Constraining method of control input acceleration by using MPC

Kim Yong Nam, Choe Song Byok, Ri Kyong Hyok, Choe Jong Hyok
Faculty of Automatics, Kim Chaek University of Technology,
Pyongyang, Democratic People's Republic of Korea

Correspondence
Kim Yong Nam, Faculty of Automatics, Kim Chaek University of Technology,
Pyongyang, Democratic People's Republic of Korea.
Email: yn.Kim223@star-co.net.kp

Summary

If we account the constraints on the incremental acceleration as well as the incremental rate of the control input in the control system, the physical limit of the actuator can be considered and the stress of the system can be reduced.

This paper deals with the design of the model predictive controller with the constraint on the incremental acceleration of the control input. The discrete time state space model of the plant is extended to the augmented model which has the incremental acceleration of the control input as the new input. The quadratic cost function, which is based on the control error and the incremental acceleration of the control input, is adopted. Constraints on the plant output, the control input, its incremental rate and acceleration are accounted. This constrained MPC problem can be solved by the quadratic programming procedure.

To select the reasonable sampling period and prediction horizon, the lower bound of the settling time is derived for the proposed MPC system. Numerical examples illustrate the validity and effectiveness of the proposed method.

KEYWORDS

model predictive control, incremental acceleration, constraint, settling time, discrete time system

1. INTRODUCTION

In practice, the control systems are operated under the various constraints. The actuator has its own physical limits on the operating range and speed, and consequently the control input is constrained. Some systems introduce the constraints on the control input intentionally for the safety.[1]

The interest in the control under the constraints has been increased and the different control methods have been studied to cope with the constraints, given the importance for practical applications.[2]

For the discrete-time linear systems subject to control input constraints, the stabilizing dynamic controller was studied based on the polynomial approach.[3] The stabilizing linear state feedback was studied such that the closed-loop system remains asymptotically stable respecting constraints on both the control input and its increment rate.[4] General framework for the design of linear controllers was developed to cope with time-varying constraints on the inputs, states and outputs, and the design procedure was proposed concerning the step response characteristics such as the steady state errors, settling times and overshoots.[5] For the saturated control input the robust feedforward compensator was designed with the gain and phase specifications.[6] The reference governors have been studied to cope with the state and control constraints by modifying the reference commands designed for the unconstrained closed-loop system.[2] For a chain of three integrators with the bounded control input, a discrete time nonlinear control law was studied to guarantee the minimum time global stabilization, which could be used as the nonlinear filter for the trajectory generation. [7]

The model predictive control is the outstanding one among the kinds of control methods accounting the constraints. [8,9,10]

The discrete time model predictive controller repeats the process, at each sampling period, in which the sequence of the control input optimizing the cost function subject to the constraints is found based on the predicted behavior of the plant and then the first step of the sequence is output to the plant. Usually the cost function has the quadratic form on both the control error and the control input or its incremental rate, and the constraints involve the ones on the plant output, control input and its incremental rate. [11,12,13] There are several methods ready to solve the constrained optimization problem for the model predictive control.[11]

The model predictive controller with the constraint on the incremental rate of the control input was studied based on the augmented discrete time state space model with the incremental rate of the control input as the new input. [11] In the same manner, based on the augmented model with the incremental acceleration of control input as the new input, model predictive control with the constraint on the incremental acceleration of the control input can be designed.

In fact, the real actuators cannot change their operating speed instantly and they work within the certain acceleration limit.

Constrained on the incremental acceleration of the control input, the incremental rate of the control input will vary smoothly and the system will experience less stress, but settle down slowly.

The longer settling time can degrade the prediction performance in the model predictive control system. Therefore, in the model predictive control system with the constraints, the information on the settling time is valuable for the selection of the sampling period and the prediction horizon to guarantee the prediction performance.

To the best our knowledge there are no studies on the model predictive control with the constraint on the incremental acceleration of the control input.

The aim of our work is to design the model predictive controller dealing with the constraint on the incremental acceleration of the control input. To do that the discrete time state space model of the plant is extended to the augmented model which has the incremental acceleration of the control input as the new input. And the quadratic cost function is formulated based on the control error and the control input incremental acceleration in the prediction horizon. Constraints on the plant output, control input, its incremental rate and acceleration are setup and represented in terms of the incremental acceleration of the control input. The quadratic programming problem in the proposed model predictive controller can be solved by the various methods such as Hildreth's procedure.[11]

And then, for the SISO plant, the lower bound of the settling time of the proposed model predictive control system is derived to select the reasonable sampling period and prediction horizon.

The paper is organized as follows. In Section 2, the model predictive control problem with the constraints on the incremental acceleration of the control input is formulated. In Section 3, the cost function and the constraints of the MPC problem are arranged in terms of the incremental acceleration of the control input so that it can be solved with the quadratic programming procedures. In Section 4, the lower bound of settling time in the proposed model predictive control system are derived. In Section 5, the numerical examples illustrate the validity and effectiveness of the proposed model predictive controller. In Section 6, the conclusion is given.

2. PROBLEM FORMULATION

Let's consider the discrete time state space model of the plant

$$\begin{cases} x_0(k+1) = A_0 x_0(k) + B_0 u(k) + B_{d0} d(k) \\ y(k) = C_0 x_0(k) \end{cases}, \tag{1}$$

where $x_0(k) \in \mathrm{R}^{n_0}, u(k) \in \mathrm{R}^m, y(k) \in \mathrm{R}^q, d(k) \in \mathrm{R}^p$ are the state vector, control input vector (control variables), plant output vector (controlled variables) and disturbance vector respectively at the step k, and it is assumed that $m > q$.

It is assumed that the disturbance $d(k)$ is generated from the white noise $\epsilon(k)$ by the equations

$$d(k) - d(k-1) = \Delta d(k), \Delta d(k) - \Delta d(k-1) = \epsilon(k). \tag{2}$$

1) Augmented state space model

Let's define the incremental rate of $u(k)$:

$$\Delta u(k) = u(k) - u(k-1) \tag{3}$$

and the incremental acceleration of $u(k)$:

$$\Delta^2 u(k) = \Delta u(k) - \Delta u(k-1). \tag{4}$$

And $\Delta x_0(k), \Delta^2 x_0(k), \Delta y(k), \Delta^2 y(k)$ are also defined as follows.

$\Delta x_0(k) = x_0(k) - x_0(k-1),$

$\Delta^2 x_0(k) = \Delta x_0(k) - \Delta x_0(k-1),$

$$\Delta y(k) = y(k) - y(k-1),$$
$$\Delta^2 y(k) = \Delta y(k) - \Delta y(k-1)$$

Based on the plant model (1), let's construct the augmented state space model

$$\begin{cases} x(k+1) = Ax(k) + B\Delta^2 u(k) + B_\epsilon \epsilon(k) \\ y(k) = Cx(k) \end{cases}, \tag{5}$$

$$x(k) = \begin{bmatrix} \Delta^2 x_0(k) \\ \Delta y(k) \\ y(k) \end{bmatrix}, \tag{6}$$

where the matrices

$$A = \begin{bmatrix} A_0 & 0 & 0 \\ C_0 A_0 & I_{q\times q} & 0 \\ C_0 A_0 & I_{q\times q} & I_{q\times q} \end{bmatrix}, B = \begin{bmatrix} B_0 \\ C_0 B_0 \\ C_0 B_0 \end{bmatrix}, B_\epsilon = \begin{bmatrix} B_{d0} \\ C_0 B_{d0} \\ C_0 B_{d0} \end{bmatrix}, C = [0_{q\times n} \quad 0_{q\times q} \quad I_{q\times q}]. \tag{7}$$

The Matrices A, B, C can be calculated from the following equations.

$$\begin{aligned} \Delta^2 x_0(k+1) &= \Delta x_0(k+1) - \Delta x_0(k) \\ &= A_0 \Delta x_0(k) + B_0 \Delta u(k) + B_{d0} \Delta d(k) - A_0 \Delta x_0(k-1) \\ &\quad - B_0 \Delta u(k-1) - B_{d0} \Delta d(k-1) \\ &= A_0 \Delta^2 x_0(k) + B_0 \Delta^2 u(k) + B_{d0} \epsilon(k), \\ \Delta y(k+1) &= y(k+1) - y(k) = C_0 \Delta x_0(k+1) \\ &= C_0 \Delta^2 x_0(k+1) + C_0 \Delta x_0(k) \\ &= C_0 A_0 \Delta^2 x_0(k) + C_0 B_0 \Delta^2 u(k) + C_0 B_{d0} \epsilon(k) + \Delta y(k), \end{aligned}$$

$$\begin{aligned} y(k+1) &= y(k) + \Delta y(k+1) = y(k) + \Delta y(k) + \Delta^2 y(k+1) \\ &= y(k) + \Delta y(k) + C_0 \Delta^2 x_0(k+1). \end{aligned}$$

The input of the augmented model (5) is the incremental acceleration of control input $\Delta^2 u(k)$.

2) Cost function and constraints in MPC

Given the state $x(k_i)$ and the incremental acceleration sequence of the control input $\Delta^2 u(k_i + k), k = 0, \ldots N_c - 1$, let's suppose that $y(k_i + k \mid k_i), k = 1, \ldots, N_p$ are the k step predicted values of the plant output $y(k_i)$, calculated at the step k_i.

Here it is assumed that $N_c < N_p$ and $\Delta^2 u(k_i + k) = 0, k = N_c, \ldots, N_p$.

Let's suppose that $r(k) \in R^q$ is the reference vector for the plant output $y(k)$.

The cost function is formulated as follows.

$$J = \sum_{j=1}^{Np} \| r(k_i + j) - y(k_i + j | k_i) \|_2 + \sum_{j=0}^{N_c - 1} \Delta^2 u(k_i + j)^T R_w \Delta^2 u(k_i + j) \tag{8}$$

The cost function is constructed with the 2-norms of the control error vectors and the R_w weighted 2-norms of the incremental acceleration vectors of the control input during the prediction horizon N_p, N_c.

The control input u, its incremental rate Δu and acceleration $\Delta^2 u$, and the plant output y are constrained by the limit vectors $\Delta^2 u_{max}, \Delta u_{max}, u_{min}, u_{max}, y_{min}, y_{max}$ as follows.

$$-\Delta^2 u_{max} \le \Delta^2 u(k_i + k) \le \Delta^2 u_{max}, \; k = 0, \ldots, N_c - 1, \tag{9}$$

$$-\Delta u_{max} \le \Delta u(k_i + k) \le \Delta u_{max}, \; k = 0, \ldots, N_c - 1, \tag{10}$$

$$u_{min} \le u(k_i + k) \le u_{max}, \; k = 0, \ldots, N_c - 1, \tag{11}$$

$$y_{min} \le y(k_i + k \mid k_i) \le y_{max}, \; k = 1, \ldots, N_p. \tag{12}$$

The model predictive control problem with the constraint on the incremental acceleration of the control input is to determine the optimal incremental acceleration of the control input $\Delta^2 \mathbf{u}(k_i)$

minimizing the cost function (8) under the constraints (9-12) for the augmented state space model (5-7).

3. Design of the model predictive controller with the constraint on the incremental acceleration of the control input

In this section the cost function and the constraints are arranged in terms of $\Delta^2 u$ so that the formulated MPC problem can be solved by the quadratic programming procedure.

Let's define the column vectors $Y(k_i) = \left[y(k_i + 1 \mid k_i)^T, \ldots, y(k_i + N_p \mid k_i)^T\right]^T$ and $\Delta^2 U(k_i) = [\Delta^2 u(k_i)^T, \ldots, \Delta^2 u(k_i + N_c - 1)^T]^T$.

The predicted output sequence vector Y can be represented as

$$Y(k_i) = Fx(k_i) + \Phi \Delta^2 U(k_i), \tag{13}$$

where the matrices F, Φ are

$$F = \begin{bmatrix} CA \\ CA^2 \\ \ldots \\ CA^{N_p} \end{bmatrix}, \ \Phi = \begin{bmatrix} CB & 0 & \ldots & 0 \\ CAB & CB & \ldots & 0 \\ \ldots & \ldots & \ldots & 0 \\ CA^{N_p-1}B & CA^{N_p-2}B & \ldots & CA^{N_p-N_c}B \end{bmatrix}. \tag{14}$$

[12]

Let's define the column vector of the reference sequence $R_s(k_i) = \left[r(k_i + 1)^T, \ldots, r(k_i + N_p)^T\right]^T$ and the weight matrix

$$\bar{R} = diag(\overbrace{R_w, \ldots, R_w}^{N_c}). \tag{15}$$

The cost function (8) can be written as follows.

$$J = \left(R_s(k_i) - Y(k_i)\right)^T \left(R_s(k_i) - Y(k_i)\right) + \Delta^2 U(k_i)^T \bar{R} \Delta^2 U(k_i) \tag{16}$$

In the unconstrained case the optimal incremental acceleration sequence of the control input $\Delta^2 U^*(k_i) = \mathrm{argmin}\, J$ can be derived as follows.

$$J = \left(R_s(k_i) - Fx(k_i)\right)^T \left(R_s(k_i) - Fx(k_i)\right) - 2\Delta^2 U(k_i)^T \Phi^T \left(R_s(k_i) - Fx(k_i)\right) + \Delta^2 U(k_i)^T (\Phi^T \Phi + \bar{R}) \Delta^2 U(k_i). \tag{17}$$

$$\frac{\partial J}{\partial \Delta^2 U(k_i)} = -2\Phi^T \left(R_s(k_i) - Fx(k_i)\right) + 2(\Phi^T \Phi + \bar{R}) \Delta^2 U(k_i) = 0.$$

$$\Delta^2 U^*(k_i) = (\Phi^T \Phi + \bar{R})^{-1} \Phi^T \left(R_s(k_i) - Fx(k_i)\right). \tag{18}$$

$\Delta^2 u^*(k_i)$, which is the optimal incremental acceleration of the control input at the step k_i, can be found in the first m elements of $\Delta^2 U^*(k_i)$.

In the cost function (17) the first term does not depend on the $\Delta^2 U(k_i)$ and then the cost function can be written as

$$J = -2\Delta^2 U(k_i)^T \Phi^T \left(R_s(k_i) - Fx(k_i)\right) + \Delta^2 U(k_i)^T (\Phi^T \Phi + \bar{R}) \Delta^2 U(k_i). \tag{19}$$

Now the constraints (9)~(12) is represented in terms of $\Delta^2 U(k_i)$ and arranged into the matrix inequality form.

At first, the constraint on the incremental acceleration of the control input (9) is represented as

$$M_{\Delta^2 u} \Delta^2 U(k_i) \le N_{\Delta^2 u}, \tag{20}$$

where the matrix $M_{\Delta^2 u}$ and the vector $N_{\Delta^2 u}$ are

$$M_{\Delta^2 u} = \begin{bmatrix} -I_{mN_c \times mN_c} \\ I_{mN_c \times mN_c} \end{bmatrix}, \ N_{\Delta^2 u} = \begin{bmatrix} \Delta^2 U_{max} \\ \Delta^2 U_{max} \end{bmatrix}, \tag{21}$$

where the column vector $\Delta^2 U_{max}$ is

$$\Delta^2 U_{max} = \begin{bmatrix} \overbrace{\Delta^2 u_{max}^T \quad \ldots \quad \Delta^2 u_{max}^T}^{N_c} \end{bmatrix}^T. \tag{22}$$

Next, the constraint on the incremental rate of the control input (10) can be represented as

$$M_{\Delta u}\Delta^2 U(k_i) \le N_{\Delta u}, \tag{23}$$

where the matrix $M_{\Delta u}$ and the vector $N_{\Delta u}$ are

$$M_{\Delta u} = \begin{bmatrix} -C_0^{\Delta u} \\ C_0^{\Delta u} \end{bmatrix},\ N_{\Delta u} = \begin{bmatrix} \Delta U_{max} + C_1^{\Delta u} u(k_i - 1) \\ \Delta U_{max} - C_1^{\Delta u} u(k_i - 1) \end{bmatrix}, \tag{24}$$

where column vector ΔU_{max} and the matrices $C_1^{\Delta u}, C_0^{\Delta u}$ are

$$\Delta U_{max} = \left[\overbrace{\Delta u_{max}^T \quad \dots \quad \Delta u_{max}^T}^{N_c}\right]^T, \tag{25}$$

$$C_1^{\Delta u} = \begin{bmatrix} I \\ I \\ \vdots \\ I \end{bmatrix},\ C_0^{\Delta u} = \begin{bmatrix} I & 0 & \cdots & 0 \\ I & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ I & I & \cdots & I \end{bmatrix}. \tag{26}$$

This can be calculated from the fact that the $\Delta U(k_i)$ can be written as

$$\Delta U(k_i) = C_1^{\Delta u}\Delta u(k_i - 1) + C_0^{\Delta u}\Delta^2 U(k_i)$$

The constraints on the control input (11) are arranged.

The control inputs $u(k_i), u(k_i + 1), u(k_i + 2), \dots$ can be written as follows.

$$\begin{aligned}
u(k_i) &= \Delta u(k_i) + u(k_i - 1) = \Delta^2 u(k_i) + \Delta u(k_i - 1) + u(k_i - 1), \\
u(k_i + 1) &= \Delta^2 u(k_i + 1) + \Delta u(k_i) + u(k_i) \\
&= \Delta^2 u(k_i + 1) + \Delta^2 u(k_i) + \Delta u(k_i - 1) + \Delta^2 u(k_i) + \Delta u(k_i - 1) \\
&\quad + u(k_i - 1) \\
&= \Delta^2 u(k_i + 1) + 2\Delta^2 u(k_i) + 2\Delta u(k_i - 1) + u(k_i - 1), \\
u(k_i + 2) &= \Delta^2 u(k_i + 2) + \Delta u(k_i + 1) + u(k_i + 1) \\
&= \Delta^2 u(k_i + 2) + \Delta^2 u(k_i + 1) + \Delta u(k_i) + u(k_i + 1) \\
&= \Delta^2 u(k_i + 2) + \Delta^2 u(k_i + 1) + \Delta^2 u(k_i) + \Delta u(k_i - 1) \\
&\quad + u(k_i + 1) \\
&= \Delta^2 u(k_i + 2) + \Delta^2 u(k_i + 1) + \Delta^2 u(k_i) + \Delta u(k_i - 1) \\
&\quad + \Delta^2 u(k_i + 1) + 2\Delta^2 u(k_i) + 2\Delta u(k_i - 1) + u(k_i - 1) \\
&= \Delta^2 u(k_i + 2) + 2\Delta^2 u(k_i + 1) + 3\Delta^2 u(k_i) + 3\Delta u(k_i - 1) \\
&\quad + u(k_i - 1),
\end{aligned}$$

..., ...

Hence $U(k_i)$ can be written as

$$U(k_i) = C_2^u u(k_i - 1) + C_1^u \Delta u(k_i - 1) + C_0^u \Delta^2 U(k_i),$$

where

$$\mathrm{C}_2^{\mathrm{u}} = \begin{bmatrix} I \\ I \\ \vdots \\ I \end{bmatrix},\ \mathrm{C}_1^{\mathrm{u}} = \begin{bmatrix} I \\ 2I \\ \vdots \\ N_c I \end{bmatrix},\ \mathrm{C}_0^{\mathrm{u}} = \begin{bmatrix} I & 0 & \cdots & 0 \\ I & 2I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ I & 2I & \cdots & N_c I \end{bmatrix}. \tag{27}$$

Therefore, the constraint (11) can be represented as

$$M_u \Delta^2 U(k_i) \le N_u, \tag{28}$$

where the matrix M_u and the vector N_u are

$$M_u = \begin{bmatrix} -C_0^u \\ C_0^u \end{bmatrix},\ N_u = \begin{bmatrix} -U_{min} + C_2^u u(k_i - 1) + C_1^u \Delta u(k_i - 1) \\ U_{max} - C_2^u u(k_i - 1) - C_1^u \Delta u(k_i - 1) \end{bmatrix}, \tag{29}$$

where

$$U_{max} = \left[\overbrace{u_{max}^T \quad \dots \quad u_{max}^T}^{N_c}\right]^T, U_{min} = \left[\overbrace{u_{min}^T \quad \dots \quad u_{min}^T}^{N_c}\right]^T. \tag{30}$$

Considering (13), the constraint on the plant output (12) can be represented as

$$M_y \Delta^2 U(k_i) \le N_y \tag{31}$$

where the matrix M_y and the vector N_y are

$$M_y = \begin{bmatrix} -\Phi \\ \Phi \end{bmatrix},\ N_y = \begin{bmatrix} -Y_{min} + Fx(k_i) \\ Y_{max} - Fx(k_i) \end{bmatrix}, \tag{32}$$

where

$$Y_{max} = \Bigg[\overbrace{y_{max}^T \quad \cdots \quad y_{max}^T}^{N_p}\Bigg]^T, Y_{min} = \Bigg[\overbrace{y_{min}^T \quad \cdots \quad y_{min}^T}^{N_p}\Bigg]^T. \tag{33}$$

At last all constraints (20), (23), (28), (31) form the final matrix inequality equation

$$M \cdot \Delta^2 U(k_i) \le N. \tag{34}$$

where

$$M = \begin{bmatrix} M_{\Delta^2 u} \\ M_{\Delta u} \\ M_u \\ M_y \end{bmatrix} \in R^{(6mN_C+2qN_p)\times(mN_c)},\quad N = \begin{bmatrix} N_{\Delta^2 u} \\ N_{\Delta u} \\ N_u \\ N_y \end{bmatrix} \in R^{(6mN_C+2qN_p)\times 1}. \tag{35}$$

Now the model predictive control problem with the constraint on the incremental acceleration of the control input can be reformulated as

$$\min\left[-2\Delta^2 U(k_i)^T\ \Phi^T\ \big(R_s(k_i) - Fx(k_i)\big) + \Delta^2 U(k_i)^T\ (\Phi^T\Phi + \bar{R})\Delta^2 U(k_i)\right] \tag{36}$$

subject to

$$M \cdot \Delta^2 U(k_i) \le N.$$

This quadratic programming problem can be solved by the various methods such as Hildreth's procedure.

4. Lower bound of settling time

Assumption 1. The proposed model predictive controller controls the stable SISO plant of which the steady state gain is K_{ss}. The control system at the zero initial states is tested with the step reference signal $r(k) = A_r, k \ge 0$.

□

When the Assumption 1 is satisfied the control error is zero and the control input has the value of $\frac{A_r}{K_{ss}}$ in the steady state.

The following Lemma can be said without proof.

Lemma 1. Assume that Assumption 1 is satisfied.

For $\forall k \ge k_s, y(k) = A_r, \Delta\ y(k) = 0$, the following equation should be satisfied.

$$\forall k \ge k_s, u(k) = \frac{A_r}{K_{ss}}, \Delta\ u(k) = 0 \tag{37}$$

□

Remark 1. In Lemma 1, the control system is settled at steady state after the k_s steps, therefore k_s can be regarded as the settling time.

□

Definition 1. For any real number a, $ceil(a)$ represents the nearest integer greater than or equal to a.

□

The following theorems tell the lower bound of the settling time to the step reference.

Theorem 1. Assume that Assumption 1 is satisfied and the incremental rate of the control input $\Delta\ u(k)$ is constrained by (10).

To be $\forall k \ge k_s,\ y(k) = A_r, \Delta\ y(k) = 0$, k_s should satisfy

$$k_s \ge k_f^{\Delta u},$$

where

$$k_f^{\Delta u} = ceil\left(\frac{A_r}{K_{ss}\cdot\Delta\, u_{max}}\right). \tag{38}$$

Proof. Lemma 1 should be satisfied.

For any integer $k_f > 0$, the following sequence of $\Delta u(k)$ maximizes $u(k_f)$ subject to $\Delta\, u(k) = 0, k \geq k_f$.

$$\Delta u(k) = \begin{cases} \Delta\, u_{max},\ 0 < k \leq k_f \\ 0,\ elsewise \end{cases} \tag{39}$$

Given $\Delta u(k)$ of (39), if $k_f = k_f^{\Delta u}$, then $\forall k > k_f, u(k) = \sum_{k=1}^{k_f} \Delta u(k) \geq \frac{A_r}{K_{ss}}, \Delta u(k) = 0$. Else if $k_f = k_f^{\Delta u} - 1$, then $\forall k > k_f, u(k) = \sum_{k=1}^{k_f} \Delta u(k) < \frac{A_r}{K_{ss}}, \Delta u(k) = 0$. Therefore $k_f^{\Delta u}$ is the minimum integer which can satisfy Equation (37) under the constraint (10).

□

Remark 2. In theorem 1,

$$\Delta u(k) = \begin{cases} \Delta\, u_{max},\ 0 < k < k_f^{\Delta u} \\ \frac{A_r}{K_{ss}} - \left(k_f^{\Delta u} - 1\right)\Delta\, u_{max},\ k = k_f^{\Delta u} \\ 0,\ elsewise \end{cases} \tag{40}$$

satisfies Equation (37).

□

Theorem 2. Assume that Assumption 1 is satisfied and the incremental acceleration of the control input $\Delta^2\, u(k)$ is constrained by (9).

To satisfy $\forall k \geq k_s,\ y(k) = A_r, \Delta\, y(k) = 0$, k_s should satisfy

$$k_s \geq k_f^{\Delta^2 u},$$

where

$$k_f^{\Delta^2 u} = ceil\left(2\sqrt{\frac{A_r}{K_{ss}\cdot\Delta^2\, u_{max}}}\right) - 1. \tag{41}$$

Proof. Lemma 1 should be satisfied.

For any even number $k_f > 0$, the following $\Delta\,^2 u(k)$ maximizes $u(k_f)$ subject to $\Delta\, u(k) = 0, k \geq k_f$.

$$\Delta\,^2 u(k) = \begin{cases} \Delta^2\, u_{max},\ 0 < k \leq \frac{k_f}{2} \\ -\Delta^2\, u_{max},\ \frac{k_f}{2} < k \leq k_f \\ 0,\ elsewise \end{cases} \tag{42}$$

Given $\Delta\,^2 u(k)$ of (44), $\Delta u(k)$ and $u(k)$ are as follows.

$$\Delta u(k) = \begin{cases} k\cdot\Delta^2\, u_{max},\ 0 < k \leq \frac{k_f}{2} \\ \frac{k_f}{2}\cdot\Delta^2\, u_{max} - \left(k - \frac{k_f}{2}\right)\cdot\Delta^2\, u_{max},\ \frac{k_f}{2} < k < k_f \\ 0,\ elsewise \end{cases} \tag{43}$$

$$u(k_f) = \sum_{i=0}^{k_f} \Delta u(i) = \frac{k_f}{2}\cdot\frac{k_f}{2}\,\Delta^2\, u_{max} \tag{44}$$

If $k_f = 2\cdot ceil\left(\sqrt{\frac{A_r}{K_{ss}\cdot\Delta^2\, u_{max}}}\right)$, then $\forall k \geq k_f, u(k) \geq \frac{A_r}{K_{ss}}, \Delta\, u(k) = 0$. Else if $k_f = 2\cdot ceil\left(\sqrt{\frac{A_r}{K_{ss}\cdot\Delta^2\, u_{max}}}\right) - 2$, then $\forall k \geq k_f, u(k) < \frac{A_r}{K_{ss}}, \Delta\, u(k) = 0$.

For any odd number $k_f > 0$ which satisfies $k_f \leq 2\cdot ceil\left(\sqrt{\frac{A_r}{K_{ss}\cdot\Delta^2\, u_{max}}}\right) - 2$, it is impossible that $\forall k \geq k_f, u(k) = \frac{A_r}{K_{ss}}, \Delta u(k) = 0$. Therefore $k_f^{\Delta^2 u}$ is the minimum integer which can satisfy Equation (37) under the constraint (9).

□

Theorem 3. Assume that Assumption 1 is satisfied and $\Delta^2 u(k), \Delta u(k)$ are both constrained by (9) and (10), respectively.

To satisfy $\forall k \geq k_s,\ y(k) = A_r, \Delta y(k) = 0$, k_s should satisfy

$$k_s \geq k_f^{\Delta u, \Delta^2 u},$$

where

$$k_f^{\Delta u, \Delta^2 u} = ceil\left(\frac{A_r}{K_{ss} \cdot ceil\left(\frac{\Delta u_{max}}{\Delta^2 u_{max}}\right) \cdot \Delta^2 u_{max}}\right) + ceil\left(\frac{\Delta u_{max}}{\Delta^2 u_{max}}\right). \tag{45}$$

Proof. Lemma 1 should be satisfied.

The transient response time is so long enough that the incremental rate of the control input $\Delta u(k)$ can reach its limit value Δu_{max}.

For any integers k_{uf}, k_f, which satisfies $2k_{uf} \leq k_f$, let's form

$$\Delta^2 u(k) = \begin{cases} \Delta^2 u_{max},\ 0 < k \leq k_{uf} \\ -\Delta^2 u_{max},\ k_f - k_{uf} < k \leq k_f \\ 0,\ elsewise \end{cases}. \tag{46}$$

Select k_{uf}, which is the length of period that the control input is accelerated or decelerated, as

$$k_{uf} = ceil\left(\frac{\Delta u_{max}}{\Delta^2 u_{max}}\right). \tag{47}$$

$\Delta^2 u(k)$ of (46) maximizes $u(k_f)$ subject to $\Delta u(k) = 0, k \geq k_f$ and the constraint $\Delta u(k) \leq k_{uf} \cdot \Delta^2 u_{max}$.

Given $\Delta^2 u(k)$ of (46), $\Delta u(k)$ and $u(k)$ are calculated as

$$\Delta u(k) = \begin{cases} k \cdot \Delta^2 u_{max},\ 0 < k \leq k_{uf} \\ k_{uf} \Delta^2 u_{max},\ k_{uf} < k \leq k_f - k_{uf} \\ k_{uf} \Delta^2 u_{max} - \left(k - (k_f - k_{uf})\right) \cdot \Delta^2 u_{max},\ k_f - k_{uf} < k \leq k_f \\ 0,\ elsewise \end{cases}, \tag{48}$$

$$u(k) = k_{uf} \cdot k_{uf} \cdot \Delta^2 u_{max} + (k_f - 2k_{uf}) k_{uf} \Delta^2 u_{max} = (k_f - k_{uf}) k_{uf} \Delta^2 u_{max}. \tag{49}$$

If $k_f = ceil\left(\frac{A_r}{K_{ss} \cdot k_{uf} \cdot \Delta^2 u_{max}}\right) + k_{uf}$ then $\forall k \geq k_f, u(k) \geq \frac{A_r}{K_{ss}}, \Delta u(k) = 0$. Else if $k_f = ceil\left(\frac{A_r}{K_{ss} \cdot k_{uf} \cdot \Delta^2 u_{max}}\right) + k_{uf} - 1$, then $\forall k \geq k_f, u(k) < \frac{A_r}{K_{ss}}, \Delta u(k) = 0$. If $\Delta u(k)$ is further constrained by Δu_{max} rather than $k_{uf} \cdot \Delta^2 u_{max}$, then for $k_f < ceil\left(\frac{A_r}{K_{ss} \cdot k_{uf} \cdot \Delta^2 u_{max}}\right) + k_{uf}$, it is impossible that $\forall k \geq k_f, u(k) \geq \frac{A_r}{K_{ss}}, \Delta u(k) = 0$.

For $k_f = ceil\left(\frac{A_r}{K_{ss} \cdot k_{uf} \cdot \Delta^2 u_{max}}\right) + k_{uf}$, the condition $2k_{uf} \leq k_f$ can be represented as

$$2k_{uf} \leq ceil\left(\frac{A_r}{K_{ss} \cdot k_{uf} \cdot \Delta^2 u_{max}}\right) + k_{uf},$$

$$\frac{A_r}{K_{ss} \cdot k_{uf} \cdot \Delta^2 u_{max}} \geq \frac{\Delta u_{max}}{\Delta^2 u_{max}},$$

$$\frac{A_r}{K_{ss}} \geq ceil\left(\frac{\Delta u_{max}}{\Delta^2 u_{max}}\right) \cdot \Delta u_{max}. \tag{50}$$

□

Remark 3. In theorem 3, the condition (50) should be satisfied to make both constraints active. If not, the control input will decelerate before its incremental rate reaches at its limit value Δu_{max}. This is regarded as the situation in the theorem 2.

□

Remark 4. If $\Delta u_{max} < \Delta^2 u_{max}$ then the constraint on $\Delta^2 u$ is not activated while Δu is only constrained. This is regarded as the situation in theorem 1.

□

Remark 5. $k_f^{\Delta u}, k_f^{\Delta^2 u}$ and $k_f^{\Delta u, \Delta^2 u}$ mean the lower bound of the "ideal" settling time after that the plant output is at the steady state. In the practical analyzing of the control system, 2% settling time is widely used. In the step response the plant output settles within 2% of its steady state value after 2% settling time. It should be noticed that 2% settling time measured could be shorter than the lower bound of the "ideal" settling time.

□

5. Numerical Example

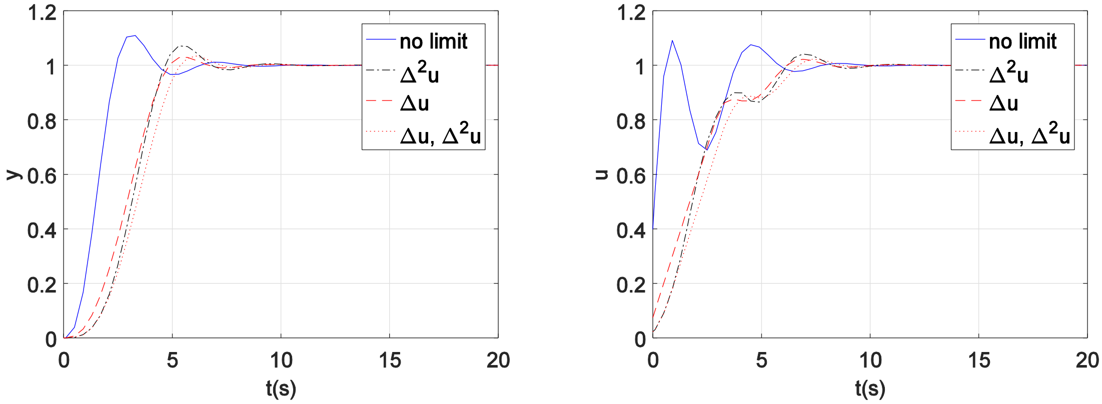
To illustrate the proposed model predictive controller the plant $P(s) = \frac{1}{s^2+s+1}$ is examined.

The plant transfer function is realized in the discrete time state space model to design the model predictive controller and the state observer is used. The plant output y and the control input u is not constrained, while the incremental rate and acceleration of the control input $\Delta u, \Delta^2 u$ are constrained. Hildreth's procedure is used to solve the quadratic programming problem. $\bar{\mathrm{R}} = 1, N_p = 20, N_c = 10$ are used.

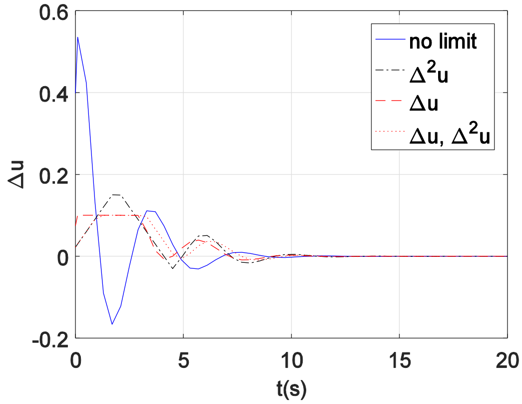
Example 1.

For the sampling period $T_s = 0.4s$, $\Delta u_{max} = 0.1$, $\Delta^2 u_{max} = 0.03$ the plant is controlled in the 4 cases constrained differently. In first case there are no active constraints. In second case $\Delta^2 u$ is only constrained. In third case Δu is only constrained. In the fourth case $\Delta u, \Delta^2 u$ are both constrained.

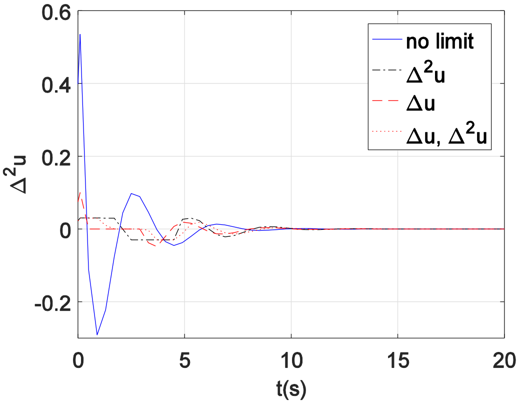
Simulation results are shown in the Figure 1. In cases of constrained $\Delta^2 u$, the inflection point on the response of u, which appeared in early period, reflects the smoothness. But the iteration number in the calculation of Hildreth's procedure, increased, meaning more computation loads.



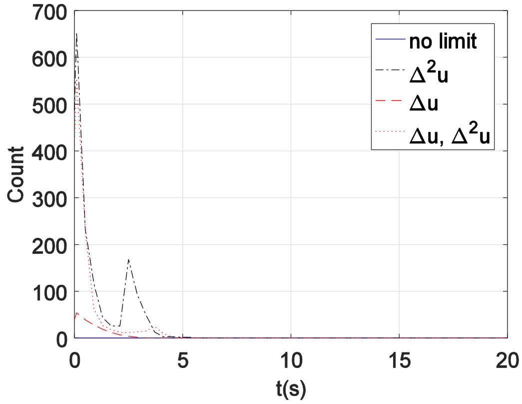
(A) (B)



(C)



(D)



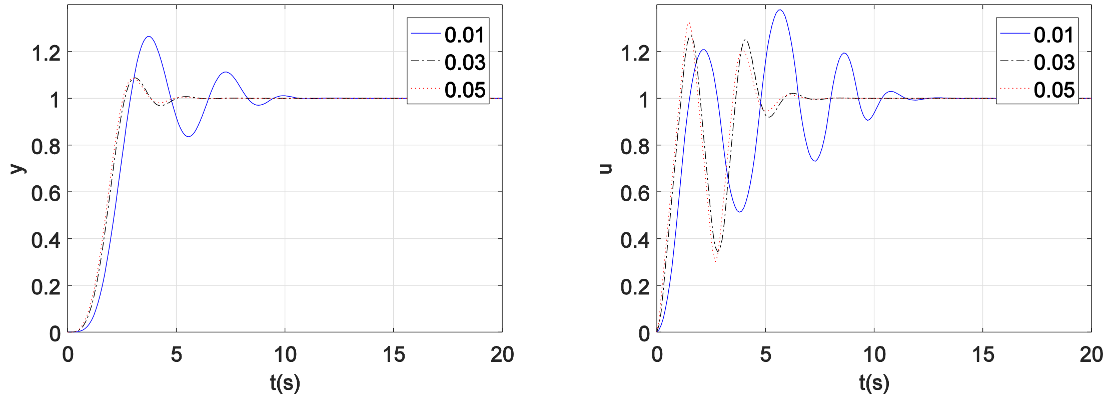
(E)

Figure 1 The graphs of $y, u, \Delta u, \Delta^2 u$ and Count (iteration number) in case of no constraints(-no limit), and $\Delta^2 u$ constrained case ($- \cdot - \Delta^2 u$) and Δu constrained case ($- - \Delta u$) and $\Delta u, \Delta^2 u$ constrained case ($\cdots\cdots \Delta u, \Delta^2 u$). A, y; B, u; C, Δu; D, $\Delta^2 u$; E, Iteration number

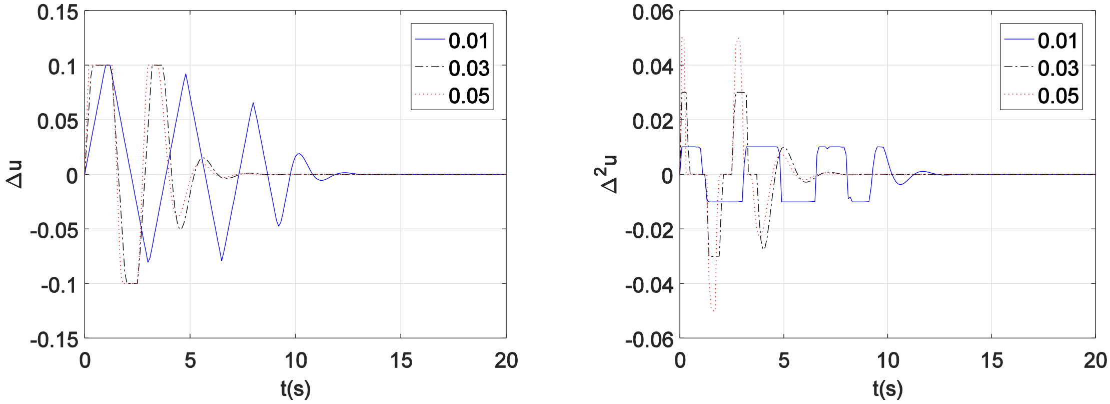
Example 2.

For $T_s = 0.1s$, the control system is simulated under the different constraints on $\Delta^2 u$ with $\Delta^2 u_{max} = 0.01, 0.03, 0.05$ and the constraint on Δu with $\Delta u_{max} = 0.1$.

When $\Delta^2 u_{max} = 0.01, 0.03, 0.05$, the lower bounds of the settling times are calculated as $k_f^{\Delta u, \Delta^2 u} = 19, 13, 12$, respectively, using theorem 3. When $\Delta^2 u_{max} = 0.01$, prediction horizons $N_p = 20$ and $N_c = 10$ are not so long enough to predict the response of the plant output and the control input during the settling period. Simulation results in Figure 2 show that the responses of $\Delta^2 u_{max} = 0.01$ are worse than the ones of $\Delta^2 u_{max} = 0.03, 0.05$.



(A) (B)



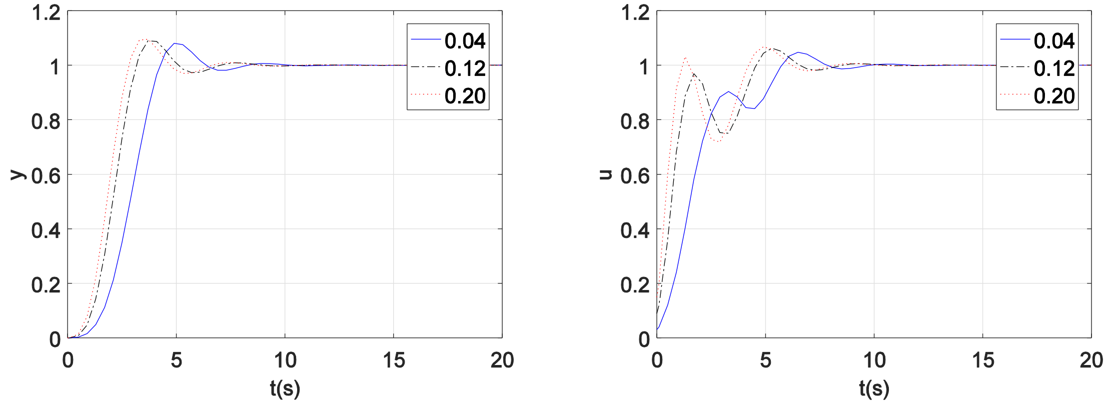
(C) (D)

Figure 2 The graphs of $y, u, \Delta u, \Delta^2 u$ in different cases of $\Delta^2 u$ constraints with $\Delta^2 u_{max} = 0.01$ (- 0.01), $\Delta^2 u_{max} = 0.03(-\cdot-0.03)$ and $\Delta^2 u_{max} = 0.05(\cdots\cdots 0.05)$, for $T_s = 0.1s$. A, y; B, u; C, Δu; D, $\Delta^2 u$

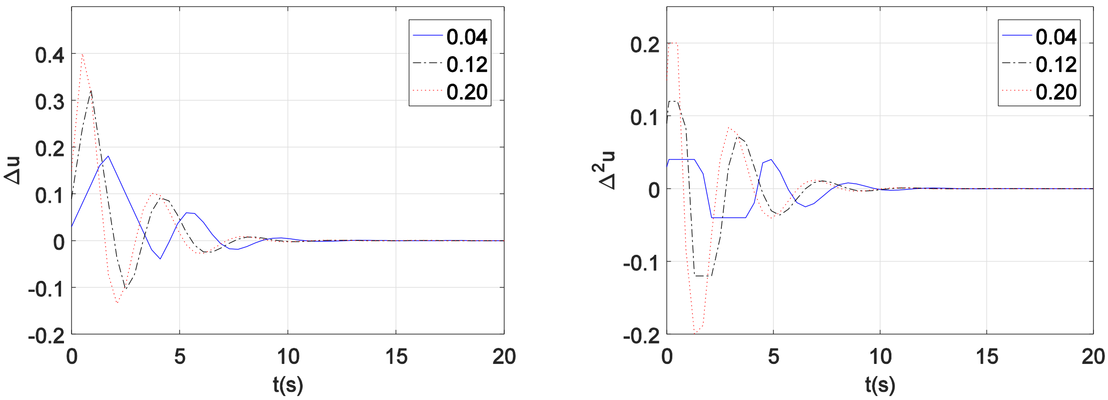
Example 3.

$T_s = 0.4s$ is used. Δu_{max} and $\Delta^2 u_{max}$ are also increased four times as large as the ones in Example 2 because the sampling period is quadrupled.

The control system is simulated under the different $\Delta^2 u$ constraints with $\Delta^2 u_{max} = 0.04, 0.12, 0.20$ and Δu constraint with $\Delta u_{max} = 0.4$. When $\Delta^2 u_{max} = 0.04, 0.12, 0.20$, $k_f^{\Delta u, \Delta^2 u} = 9{,}5{,}5$ respectively. Simulation results in Figure 3, show better responses than the ones in example 2.



(A) (B)



(C) (D)

Figure 3 The graphs of $y, u, \Delta u, \Delta^2 u$ in different cases of $\Delta^2 u$ constraints with $\Delta^2 u_{max} = 0.04$ (- 0.04), $\Delta^2 u_{max} = 0.12(-\cdot-0.12)$ and $\Delta^2 u_{max} = 0.20(\cdots\cdots 0.20)$ with $T_s = 0.4s$. A, y; B, u; C, Δu; D, $\Delta^2 u$

Simulation results show that the proposed control system works well when the prediction horizon of the plant output is longer than two times of the lower bound of settling time and the prediction horizon of the control input is longer than the lower bound of settling time.

6. CONCLUSION

In this paper the model predictive control with the constraint on the incremental acceleration of the control input is provided to deal with the physical limit of the actuator and reduce the stress to the system.

The discrete time state space model of the plant was extended to the augmented model which had the incremental acceleration of the control input as the new input. The quadratic cost function on the control error and the incremental acceleration of the control input was adopted to be minimized subject to the constraints on the plant output, the control input, its incremental rate and acceleration.

The proposed model predictive control problem is the quadratic programming problem which can be solved with the various methods.

When the model predictive control system is operated under the various constraints, the settling time is lengthened so that the prediction of the control system behavior can experience the troubles.

The lower bound of the settling time was calculated to select the reasonable sampling period and prediction horizon. Numerical examples verified the effectiveness of the proposed method.

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