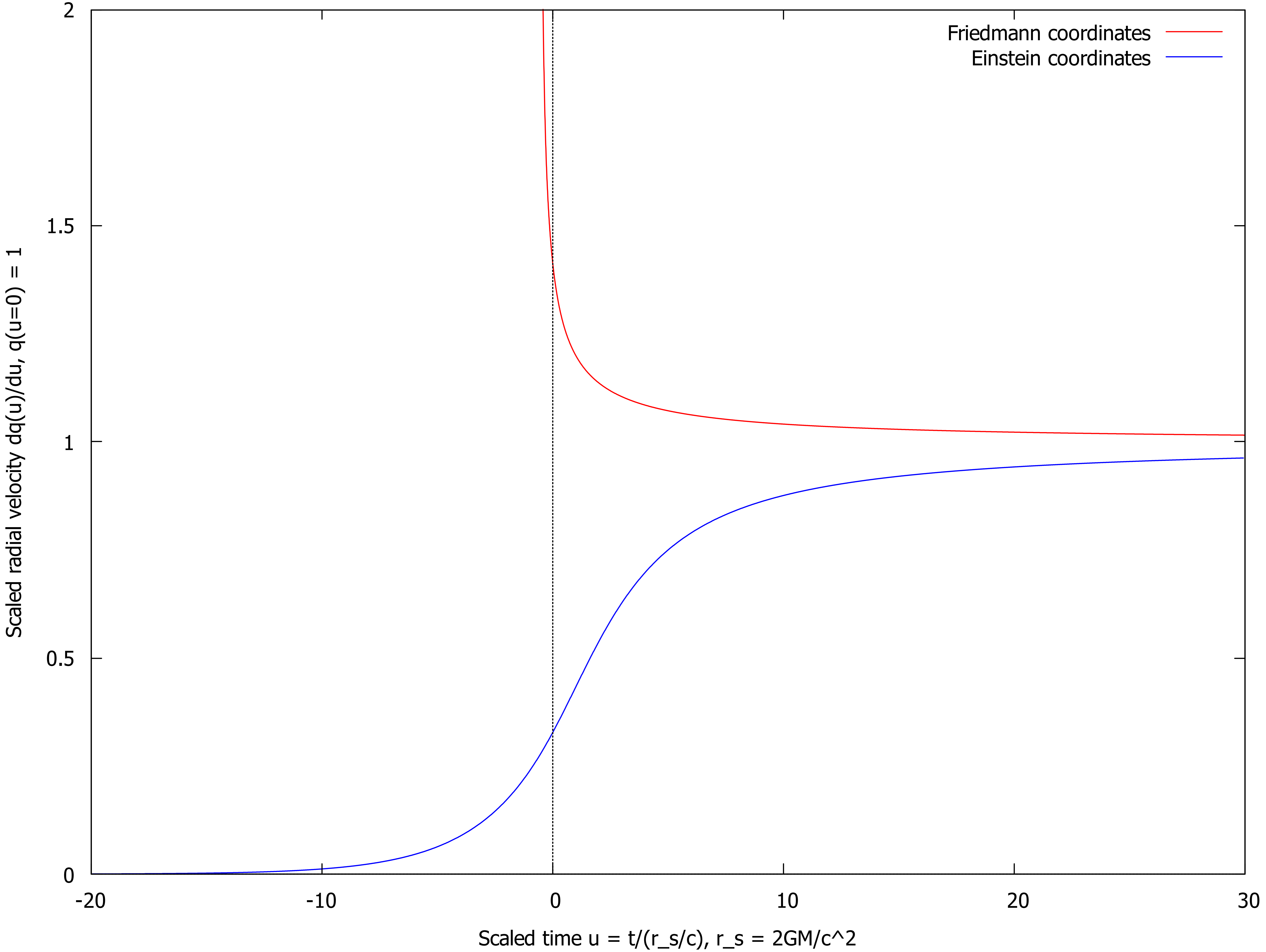
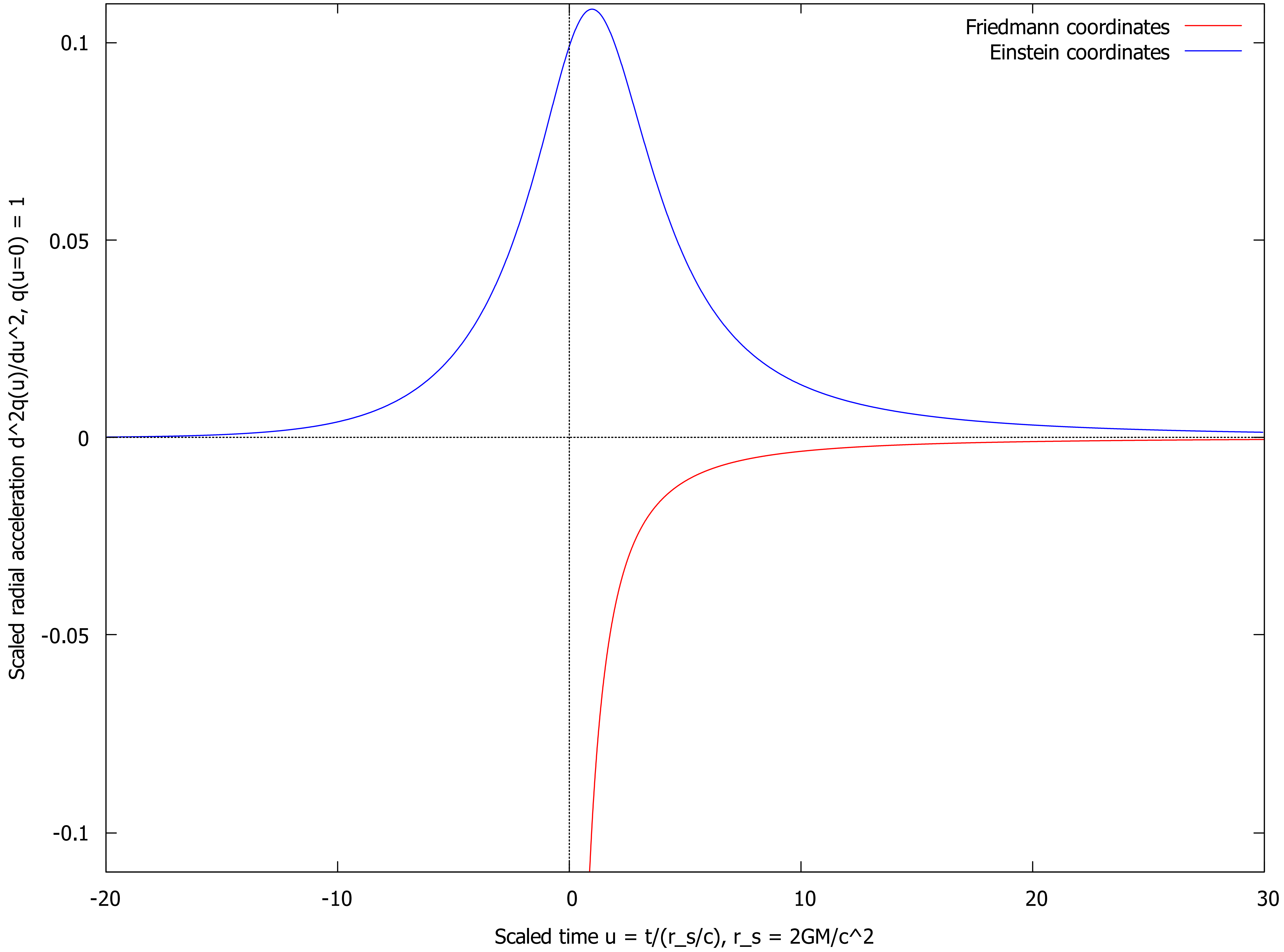


Figure 3 The simplest model universe's scaled radial acceleration



Friedmann versus Einstein Coordinates for Cosmology

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Abstract The James Webb Space Telescope has discovered a large population of bright compact galaxies in the early universe. Their abundance suggests that the early universe may not have expanded as explosively as Big Bang cosmology implies, that it may have been relatively more compact for a longer period of time. It is plausible that the physical issue with the Robertson-Walker metric form in this regard is Friedmann's 1922 coordinate condition, which makes gravity effectively Newtonian, devoid of gravitational time dilation. Einstein's successful 1915 coordinate condition in contrast permits the metric to be Lorentz covariant and compels it to always have a matrix inverse, a constraint which the Big Bang flouts. We exhibit a transformation of the Robertson-Walker metric form to Einstein coordinates, and we study in detail the radial evolution, in respectively Friedmann and Einstein coordinates, of the very simplest expanding-dust-sphere cosmology model. The deceleration of cosmic expansion in Friedmann coordinates is changed in Einstein coordinates to its acceleration, and the Big Bang in Friedmann coordinates is swapped in Einstein coordinates for a peak in that inflation.

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1. Introduction and a review of the simplest cosmology model in Friedmann coordinates

The James Webb Space Telescope (JWST) has discovered a large population of bright compact galaxies in the early universe (i.e., having redshifts $z > 10$), which are evolved enough to produce clear signatures of ionized elements as heavy as oxygen.[1] The abundance of these early bright compact galaxies suggests that the early universe may not have expanded as explosively as Big Bang cosmology implies, that it may have been relatively more compact for a longer period of time.

In fact, Friedmann's 1922 coordinate condition, $g_{00}(x) = 1$ for all x, which is *manifestly built into the Robertson-Walker metric form*,

$$(c\,d\tau)^2 = (c\,dt)^2 - (R(t))^2\big[(1/(1 - kr^2))(dr)^2 + r^2\big((d\theta)^2 + (\sin\theta\,d\phi)^2\big)\big], \tag{1.1}$$

of Big Bang cosmology, *turns out to completely eliminate gravitational time dilation* because gravitational time dilation is given by,[2]

$$[(\text{the tick rate of a clock at } x_2)/(\text{the tick rate of a clock at } x_1)] = \sqrt{g_{00}(x_2)/g_{00}(x_1)}. \tag{1.2}$$

Since Friedmann's 1922 coordinate condition $g_{00}(x) = 1$ for all x *completely eliminates gravitational time dilation*, the explosive pace of the expansion of the early universe in Big Bang cosmology *might conceivably disagree with physical reality.* The *absence* of gravitational time dilation in Big Bang cosmology *dovetails* with the fact that *its chief results follow from Newtonian gravitational dynamics* [3].

Undoubtedly the *simplest* cosmological model for the universe is an Oppenheimer-Snyder-style radially-expanding sphere of uniform-density "dust" whose constituent "dust particles" *interact only gravitationally with each other.*[4] The *Newtonian* gravitational dynamics of the *radius* $r(t)$ of such a radially expanding sphere of uniform-density dust [3, 4] is governed, *in light of the Birkhoff theorem*, by the following *two versions* of the *same* familiar Newtonian equation of a test body's exclusively-radial outward-directed motion in the gravitational field of a static point mass M fixed to the origin of coordinates,

$$d^2r/dt^2 = -GM/r^2 \text{ and } dr/dt = \sqrt{(2GM/r) + v^2}, \tag{1.3}$$

where M *is the total conserved energy divided by* c^2 *of the radially-expanding sphere of uniform-density "dust"*, and $v > 0$ is the $r \to \infty$ asymptotic outward-directed velocity of its radius.

One of *the most distant galaxies which has been observed* has a redshift z of 13.2, which corresponds to a recession speed of $0.99c$. Therefore it is reasonable to put the value of the dust-sphere model universe's asymptotic outward-directed radial velocity v to c in Eq. (1.3) above. *After having done that*, it is very convenient to *reexpress* Eq. (1.3) in terms of the dust-sphere model universe's *dimensionless scaled "time" variable* $u = t/(r_s/c)$ and in terms of that model universe's corresponding *dimensionless scaled "radius" variable* $q(u) = r(t)/r_s$, where $r_s \stackrel{\text{def}}{=} 2GM/c^2$, that model universe's Schwarzschild radius. Eq. (1.3) is thereby simplified to the two closely-related *dimensionless scaled equations*,

$$d^2q/du^2 = -1/(2q^2) \text{ and } dq/du = \sqrt{(1/q) + 1}. \tag{1.4a}$$

The solutions $q(u)$ of the second differential equation $dq/du = \sqrt{(1/q) + 1}$ of Eq. (1.4a) *can't* be expressed in terms of elementary functions, *but their inverse functions* $u(q)$ *can be so expressed*, i.e.,

$$u_{q_0}(q) = \int_{q_0}^{q} dq' \sqrt{q'/(q'+1)} =$$

$$\Big[\sqrt{q(q+1)} - \ln\big(\sqrt{q} + \sqrt{q+1}\big)\Big] - \Big[\sqrt{q_0(q_0+1)} - \ln\big(\sqrt{q_0} + \sqrt{q_0+1}\big)\Big]. \tag{1.4b}$$

For *the initial condition* $q(u = 0) = 1$, namely that the expanding dust-sphere model universe *attains its Schwarzschild radius at time zero*, the *corresponding* dimensionless scaled radius solution $q(u)$ of the second differential equation in Eq. (1.4a) *is the inverse of the* Eq. (1.4b) *function* $u_{q_0=1}(q)$ *because* $u_{q_0=1}(q = 1) = 0$. Thus the $q(u)$ *which satisfies* $q(u = 0) = 1$, while *not* directly expressible in terms of elementary functions, *is readily plotted using* Eq. (1.4b). Its plot is displayed as the red curve of Figure 1, and the corresponding plots of its dimensionless scaled radial velocity $dq(u)/du = \sqrt{(1/q(u)) + 1}$ and its dimensionless scaled radial acceleration $d^2q(u)/du^2 = -1/(2(q(u))^2)$ are displayed as the red curves of Figures 2 and 3 respectively. These three red curves depict a Big Bang dust-sphere model universe whose dimensionless scaled radius $q(u)$ *suddenly begins steeply increasing from the value zero at the finite initial dimensionless scaled time* $u = u_i \stackrel{\text{def}}{=} u_{q_0=1}(q = 0) = -\sqrt{2} + \ln(1 + \sqrt{2}) = -0.53284$ *when the Big Bang occurs; the fact that* $q(u_i) = 0$

implies that at the finite dimensionless scaled time $u = u_i$ when the Big Bang occurs the dust-sphere model universe's dimensionless scaled radial velocity $dq(u)/du|_{u=u_i} = \sqrt{(1/(q(u_i) = 0)) + 1}$ *is infinite* and its dimensionless scaled radial acceleration $d^2q(u)/du^2|_{u=u_i} = -1/(2(q(u_i) = 0)^2)$ *is infinite as well.*

Furthermore, at the finite initial dimensionless scaled time $u = u_i = -\sqrt{2} + \ln(1 + \sqrt{2}) = -0.53284$ when the Big Bang occurs the dimensionless function $R(t)$ of the corresponding Robertson-Walker metric form of Eq. (1.1) *vanishes in concert with the vanishing of the dust-sphere model universe's dimensionless scaled radius* $q(u)$, so at the finite initial time when the Big Bang occurs *the corresponding Robertson-Walker metric's matrix inverse is undefined.* Consequently, at the finite initial time when the Big Bang occurs *the corresponding Robertson-Walker metric's affine connection, namely its gravitational field, is undefined, as are its curvature tensors.*

In *addition*, as we see from the red curve of Figure 2, *for all* $u \geq u_i = -\sqrt{2} + \ln(1 + \sqrt{2}) = -0.53284$, $dq(u)/du > 1$, *i.e., from the time of the Big Bang onward, the dust-sphere model universe's radial expansion velocity exceeds* c, which contradicts the very well-established basic physical principle that entities such as "dust" (or even light) *don't travel at speeds exceeding* c [5].

It is thus apparent that Big Bang cosmology, which *incorporates* Friedmann's 1922 coordinate condition $g_{00}(x) = 1$ for all x in its Eq. (1.1) Robertson-Walker metric form, *not only* may be incompatible with the early universe of abundant bright compact galaxies revealed by JWST, but that it *as well* challenges well-established physical principles.

Friedmann's 1922 coordinate condition $g_{00}(x) = 1$ for all x *doesn't* guarantee the existence of the matrix inverse of the metric $g_{\mu\nu}(x)$ at all x, so it *doesn't* guarantee the existence of the affine connection (the gravitational field) or the curvature tensors at all x; we noted above that for the simplest model cosmology the corresponding Robertson-Walker metric has no matrix inverse at the time of the Big Bang. Furthermore, Friedmann's 1922 coordinate condition $g_{00}(x) = 1$ for all x *manifestly doesn't permit* $g_{\mu\nu}(x)$ *to be Lorentz covariant;* the red curve of Figure 2 shows that for the simplest model cosmology the corresponding Robertson-Walker metric *engenders arbitrarily-large violations of the Lorentzian speed limit* c.

In *contrast* to these issues of physical principle regarding Friedmann's 1922 coordinate condition $g_{00}(x) = 1$ for all x, the coordinate condition $\det(g_{\mu\nu}(x)) = -1$ for all x which Einstein *successfully* applied to Mercury's remnant perihelion shift and also to the deflection of starlight by the sun's gravity in his landmark November 18, 1915 paper [6] *compels* $g_{\mu\nu}(x)$ *to have a well-defined matrix inverse for all* x *and permits* $g_{\mu\nu}(x)$ *to be Lorentz covariant.* Therefore in the next section we work out *a coordinate transformation* of the Eq. (1.1) Robertson-Walker metric form—whose *virtue* of course is that it is appropriate for Einstein equations whose gravitational sources *are spherically-symmetric and homogeneous* [7]—such that *the coordinate-transformed Robertson-Walker metric form* satisfies Einstein's 1915 coordinate condition $\det(g_{\mu\nu}(x)) = -1$ for all x *instead of satisfying* Friedmann's 1922 coordinate condition $g_{00}(x) = 1$ for all x, as the Eq. (1.1) Robertson-Walker metric form *itself* does. We thereby *take advantage* of the fact that *a coordinate transformation* of a metric solution of an Einstein equation *is also a metric solution of the same Einstein equation.*

2. Transformation of the Robertson-Walker metric form to Einstein coordinates

Following *arbitrary* non-interdependent transformations $t'(t)$ and $r'_k(r)$ of its time t and radius r coordinates, the Eq. (1.1) Robertson-Walker metric form becomes,

$$(c\,d\tau)^2 = (dt(t')/dt')^2(c\,dt')^2 -$$
$$(R(t(t')))^2\big[\big(1/\big(1 - k(r(r'_k))^2\big)\big)(dr(r'_k)/dr'_k)^2(dr'_k)^2 + (r(r'_k)/r'_k)^2(r'_k)^2\big((d\theta)^2 + (\sin\theta\,d\phi)^2\big)\big], \tag{2.1}$$

whose determinant will be -1, which is *required* of metrics expressed in Einstein coordinates, *if both,*

$$(dt(t')/dt')^2(R(t(t')))^6 = 1 \;\; \textit{and} \;\; \big(1/\big(1 - k(r(r'_k))^2\big)\big)(dr(r'_k)/dr'_k)^2(r(r'_k)/r'_k)^4 = 1 \;\; \textit{are satisfied.} \tag{2.2}$$

The *first* Eq. (2.2) requirement implies that,

$$(dt(t')/dt')^2 = (R(t(t')))^{-6}, \tag{2.3}$$

and it furthermore implies the following unique time transformation $t'(t)$ *that satisfies* $t'(t = 0) = 0$,

$$t'(t) = \textstyle\int_0^t |R(w)|^3 dw, \tag{2.4}$$

while the *second* Eq. (2.2) requirement implies that,

$$(1/(1-k(r(r'_k))^2))(dr(r'_k)/dr'_k)^2 = (r'_k/r(r'_k))^4, \tag{2.5}$$

and it furthermore implies the following unique radius transformation $r'_k(r)$ *that satisfies* $r'_k(r=0)=0$,

$$r'_k(r) = \left(3\int_0^r \left(1-ks^2\right)^{-\frac{1}{2}} s^2\, ds\right)^{\frac{1}{3}}. \tag{2.6}$$

Inserting the results given by Eqs. (2.3) and (2.5) into Eq. (2.1) yields the following transformation of the Eq. (1.1) Robertson-Walker metric form to Einstein coordinates,

$$(c\,d\tau)^2 = (R(t(t')))^{-6}(c\,dt')^2 - (R(t(t')))^2\left[(r'_k/r(r'_k))^4(dr'_k)^2 + (r(r'_k)/r'_k)^2(r'_k)^2\left((d\theta)^2 + (\sin\theta\, d\phi)^2\right)\right], \tag{2.7}$$

where the function $t(t')$ in Eq. (2.7) is the *inverse* of the specific time transformation $t'(t)$ which is explicitly given by Eq. (2.4), while the function $r(r'_k)$ in Eq. (2.7) is the *inverse* of the specific radius transformation $r'_k(r)$ which is explicitly given by Eq. (2.6). The Eq. (2.7) transformation of the Eq. (1.1) Robertson-Walker metric form to Einstein coordinates is readily verified to satisfy the Einstein coordinate condition $\det(g_{\mu\nu}(x)) = -1$ for all x.

However, to work out the motion $r(t)$ in Einstein coordinates *of only the radius* of an Oppenheimer-Snyder-style radially-expanding sphere of uniform-density "dust" whose constituent "dust particles" interact only gravitationally with each other, i.e., the *simplest* cosmological model, *it definitely isn't necessary* to actually insert the metric form described by Eqs. (2.7), (2.4) and (2.6) above into the appropriate Einstein equation. *In light of the Birkhoff theorem*, one can *instead use the gravitational geodesic equation* to work out a test body's exclusively-radial motion *in the Einstein-coordinate static metric for a static point mass* M *fixed to the origin of coordinates*, where M is equal to the total conserved energy divided by c^2 of the Oppenheimer-Snyder-style radially-expanding sphere of uniform-density "dust" whose constituent "dust particles" interact only gravitationally with each other, i.e., the *simplest* cosmological model.

We next briefly touch on the selection *of the appropriate Einstein-coordinate static metric for a static point mass* M *fixed to the origin of coordinates*, following which *we use the gravitational geodesic equation* to work out a test body's *exclusively-radial motion in that static metric.*

3. The appropriate Einstein-coordinate static metric for a static point mass

In his November 18, 1915 paper [6] Einstein obtained for the coordinate condition $\det(g_{\mu\nu}(x)) = -1$ for all x *a unique second-order approximation* to the static, spherically-symmetric metric of a static point mass (the sun) fixed to $r = 0$. However, in his January 13, 1916 paper [8] Karl Schwarzschild obtained for the coordinate condition $\det(g_{\mu\nu}(x)) = -1$ for all x *a one-parameter family* of static, spherically-symmetric *seemingly exact metric solutions* for a static point mass M fixed to $r = 0$. This *one-parameter family* of seemingly exact static, spherically-symmetric metric solutions for the coordinate condition $\det(g_{\mu\nu}(x)) = -1$ for all x for a static point mass M fixed to $r = 0$ can be presented as,

$$(c\,d\tau)^2 = \left(1 - r_s/\left(r^3 - r_a^3 + r_s^3\right)^{\frac{1}{3}}\right)(c\,dt)^2 - \left(1/\left(1 - r_s/\left(r^3 - r_a^3 + r_s^3\right)^{\frac{1}{3}}\right)\right)\left(r/\left(r^3 - r_a^3 + r_s^3\right)^{\frac{1}{3}}\right)^4(dr)^2 - \left(\left(r^3 - r_a^3 + r_s^3\right)^{\frac{1}{3}}/r\right)^2 r^2\left((d\theta)^2 + (\sin\theta\, d\phi)^2\right), \tag{3.1}$$

where $r_s \stackrel{\text{def}}{=} 2GM/c^2$ is the Schwarzschild radius for a static point mass M, and r_a is the metric-family parameter, which has the dimension of length.

If the static point mass M fixed to $r = 0$ is replaced by a static, spherically-symmetric smooth mass distribution of nonzero extent and total mass M centered at $r = 0$, the resulting static, spherically-symmetric metric is expected to be nonsingular everywhere (as is the case for the corresponding static Newtonian gravitational potential). As that static, spherically-symmetric smooth mass distribution of nonzero extent and total mass M centered at $r = 0$ *shrinks to the static point mass* M *fixed to* $r = 0$, the resulting static, spherically-symmetric metric *is expected to develop a singularity at* $r = 0$ (exactly as occurs for the corresponding static Newtonian gravitational potential), but *is expected to still be nonsingular at all* $r > 0$ (also exactly as occurs for the corresponding static Newtonian gravitational potential). The *only* member of the Eq. (3.1) metric family *which has those two properties* is the one with $r_a = 0$, namely,

$$(c\,d\tau)^2 = \left(1 - r_s/\left(r^3 + r_s^3\right)^{\frac{1}{3}}\right)(c\,dt)^2 - \left(1/\left(1 - r_s/\left(r^3 + r_s^3\right)^{\frac{1}{3}}\right)\right)\left(r/\left(r^3 + r_s^3\right)^{\frac{1}{3}}\right)^4(dr)^2 - \left(\left(r^3 + r_s^3\right)^{\frac{1}{3}}/r\right)^2 r^2\left((d\theta)^2 + (\sin\theta\, d\phi)^2\right), \tag{3.2}$$

which has been shown to be *the appropriate exact static metric solution for a static point mass M fixed to* $r = 0$ [9], and *is precisely the metric Karl Schwarzschild selected in his January 13, 1916 paper* [8].

For both Mercury's remnant perihelion shift and the deflection of starlight by the sun's gravity the results of Schwarzschild's January 13, 1916 Eq. (3.2) metric *are only negligibly different* from the results of Einstein's November 18, 1915 *second-order approximation* to that metric.

Although the parameter value $r_a = 0$ in Eq. (3.1) produces *the appropriate exact static metric solution for a static point mass M fixed to* $r = 0$ [9], the parameter value $r_a = r_s$ in Eq. (3.1) produces *by far the algebraically-simplest metric form of the* Eq. (3.1) *metric family*, namely,

$$(c\,d\tau)^2 = (1 - r_s/r)(c\,dt)^2 - (1/(1 - r_s/r))(dr)^2 - r^2\big((d\theta)^2 + (\sin\theta\,d\phi)^2\big). \tag{3.3}$$

The Eq. (3.3) *inappropriate* seemingly exact static metric solution for a static point mass M fixed to $r = 0$, *which has an inappropriate and entirely unnecessary event-horizon singularity at* $r = r_s$, was explicitly exhibited for the first time in a May 27, 1916 paper by J. Droste [10], who apparently was attracted to it because of its relative algebraic simplicity. Its algebraic simplicity relative *specifically* to Schwarzschild's January 13, 1916 Eq. (3.2) *appropriate exact static metric solution for a static point mass M fixed to* $r = 0$ [9] also appealed to the mathematician David Hilbert, who strongly promoted it, with *the consequence* that textbooks almost *universally* very prominently feature the *inappropriate* Eq. (3.3) seemingly exact static metric solution for a static point mass M fixed to $r = 0$ *which has an inappropriate and entirely unnecessary event-horizon singularity at* $r = r_s$. Those textbooks *furthermore very mistakenly attribute the* Eq. (3.3) *metric to Schwarzschild instead of to Droste or Hilbert.* The *upshot* of this *confluence of two severe mistakes in virtually all textbooks* was that Schwarzschild's January 13, 1916 Eq. (3.2) *appropriate exact static metric solution for a static point mass M fixed to* $r = 0$ [9] *fell into almost complete obscurity* [8].

Disregarding these two ubiquitous textbook gaffes, we next *use the gravitational geodesic equation* to work out a test body's exclusively-radial motion in Schwarzschild's January 13, 1916 Eq. (3.2) *appropriate exact static metric solution for a static point mass M fixed to* $r = 0$ [9]. *In conjunction with the Birkhoff theorem*, that will permit us to *replace* the second Eq. (1.4a) dimensionless scaled Newtonian gravitational equation of motion $dq/du = \sqrt{(1/q) + 1}$ for the dimensionless scaled radius $q(u)$ of an Oppenheimer-Snyder-style uniform-density expanding dust-sphere model universe *in Friedmann coordinates* by *the corresponding* dimensionless scaled equation of motion for the dimensionless scaled radius $q(u)$ of an Oppenheimer-Snyder-style uniform-density expanding dust-sphere model universe *in Einstein coordinates.*

4. Radial test-body motion in the actual Schwarzschild static metric for a static point mass

To conveniently apply the gravitational geodesic equation to radial test-body motion in Schwarzschild's January 13, 1916 Eq. (3.2) metric, it is very useful to initially reexpress that metric in Schwarzschild's compact representation [8]. Given the three abbreviations $R(r) \stackrel{\text{def}}{=} \big(r^3 + r_s^3\big)^{\frac{1}{3}}$, $B(R(r)) \stackrel{\text{def}}{=} 1 - r_s/R(r)$ and $A(R(r)) \stackrel{\text{def}}{=} 1/B(R(r))$, it is readily shown that $dR(r)/dr = (r/R(r))^2$, and consequently that the Eq. (3.2) metric has the compact representation,

$$(c\,d\tau)^2 = B(R(r))(c\,dt)^2 - A(R(r))(dR(r))^2 - (R(r))^2\big((d\theta)^2 + (\sin\theta\,d\phi)^2\big). \tag{4.1}$$

The Eq. (4.1) metric representation *itself* immediately yields the following *first-order* equation of test-body gravitational motion,

$$c^2 = c^2 B(R(r))(dt/d\tau)^2 - A(R(r))(dR(r)/d\tau)^2 - (R(r))^2\big((d\theta/d\tau)^2 + (\sin\theta\,(d\phi/d\tau))^2\big). \tag{4.2}$$

Since the test body we consider *moves exclusively radially*, its angular frequencies $d\theta/d\tau$ and $d\phi/d\tau$ are both equal to zero, which reduces Eq. (4.2) to,

$$c^2 = \big[c^2 B(R(r)) - A(R(r))(dR(r)/dt)^2\big](dt/d\tau)^2. \tag{4.3}$$

To turn Eq. (4.3) into an equation of test-body radial motion, i.e., a differential equation for dr/dt, we need to *evaluate* the Eq. (4.3) factor $(dt/d\tau)^2$. Doing so requires *integrating the time component* of the test body's second-order in τ *four-vector gravitational geodesic equation of motion,* [11]

$$d^2x^\kappa/d\tau^2 + (1/2)\,g^{\kappa\lambda}(x)\big[\partial g_{\lambda\nu}(x)/\partial x^\mu + \partial g_{\lambda\mu}(x)/\partial x^\nu - \partial g_{\mu\nu}(x)/\partial x^\lambda\big]\,(dx^\mu/d\tau)\,(dx^\nu/d\tau) = 0. \tag{4.4}$$

For *the particular metric form given by* Eq. (4.1), the *time component* of the Eq. (4.4) test-body gravitational geodesic equation of motion is,[12]

$$\frac{d^2t}{d\tau^2} + \frac{dt}{d\tau}\frac{dB(R(r))/dR(r)}{B(R(r))}\frac{dR(r)}{d\tau} = 0, \tag{4.5a}$$

which can be written,

$$\frac{1}{dt/d\tau}\frac{d(dt/d\tau)}{d\tau} + \frac{dB(R(r))/dR(r)}{B(R(r))}\frac{dR(r)}{d\tau} = 0, \tag{4.5b}$$

which in turn can be written,

$$d\big(\ln\big(dt/d\tau\big) + \ln\big(B(R(r))\big)\big)/d\tau = 0, \tag{4.5c}$$

which implies that,

$$\ln\big((dt/d\tau)(B(R(r)))\big) = -C, \tag{4.5d}$$

where C is an arbitrary dimensionless constant. Eq. (4.5d) implies that,

$$dt/d\tau = 1/(KB(R(r))), \tag{4.5e}$$

where $K = \exp(C)$ is an arbitrary dimensionless positive constant. Inserting Eq. (4.5e) into Eq. (4.3) yields,

$$\big(A(R(r))/(B(R(r)))^2\big)(dR(r)/dt)^2 - \big(c^2/B(R(r))\big) = -c^2K^2. \tag{4.6a}$$

The object $dR(r)/dt$ in Eq. (4.6a) is of course equal to $(dR(r)/dr)(dr/dt)$, and we have pointed out above Eq. (4.1) that $dR(r)/dr = (r/R(r))^2$. Inserting this result along with $B(R(r)) = 1 - r_s/R(r)$ and $A(R(r)) = 1/B(R(r))$ into Eq. (4.6a) yields,

$$\big((r/R(r))^4/(1 - r_s/R(r))^3\big)(dr/dt)^2 - \big(c^2/(1 - r_s/R(r))\big) = -c^2K^2, \tag{4.6b}$$

where $R(r) = (r^3 + r_s^3)^{\frac{1}{3}}$ and $r_s \stackrel{\text{def}}{=} (2GM/c^2)$. We are now able to write down the Einstein-gravity analog of the Newtonian-gravity equation of motion $(dr/dt)^2 = (2GM/r) + v^2$ of a test body *which moves only radially relative to a static point mass M that is fixed to $r = 0$*,

$$(dr/dt)^2 = c^2\big((R(r)/r)^2(1 - r_s/R(r))\big)^2\left[1 - K^2(1 - r_s/R(r))\right], \tag{4.6c}$$

where $R(r) = (r^3 + r_s^3)^{\frac{1}{3}}$. Defining the dimensionless variable q as $q \stackrel{\text{def}}{=} (r/r_s)$, we note that $(R(r)/r) = \big((q^3 + 1)^{\frac{1}{3}}/q\big)$ and $(1 - r_s/R(r)) = \big(1 - \big(1/(q^3 + 1)^{\frac{1}{3}}\big)\big)$, so Eq. (4.6c) becomes,

$$(dr/dt)^2 = c^2\big(\big((q^3 + 1)^{\frac{1}{3}}/q\big)^2\big(1 - \big(1/(q^3 + 1)^{\frac{1}{3}}\big)\big)\big)^2\left[1 - K^2\big(1 - \big(1/(q^3 + 1)^{\frac{1}{3}}\big)\big)\right], \tag{4.6d}$$

where $q \stackrel{\text{def}}{=} (r/r_s)$. In the Newtonian-gravity case where $(dr/dt)^2 = (2GM/r) + v^2$, *$(dr/dt)^2$ grows without bound as $r \to 0$*. Indeed, in the Newtonian-gravity case, *$|dr/dt|$ is asymptotic to $\sqrt{2GM/r}$ as $r \to 0$*.

To work out the asymptotic behavior of $(dr/dt)^2$ as $q \to 0$ in Eq. (4.6d), we note that as $q \to 0$, $(q^3+1)^{\frac{1}{3}}/q \simeq 1/q$ and $\big(1-\big(1/(q^3+1)^{\frac{1}{3}}\big)\big) \simeq q^3/3$, so $\big((q^3+1)^{\frac{1}{3}}/q\big)^2\big(1-\big(1/(q^3+1)^{\frac{1}{3}}\big)\big) \simeq q/3$, which together with Eq. (4.6d) yields that $((dr/dt)/c)^2 \simeq (q/3)^2$ as $q \to 0$. Thus $|dr/dt| \simeq c(q/3)$ as $q \to 0$, so,

$$\textit{the test body's radial speed } |dr/dt| \text{ is asymptotic to } (c/(3r_s))r \text{ as } r \to 0, \tag{4.6e}$$

which is precisely the opposite of the unbounded speed of the test body as $r \to 0$ in the Newtonian-gravity case. In the Einstein-gravity case, *the gravitational time-dilation effect of very strong gravity reduces speeds.*

We next verify that $(dr/dt)^2 < c^2$. We *first* show that $d\big(\big((q^3 + 1)^{\frac{1}{3}}/q\big)^2\big(1 - \big(1/(q^3 + 1)^{\frac{1}{3}}\big)\big)\big)/dq > 0$ when $q > 0$. Since $d\big(\big((q^3 + 1)^{\frac{1}{3}}/q\big)^2\big(1 - \big(1/(q^3 + 1)^{\frac{1}{3}}\big)\big)\big)/dq = \left[2 + q^3 - 2(q^3 + 1)^{\frac{1}{3}}\right]/\left[q^3(q^3 + 1)^{\frac{2}{3}}\right]$, *we must show that $2 + q^3 > 2(q^3 + 1)^{\frac{1}{3}}$* when $q > 0$. We do so by exhibiting a chain of inequalities *which are logically equivalent to $2 + q^3 > 2(q^3 + 1)^{\frac{1}{3}}$, where the final inequality in the chain is clearly valid when $q > 0$*,

$$2 + q^3 > 2(q^3 + 1)^{\frac{1}{3}} \iff 1 + (q^3/2) > (1 + q^3)^{\frac{1}{3}} \iff 1 + 3(q^3/2) + 3(q^3/2)^2 + (q^3/2)^3 > 1 + q^3$$
$$\iff (1/2)q^3 + (3/4)q^6 + (1/8)q^9 > 0 \ \text{ when } \ q > 0. \tag{4.6f}$$

Therefore $\big(\big((q^3 + 1)^{\frac{1}{3}}/q\big)^2\big(1 - \big(1/(q^3 + 1)^{\frac{1}{3}}\big)\big)\big)$ is a strictly increasing function of q when $q > 0$, so when $q > 0$, it is less than its $q \to \infty$ limit, which has the value unity. Consequently, from Eq. (4.6d), $(dr/dt)^2 < c^2\left[1 - K^2\big(1 - \big(1/(q^3 + 1)^{\frac{1}{3}}\big)\big)\right] < c^2$ when $q > 0$, because $K^2 > 0$ and $\big(1 - \big(1/(q^3 + 1)^{\frac{1}{3}}\big) > 0$ when

$q > 0$. Thus, $(dr/dt)^2 < c^2$ when $q > 0$, and, when $q = 0$, Eq. (4.6e) implies that $(dr/dt)^2 = 0$, so $(dr/dt)^2 < c^2$ *under all circumstances; the test body never has a speed as great as* c. This gravitational result *in Einstein coordinates* is physically sensible [5]; *the speeds exceeding* c which the Newtonian gravity equation $(dr/dt)^2 = (2GM/r) + v^2$ permits *in Friedmann coordinates* aren't physically sensible [5].

We next investigate the asymptotic radial speed $|dr/dt|$ of the test body as $r \to \infty$. From Eq. (4.6d) we see that as $q \to \infty$, $(dr/dt)^2 \to c^2\big(1 - K^2\big)$. Therfore,

$$|dr/dt| \to c\sqrt{1-K^2} \;\text{ as }\; r \to \infty, \tag{4.6g}$$

so $K^2 = \big(1 - (v/c)^2\big)$, where $v \geq 0$ is the test body's asymptotic radial speed. Upon inserting $K^2 = \big(1 - (v/c)^2\big)$ into Eq. (4.6d), it becomes,

$$(dr/dt)^2 = c^2\Big(\big((q^3+1)^{\frac{1}{3}}/q\big)^2\big(1 - \big(1/(q^3+1)^{\frac{1}{3}}\big)\big)\Big)^2\left[1 - \big(1-(v/c)^2\big)\big(1-\big(1/(q^3+1)^{\frac{1}{3}}\big)\big)\right]. \tag{4.6h}$$

To apply Eq. (4.6h) to the motion of the radius of the Oppenheimer-Snyder-style uniform-density expanding dust-sphere model universe (the *simplest* model universe) *via the Birkhoff theorem*, we again note that one of most distant known galaxies has a redshift z of 13.2, whose corresponding recession speed is $0.99c$, so again it is reasonable to put v to c in Eq. (4.6h), which yields,

$$(dr/dt)^2/c^2 = \Big(\big((q^3+1)^{\frac{1}{3}}/q\big)^2\big(1 - \big(1/(q^3+1)^{\frac{1}{3}}\big)\big)\Big)^2, \tag{4.7a}$$

Since this model universe *is expanding, not contracting*, Eq. (4.7a) becomes.

$$(dr/dt)/c = +\big((q^3+1)^{\frac{1}{3}}/q\big)^2\big(1 - \big(1/(q^3+1)^{\frac{1}{3}}\big)\big) = (1+q^3)^{\frac{1}{3}}\big((1+q^3)^{\frac{1}{3}} - 1\big)/q^2, \tag{4.7b}$$

and since the dimensionless scaled time u is $t/(r_s/c)$, and $q = r/r_s$ is the dimensionless scaled radius, then in terms of the dimensionless scaled radius q and the dimensionless scaled time u Eq. (4.7b) becomes,

$$dq/du = (1+q^3)^{\frac{1}{3}}\big((1+q^3)^{\frac{1}{3}} - 1\big)/q^2. \tag{4.7c}$$

Eq. (4.7c) is the Einstein-coordinate *replacement* of the Friedmann-coordinate *second* differential equation of model-universe radius motion $dq/du = \sqrt{(1/q)+1}$ of Eq. (1.4a). Here we *again* see that as $q \to 0$, the model-universe dimensionless scaled radial velocity $dq/du \to +\infty$ in Friedmann coordinates, but that as $q \to 0$, the model-universe dimensionless scaled radial velocity $dq/du \to 0$ in Einstein coordinates, which reflects the *absence* of gravitational time dilation in Friedmann coordinates, and the *dominance* of gravitational time dilation in Einstein coordinates when gravity is sufficiently strong.

The Einstein-coordinate *replacement* of the Friedmann-coordinate *first* equation $d^2q/du^2 = -1/(2q^2)$ of Eq. (1.4a) is obtained from the Eq. (4.7c) Einstein-coordinate expression for dq/du as follows,

$$\begin{aligned} d^2q/du^2 &= \{d[(dq/du)]/dq\}\,(dq/du) = \\ &\big(q^3 - 2\big((1+q^3)^{\frac{1}{3}} - 1\big)\big)\big((1+q^3)^{\frac{1}{3}} - 1\big)/\big(q^5(1+q^3)^{\frac{1}{3}}\big). \end{aligned} \tag{4.7d}$$

Analysis of Eq. (4.7d) shows that the radial acceleration d^2q/du^2 of the model universe is *positive* for all $q > 0$ in Einstein coordinates, *in contrast to the negative values* $-1/(2q^2)$ of the radial acceleration d^2q/du^2 of the model universe for all $q > 0$ in Friedmann coordinates. Such an unexpected *positive* acceleration of the universe's expansion *has indeed been observed*, and its discoverers awarded a Nobel prize. Here we see that *positive* acceleration of the universe's expansion *is an entirely natural gravitational phenomenon in Einstein coordinates*, which *doesn't* require ad hoc insertion of a "dark energy" cosmological-constant term $\lambda g_{\mu\nu}(x)$ into the Einstein equation.

In Einstein coordinates the evolution in dimensionless scaled time u of the model universe's dimensionless scaled radius q is given by the solution $q(u)$ of the Eq. (4.7c) equation of motion. For the initial condition $q(u=0) = 1$, i.e., that the model universe attains its Schwarzschild radius at time zero, the numerical solution $q(u)$ of the Eq. (4.7c) equation of motion is displayed as the blue curve of Figure 1. Its corresponding dimensionless scaled radial velocity $dq(u)/du$ and dimensionless scaled radial acceleration $d^2q(u)/du^2$ in Einstein coordinates are displayed as the blue curves of Figures 2 and 3 respectively. These three blue curves show that in Einstein coordinates the model universe *exists at all values of the dimensionless scaled time* u, but the blue curve of Figure 1 shows that at dimensionless scaled times u which are much less than -1 the model universe in Einstein coordinates *is exponentially small relative to its Schwarzschild radius*, and correspondingly has its physical processes and radiation frequencies *so greatly gravitationally time-dilated* that it can aptly be colloquially described as being in a state of "suspended animation". The fact that in Einstein coordinates the model universe *has always existed* removes the need to account for the universe's excess of particles over antiparticles, which has been an awkward issue for the Big-Bang *origin at a finite time* of the model universe in Friedmann coordinates.

The blue curve of Figure 2 shows that the radial expansion velocity of the model universe in Einstein coordinates *never exceeds* c, in accord with the very thoroughly tested precepts of Lorentzian relativity [5], whereas the red curve of Figure 2 shows that the radial expansion velocity of the model universe in Friedmann coordinates *is unbounded and always exceeds* c, in extreme violation of those precepts [5].

The red curve of Figure 3 shows *unbounded deceleration* of the expansion of the model universe in Friedmann coordinates, whereas the blue curve of Figure 3 *in contrast* shows *perpetual acceleration* of the expansion of the model universe in Einstein coordinates, *with a pronounced peak in that inflation* near $u = 1$.

We now reflect on the reasons why Einstein-coordinate gravity, which *doesn't* suppress gravitational time dilation and *respects* the test-body speed limit c, may cause acceleration *opposite* to that caused by Friedmann-coordinate Newtonian gravity, as evidenced *by the stark contrast* between the blue and red curves of Figure 3. The effect of *gravitational time dilation* is to *decrease* the speed of a test body *which is moving toward* a gravitational center (e.g., a static point mass), and to *increase* the speed of a test body *which is moving away from* that gravitational center. This type of acceleration is indeed *opposite* to that caused by Newtonian gravity; it becomes important for gravitational fields which are *so strong* that this effect of their gravitational time dilation *dominates the opposite effect of their Newtonian gravity.*

Even a *weak* gravitational field, however, turns out, in Einstein coordinates, to *repel* rather than *attract* a test body *which is moving close enough to the speed* c. Consider a light packet which is moving toward a gravitational center, but is distant enough from that center that the gravitational field at the light packet's position is weak. If the light packet *is accelerated toward the gravitational center the same way that a test body at rest at the light packet's position is*, then the light packet's speed *immediately exceeds* c! In fact, in Einstein coordinates such a light packet *is accelerated away from the gravitational center twice as strongly as a test body at rest at the light packet's position is accelerated toward the gravitational center.*

In Einstein coordinates the presence of gravity *always slows light to a speed less than* c, so the effect of gravity on light *is refractive.* Astronomers cleverly exploit *the refractive effect of gravity on light* by using the gravitational field of a foreground galaxy as a *lens* which *magnifies* the image *of a more distant galaxy.*

In Einstein coordinates *it isn't only light* which is accelerated *away* from a gravitational center; a test body which is moving at a speed greater than $c/\sqrt{3} = 0.57735c$ *is also accelerated away from a gravitational center*, although to a lesser extent. Since the speed $c/\sqrt{3}$ corresponds to a redshift z of approximately 1, in Einstein coordinates there exist *many* telescope-visible galaxies which are gravitationally accelerated *away from an observer* by the sphere of cosmic matter whose radius is the observer's distance to such a galaxy.

These nonintuitive aspects of radial gravitational acceleration in Einstein coordinates can be worked out from Eq. (4.6h), which it is convenient to reexpress for that purpose as,

$$(dr/dt)^2 = c^2\big(\chi(q) - \big(1 - (v/c)^2\big)\,\xi(q)\big), \tag{4.8a}$$

where,

$$\chi(q) = \big((q^3+1)^{\frac{1}{3}}/q\big)^4\big(1 - \big(1/(q^3+1)^{\frac{1}{3}}\big)\big)^2 \;\text{ and }\; \xi(q) = \big((q^3+1)^{\frac{1}{3}}/q\big)^4\big(1 - \big(1/(q^3+1)^{\frac{1}{3}}\big)\big)^3. \tag{4.8b}$$

Differentiating both sides of Eq. (4.8a) with respect to t yields,

$$2(dr/dt)(d^2r/dt^2) = c^2\big(d\chi(q)/dq - (1-(v/c)^2)d\xi(q)/dq\big)\,(dq/dr)(dr/dt), \tag{4.8c}$$

where,

$$\begin{aligned} d\chi(q)/dq &= \big(4/q^5\big)\big(1 - \big(1/(q^3+1)^{\frac{1}{3}}\big)\big)\big(1 + (q^3/2) - (q^3+1)^{\frac{1}{3}}\big) \;\text{ and} \\ d\xi(q)/dq &= \big(4/q^5\big)\big(1 - \big(1/(q^3+1)^{\frac{1}{3}}\big)\big)^2\big(1 + (3q^3/4) - (q^3+1)^{\frac{1}{3}}\big). \end{aligned} \tag{4.8d}$$

Since $q = (r/r_s)$ and $r_s = 2GM/c^2$, Eq. (4.8c) can be rexpressed as follows,

$$\begin{aligned} d^2r/dt^2 &= \tfrac{1}{2}\,(c^2/r_s)\,(r_s/r)^2q^2\,\big(d\chi(q)/dq - \big(1-(v/c)^2\big)\,d\xi(q)/dq\big) = \\ &\big(GM/r^2\big)\,\big(\big(q^2\,d\chi(q)/dq\big) - \big(1-(v/c)^2\big)\,\big(q^2\,d\xi(q)/dq\big)\big). \end{aligned} \tag{4.8e}$$

From Eq. (4.8d) we see that as $q \to \infty$, $(q^2\, d\chi(q)/dq) \to 2$ and $(q^2\, d\xi(q)/dq) \to 3$, so from Eq, (4.8e),

$$\textit{the test body's radial acceleration } d^2r/dt^2 \textit{ is asymptotic to } -\big(GM/r^2\big)\big(1 - 3(v/c)^2\big) \textit{ as } r \to \infty, \tag{4.8f}$$

which agrees with the Newtonian-gravity acceleration result $d^2r/dt^2 = -(GM/r^2)$ *only when the test body's asymptotic radial velocity* $v \ll c$. On the other hand, when the test body's asymptotic radial velocity $v > c/\sqrt{3} = 0.57735c$, *the test body's asymptotic radial acceleration becomes positive.* Since the speed $c/\sqrt{3}$ corresponds to a redshift z of approximately 1, in Einstein coordinates galaxies whose redshifts exceed 1 are in the process *of increasing their redshifts* (i.e., their acceleration *is in the same direction* as their recession velocity). This is *an entirely natural gravitational phenomenon in Einstein coordinates* which *doesn't* require ad hoc insertion of a "dark energy" cosmological-constant term $\lambda g_{\mu\nu}(x)$ into the Einstein equation.

For a radially-traveling packet of light, $v = c$, so Eq. (4.8f) tells us that its asymptotic radial acceleration is $+2\big(GM/r^2\big)$, *which is opposite in direction and double in magnitude* the radial acceleration $-\big(GM/r^2\big)$ of a test body at rest (i.e., $v = 0$) at the same radius r as that packet of light. It is therefore apparent, as Einstein came to realize in his landmark November 18, 1915 paper [6], that straightforward application of the Principle of Equivalence *fails altogether for light.*

Furthermore, putting the value of v to c in Eq. (4.8a) yields,

$$(dr/dt)^2 = c^2\chi(q), \tag{4.8g}$$

where $\chi(q)$ *increases monotonically from zero at* $q = 0$ *toward unity as* $q \to \infty$, as can be verified by analyzing Eqs. (4.8b) and (4.8d). Therefore in Einstein coordinates a radially-traveling light packet's speed in the gravitational field of a point mass *is less than* c, and the *closer* the light packet is to the point mass, the *slower* its speed is. Thus the effect of gravity on light *is refractive*, as astronomers *are well aware.*

The red and blue $q(u)$ curves of Figure 1 show the growth of the expanding model universe's dimensionless scaled radius q as a function of its dimensionless scaled time u in Friedmann and Einstein coordinates respectively. Although the Einstein-coordinate blue $q(u)$ curve increases exponentially from very slightly positive values when u is much less than -1, *it nevertheless is very quickly overtaken by the Friedmann-coordinate red* $q(u)$ *curve*, which only increases from zero when u is greater than $u_i = -\sqrt{2} + \ln(1 + \sqrt{2}) = -0.53284$. That occurs *because the initial rate of increase of the Friedmann-coordinate red* $q(u)$ *curve is unbounded* (see the $dq(u)/du$ red curve of Figure 2).

The extremely gravitationally time-dilated "suspended animation" state of the model universe in Einstein coordinates *dissipates around* $u = 1$, *after* $q(u)$ passes the value 1 and its inflationary acceleration of expansion *peaks* (see the blue curve of Figure 3), immediately *following* which its radius and radial expansion velocity *are quite appreciably less than those of the model universe in Friedmann coordinates* (compare the blue to the red curves of Figures 1 and 2 during the era of values of u which are roughly between 1 and 5). This makes the model universe in Einstein coordinates more amenable to galaxy formation during that early era of values of u roughly between 1 and 5 than that model universe in Friedmann coordinates is.

Furthermore, the *nature* of the model universe in Einstein coordinates at values of u much less than -1 *is that of an extremely slowly expanding zero-angular-momentum extremely gravitationally time-dilated black hole.* (Recall, however, that *physical* black holes, no matter how extremely gravitationally time-dilated, *don't have event horizons.*) It seems *reasonable* that the inflationary-peak *breakdown at values of* u *around* 1 *of such an expanding zero-angular-momentum extremely gravitationally time-dilated black hole produces a great many nonzero-angular-momentum extremely gravitationally time-dilated black holes which are individually stable*, along with a considerably *lesser* amount of matter *not* organized into such nonzero-angular-momentum extremely time-dilated stable black holes. The *existence of such constituents* of a relatively compact, not-too-rapidly-expanding universe would be favorable *to early gravitational condensation of compact galaxies*, and *such compact galactic environments would have promoted the rapid birth of stars, including very short-lived ultraviolet giants, from those galaxies' considerably lesser amount of matter not organized into nonzero-angular-momentum extremely time-dilated stable black holes.* These compact early galaxies would have been bright and hot well into the ultraviolet; their frequency-downshifted black-body radiation may be the cosmic microwave background of the present era.

The *unbounded-radial-velocity explosive Big-Bang birth* of the model universe in Friedmann coordinates *contrariwise is highly unfavorable to early formation of nonzero-angular-momentum extremely gravitationally time-dilated stable black holes or consequent early gravitational condensation of compact galaxies.*

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