Numerical solution of Bagley-Torvik equation based on hybrid functions

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ABSTRACT

This paper proposes an efficient numerical solution of Bagley-Torvik equation which shows vibrating process of a rigid plate immersed in a Newtonian fluid. The proposed approaches are proved to get numerical solution with high accuracy for fractional order differential equation of a version of different Bagley-Torvik equation by simulation.

Keywords: system identification, non-parameter model, external excitation

1. Introduction

Fractional calculus, firstly proposed by Leibniz about 300 years ago, is now being widely used in many aspects of engineering, sociology, etc. at the moment [1-4]. It performs a function to model and analyze auto-control, signal processing, virus infection model, heat transmitting properties of porous media, etc. The object's vibration property joggling through newton viscous fluid can be modeled by fractional derivative. This function is Bagley-Torvik equation [5-8]. Worldwide, there exist a number of solutions for that function [9-15]. The paper also proposes hybridization of basic function which can improve different numerical solution of fractional calculus, its fractional operational matrix and efficient solution. Simulation result proves high accuracy of numerical value of the proposed approach. It is worth mentioning that Hybrid functions of orthonormal Bernoulli polynomials and block impulse functions has not been used in solving the Bagley-Torvik equation.

The paper is as follows: Section 2 covers fundamental conception. Section 3 covers the proposed solution. Section 4 covers simulation results. Section 5 is for conclusion.

Section 2 covers preliminaries and section 3 covers the proposed non-parametric identification method. Section 4 covers the analysis on the simulation. Section 5 covers Conclusion.

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2．Preliminaries

1) Main Definitions

Definition 1 [16]

If $\Omega=[a,b],(-\infty<a<b<+\infty)$ is finite interval on Real axis R, Riemann-Liouville's $\alpha\in R^{+}$ order fractional integral $I_{a+}^{\alpha}f$ is defined as follows;

$$\left(I_{a+}^{\alpha}f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_{a}^{x}\frac{f(t)}{(x-t)^{1-\alpha}}dt,(x>a,\alpha>0) \tag{2.1}$$

Where $\Gamma(\cdot)$ is gamma function .

Definition 2 [16]

Riemann-Liouville's $\alpha\in R^{+}$ order fractional derivative $D_{a+}^{\alpha}y$ is defined as follows;

$$\left(D_{a+}^{\alpha}y\right)(x)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^{n}\int_{a}^{x}\frac{y(t)}{(x-t)^{\alpha-n+1}}dt,(n=[\alpha]+1,x>a) \tag{2.2}$$

If $\alpha=n\in N_{0}$,

$$\left(D_{a+}^{0}y\right)(x)=y(x),\ \left(D_{a+}^{n}y\right)(x)=y^{(n)}(x) \tag{2.3}$$

Definition 3 [16]

Caputo's Fractional order derivative $\left({}^{C}D_{a+}^{\alpha}y\right)(x)$, defined in the interval $[a,b]$ on real axis R, is defined as follows based on Riemann-Liouville's fractional derivative function $\left(D_{a+}^{\alpha}y\right)(x)$;

$$\left({}^{C}D_{a+}^{\alpha}y\right)(x)=\left(D_{a+}^{\alpha}\left[y(x)-\sum_{k=0}^{n-1}\frac{y^{(k)}(a)}{k!}(x-a)^{k}\right]\right) \tag{2.4}$$

where, $n=[\alpha]+1$.

2) Mathematical modeling of vibrating process of diving apparatus in a Newton' fluid [7]

Diving apparatus with S (mass surface), M(mass) in Newton liquid with μ (acoustic coefficient), ρ (density) is vibrating by the external force and kicking force of the spring, hanging to the massless spring. Then, the equation for dislocation is as follows;

$$F_{\mathrm{R}}=-2S\sqrt{\frac{\mu\rho}{\pi}}\int_{-\infty}^{t}\frac{dv}{dt}\frac{d\tau}{(t-\tau)^{1/2}}$$

where external force $f(t)$, dislocation $y(t)$, the velocity of apparatus is $v(t)=dy/dt$, acceleration is $a(t)=d^2y/dt^2$, force of spring to apparatus is $-ky(t)$, basset force is F_{R}.

The kinetic equation of apparatus by Newton's 2nd law of motion is

$$M\frac{d^2y}{dt^2}=F_{\mathrm{R}}-k\cdot y(t)+f(t)$$

$$=-2s\sqrt{\mu\rho}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{t}\frac{d^2y}{dt^2}\frac{d\tau}{(t-\tau)^{1/2}}-k\cdot y(t)+f(t)$$

Then,

$$M\frac{d^2y}{dt^2}+2s\sqrt{\mu\rho}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{t}\frac{d^2y}{dt^2}\frac{d\tau}{(t-\tau)^{1/2}}+k\cdot y(t)=f(t)$$

When $t\le 0$, $y=0, v=0,\ dv/dt=0, f(t)=0$,

$$Ma(t)=M\frac{d^2y}{dt^2}=F_{\mathrm{R}}-k\cdot y(t)+f(t)$$

$$=-2s\sqrt{\mu\rho}\frac{1}{\sqrt{\pi}}\int_{0}^{t}\frac{d^2y}{dt^2}\frac{d\tau}{(t-\tau)^{1/2}}-k\cdot y(t)+f(t),\ \ t>0$$

Considering $m=2$, $\alpha=3/2,\ \Gamma(m-\alpha)=\Gamma(1/2)=\sqrt{\pi}$,

$$\frac{1}{\sqrt{\pi}}\int_0^t D^2y(t)\frac{1}{(t-\tau)^{1/2}}d\tau=\frac{1}{\Gamma(2-1/2)}\int_0^t D^2y(t)\frac{1}{(t-\tau)^{1/2}}d\tau=I^{2-3/2}D^2y(t)=D^{3/2}y(t)$$

The vibrating equation of diving apparatus is as follows;

$$M\frac{d^2y}{dt^2}+2S\sqrt{\mu\rho}D^{1.5}y(t)+ky(t)=f(t),\quad t>0. \qquad (2.5)$$

3. Main Results

1) Fractional operational matrix based on hybrid base function

Definition 4 [17]

Block impulse function (BPF) is defined as follows;

$$b_n(t)=\begin{cases}1, & (n-1)h\le t<nh\\ 0, & otherwise\end{cases} \qquad (3.1)$$

Where $h=T/N$ (N is the number of intervals by BPF.)

If $B_N(t)=\{b_1(t),\cdots,b_N(t)\}$, fractional derivative of $B_N(t)$ is approximated in matrix as follows;

$$I^{\alpha}B_N(t) \approx F^{\alpha}B_N(t). \tag{3.2}$$

where F^{α} is such a matrix.

$$F^{\alpha} = \frac{h^{\alpha}}{\Gamma(\alpha+2)}\begin{pmatrix} \xi_1 & \xi_2 & \xi_3 & \cdots & \xi_N \\ 0 & \xi_1 & \xi_2 & \cdots & \xi_{N-1} \\ 0 & 0 & \xi_1 & \cdots & \xi_{N-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \xi_1 \end{pmatrix}. \tag{3.3}$$

$$\xi_1 = 1, \xi_n = n^{\alpha+1} - 2(n-1)^{\alpha+1} + (n-2)^{\alpha+1},\ n = 2,\cdots,N$$

Matrix F^{α} is called as block impulse operational matrix.

Definition 5

M-order orthonormal Bernoulli polynomials is as follows [18];

$$\phi_m(t) := \sqrt{2m+1}\sum_{k=0}^{m}(-1)^k C_m^k C_{2m-k}^m t^{m-k}, m = 0,1,2\cdots \tag{3.4}$$

From orthonormality,

$$\int_0^1 \phi_i(t)\phi_j(t)dt = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases} \quad i,j = 0,1,2\cdots \tag{3.5}$$

Hybrid functions of orthonormal Bernoulli polynomials and block impulse functions are as follows;

$$\psi_{mn}(t) = \begin{cases} \phi_m\left(\frac{t}{h} - n + 1\right), & (n-1)h \le t < nh \\ 0, & otherwise \end{cases} \tag{3.6}$$

Where

$$\langle \psi_{mn}, \psi_{qp} \rangle = \int_0^T \psi_{mn}(t)\psi_{qp}(t)dt = \begin{cases} h, & m=q, n=p \\ 0, & otherwise \end{cases} \tag{3.7}$$

Absolute integrality function $y(t)$ is developed in hybrid of orthonormal Bernoulli polynomials and BPFs (HOBPBPF) as follows;

$$y(t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} y_{mn}\psi_{mn}(t) \tag{3.8}$$

Symbol $<\cdot,\cdot>$ is inner product. If it is divided into limited entry,

$$y(t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} y_{mn}\psi_{mn}(t) \approx \sum_{m=0}^{M}\sum_{n=1}^{N} y_{mn}\psi_{mn}(t) = Y^T\Psi(t) \tag{3.9}$$

where $Y=[y_{01},\cdots,y_{M1},\cdots,y_{0N},\cdots,y_{MN}]^T$, $\Psi(t)=[\psi_{01}(t),\cdots,\psi_{M1}(t),\cdots,\psi_{0N}(t),\cdots,\psi_{MN}(t)]^T$.

Fractional integral of hybrid vector $\Psi(t)$ is described in the form of algebra as follows;

$$I^{\alpha}\Psi(t)\approx M^{\alpha}_{s\times s}\Psi(t) \tag{3.10}$$

where $\mathrm{M}^{\alpha}_{s\times s}$ is s-order square matrix and is called as Fractional integral operational matrix by hybrid of block impulse function and orthonormal Bernoulli polynomials.

Actually, $\Psi(t)$ is described as

$$\Psi(t)\approx \mathrm{P}_{s\times s}\mathrm{B}_s(t). \tag{3.11}$$

where $\mathrm{B}_s(t)=[b_1(t),b_2(t),\cdots,b_s(t)]^T$ is block impulse function matrix, $P_{s\times s}$ is transformation matrix. $P_{s\times s}$ is usually block diagonal matrix for certain N or M, and is described as follows;

$$P_{s\times s}=\begin{pmatrix} A & & & & \\ & A & & & \\ & & A & & \\ & & & \ddots & \\ & & & & A \end{pmatrix} \tag{3.12}$$

Where $A=(a_{mi})_{(M+1)\times(M+1)}$ is $M+1$ -order square matrix and its element a_{mi} is described as follows;

$$a_{mi}=\sum_{k=0}^{m-1}(-1)^k\sqrt{2m+1}C^k_{m-1}C^{m-1}_{2m-k-2}\frac{[i^{m-k}-(i-1)^{m-k}]}{(m-k)(M+1)^{m-k-1}}\ ,(i=1,2,\cdots,M+1) \tag{3.13}$$

Where, $\frac{[i^{m-k}-(i-1)^{m-k}]}{(m-k)(M+1)^{m-k-1}}$ is average integral value of $y=t^{m-k}$.

With $P_{s\times s}$ in (3.11), the following expression is obtained.

$$B_s(\mathrm{t})=P^{-1}_{s\times s}\Psi(t), \tag{3.14}$$

so displace to (3.14),

$$I^{\alpha}\Psi(t)\approx \mathrm{P}_{s\times s}F^{\alpha}\mathrm{P}^{-1}_{s\times s}\Psi(t) \tag{3.15}$$

In (3.15), fractional integral operating matrix by hybrid is described as follows;

$$\mathrm{M}^{\alpha}_{s\times s}=\mathrm{P}_{s\times s}F^{\alpha}\mathrm{P}^{-1}_{s\times s} \tag{3.16}$$

2) Approximate solution of Bagley-Torvik equation

Derive numerical value solution formula based on hybrid basic function of Bagley-Torvik equation.

$$m\ddot{x}(t)+c_1D_t^{3/2}x(t)+c_2\dot{x}(t)+kx(t)=f(t) \tag{3.17}$$

Adopting 2nd order fractional integral on two sides,

$$mx(t)+c_1I_t^{1/2}x(t)+c_2I_t^1x(t)+kI_t^2x(t)=I_t^2f(t)$$

The approximation is

$$mx^T\Psi(t)+c_1x^TPF^{1/2}P^{-1}\Psi(t)+c_2x^TPF^1P^{-1}\Psi(t)+kx^TPF^2P^{-1}\Psi(t)$$
$$=f^TPF^2P^{-1}\Psi(t)$$

Or

$$mx^T\Psi(t)+c_1x^TM_1\Psi(t)+c_2x^TM_2\Psi(t)+kx^TM_3\Psi(t)=f^TM_3\Psi(t)$$

and

$$x^{\mathrm{T}}=f^TM_3\left(mI+c_1M_1+c_2M_2+kM_3\right)^{-1} \tag{3.18}$$

4 Simulation result

Example 1.

Study Bagley-Torvik equation with $m=1, c_1=0.2, c_2=0.1, k=1, f(t)=sin\,t$, initial condition $x(0)1, x'(0)=0$.

$$\ddot{x}(t)+0.2D_t^{3/2}x(t)+0.1\dot{x}(t)+x(t)=sin\,t \tag{4.1}$$

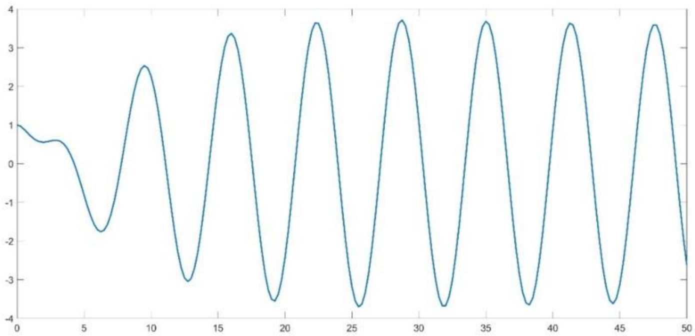

Fig 1. Solution result for Eq.(4.1)

$$\ddot{x}(t)+D_t^{3/2}x(t)+x(t)=0 \tag{4.2}$$

when $m=c=k=1, f(t)=0$, $x(0)=1, x'(0)=0$, the solution result for different α is as follows;

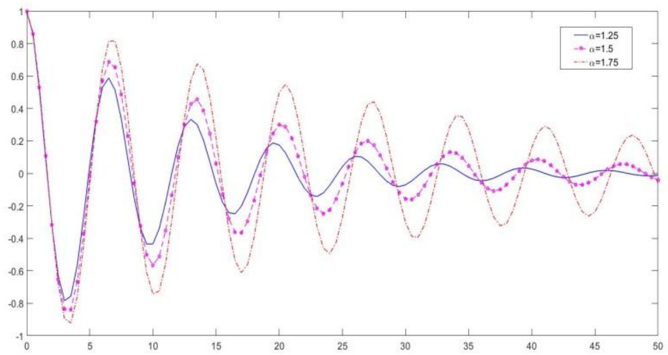

Fig2. The solution result for Eq.(4.2)

5 Conclusion

This paper proposed a new type of hybrid basic function using hybrid of orthonormal Bernoulli polynomials and block impulse function and improved the accuracy of function approximation by it. Having offered a new sort of factional integral operating matrix, it also proposed efficient algorithm for approximate solution of fractional differential equation. The proposed approach improved the accuracy of numerical value solution of Bagley-Torvik fractional differential equation, which modeled vibrating process of diving apparatus in acoustic liquid.

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