An extension of the mollifier

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Abstract

We extend the convergence for mollifiers to that for differential forms of arbitrary degrees.

1 Introduction

On a differential manifold, let c be a singular chain whose current of the integration is denoted by T_c. Let ω_ϵ be a smooth form that blows up as the real positive number $\epsilon \to 0$. G. de Rham in [2] explored the differential geometry in the current $T_c \wedge \omega_\epsilon$ for a particular situation. After decades of emerging of the new techniques, de Rham's original style is becoming old memory. However, it is still necessary in some cases. For instance, the classical smoothing process is necessary in modern cohomology theory. It is expressed as a weak limit of a convolution with a mollifier. We formulate it in the differential form of top degree as follows. Let $\mathbf{x} = (x_1, \cdots, x_n)$ be the coordinates of $\mathbb{R}^n$ with the volume form $dx_1 \wedge \cdots \wedge dx_n$ denoted by $d\mu$. Let ω_ϵ for $\epsilon > 0$ be the differential n-form,

$$\frac{1}{\epsilon^n} f(\frac{\mathbf{x}}{\epsilon}) d\mu \tag{1.1}$$

where $f(\mathbf{x})$ is a function of a mollifier, i.e. a smooth bump function around the origin such that

$$\int_{\mathbb{R}^n} f(\mathbf{x}) d\mu = 1.$$

Let c be an n-dimensional polyhedron in $\mathbb{R}^n$ that contains the origin as its interior point. Then the current $T_c \wedge \omega_\epsilon$, as $\epsilon \to 0$, converges weakly to the δ function at the origin (see chapter 3, [3]). In this paper we would like to show that if the form ω_ϵ does not have the top-degree or does not meet the de Rham's requirement, the convergence in the sense of measures still holds. This measure theoretical convergence suggests a direction other than de Rham theory. To state the convergence as a theorem, we first extends the mollifier to differential forms of lower degrees. *

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*G. de Rham did this for a particular type of ω_ϵ (see [2]).

Definition 1.1. *(blow-up forms)*

Let $\digamma_\epsilon$ for $\epsilon > 0$ be a family of smooth forms of degree r in an Euclidean space $\mathbb{R}^n$. If there are an orthogonal decomposition $\mathbb{R}^n = \mathbb{R}^r \oplus \mathbb{R}^{n-r}$ with coordinate $\mathbf{u}$ for the subspace $\mathbb{R}^r$ and a smooth form $\digamma_1(\mathbf{u})$ of degree r on $\mathbb{R}^r$ with a compact support such that

$$\digamma_\epsilon = \pi^* \digamma_1(\frac{\mathbf{u}}{\epsilon}) \tag{1.2}$$

or abbreviated as

$$\digamma_\epsilon = \digamma_1(\frac{\mathbf{u}}{\epsilon})$$

where $\pi : \mathbb{R}^n \to \mathbb{R}^r$ is the orthogonal projection, then $\digamma_\epsilon$ is called a blow-up form along $\mathbb{R}^r$ at $\mathbb{R}^{n-r}$.

Theorem 1.2. *(Main theorem) Let c be a p dimensional regular cell in $\mathbb{R}^n$. Let ω_ϵ be a blow-up form of degree $r \le p$ in $\mathbb{R}^n$. Then the current*

$$T_c \wedge \omega_\epsilon \tag{1.3}$$

converges weakly to a current as $\epsilon \to 0$.

Remark. Unlike the classical case, Main theorem does not provide a full description of the weak limit.

In the rest of paper, we give the technical detail of the proof. It consists of one lemma in set-theoretic limit and an estimate in functional analysis. The appendix includes another lemma which is mainly for the estimate in analysis. However, it extends the notion of topological degree of a map in the spirit consistent with the main theorem.

2 proof

In the following, for an Euclidean space $\mathbb{R}^l$ with a coordinate $\mathbf{z}$, we'll abuse the notation to denote the volume form of a subspace with the concordant orientation and the volume density in Lebesgue integrals by the same expression $d\mu_{\mathbf{z}}$. The argument starts with a definition and a lemma about points and sets.

Definition 2.1. *Let $W \subset \mathbb{R}^p$ be a subset in an Euclidean space with the origin $\mathbf{o}$. A point $\mathbf{a} \in \mathbb{R}^p$ is said to be a stable point of W if the line segment*

$$\{\mathbf{o} + t(\overrightarrow{\mathbf{oa}}),\ 0 < t \le 1\}$$

either lies in W completely or in W^c completely, where $\overrightarrow{\mathbf{oa}} \in T_{\mathbf{o}}\mathbb{R}^p = \mathbb{R}^p$ is the vector from $\mathbf{o}$ to $\mathbf{a}$, and W^c is the complement $\mathbb{R}^p \backslash W$. We denote the collection of stable points of W by $W_s^{\mathbf{o}}$.

Recall a regular cell c is a couple: a) oriented polyhedron $\Pi_p \subset \mathbb{R}^p$, b) a diffeomorphic embedding c of a neighborhood of Π to $\mathbb{R}^n$. Let $\mathbb{R}^r, \mathbb{R}^{p-r}, \mathbb{R}^{n-p}$ be subspaces of $\mathbb{R}^n$ with coordinates $\mathbf{u}$, $\mathbf{v}_1$ and $\mathbf{v}_2$ respectively such that

$$\mathbb{R}^n = \mathbb{R}^r \oplus \mathbb{R}^{p-r} \oplus \mathbb{R}^{n-p}. \tag{2.1}$$

Let

$$\eta : \mathbb{R}^n \quad \rightarrow \quad \mathbb{R}^p = \mathbb{R}^r \oplus \mathbb{R}^{p-r}$$

be the orthogonal projection to its subspace $\mathbb{R}^p$. Let $D_{\frac{1}{\epsilon}}$ for a positive ϵ be the linear transformation of $\mathbb{R}^n$ defined by the map

$$(\mathbf{u}, \mathbf{v}_1, \mathbf{v}_2) \quad \rightarrow \quad (\frac{\mathbf{u}}{\epsilon}, \mathbf{v}_1, \mathbf{v}_2). \tag{2.2}$$

In the context, we denote its restriction to subspaces also by $D_{\frac{1}{\epsilon}}$. All measures in the following are the Lebesgue measures on Euclidean spaces.

Lemma 2.2. *Denote $W := \eta(C)$. There exists a subset $W_{fu} \subset W$ of measure 0 such that the set-theoretic limit (§4, [1])*

$$\lim_{\epsilon \to 0} D_{\frac{1}{\epsilon}} \bigg(W \backslash W_{fu} \bigg) \tag{2.3}$$

exists †.

Proof. We denote

$$L := \mathbb{R}^{p-r}$$

The point $\mathbf{o} \in L$ should be viewed as the origin of the affine subspace $\mathbb{R}^r \oplus \mathbf{o}$ where $\mathbf{o} \in \mathbb{R}^{p-r}$ is a point, and partial scalar multiplication $D_{\frac{1}{\epsilon}}$ acts on it as the scalar multiplication. Let

$$W^{\mathbf{o}} = W \cap \bigg(\mathbb{R}^r \oplus \{\mathbf{o}\} \bigg).$$

Let $\mathcal{R}_{\mathbf{o}}$ be the ray

$$\{\mathbf{o} + t(\overrightarrow{\mathbf{oa}}) : \mathbf{a} \in W^{\mathbf{o}}, t > 0\}$$

that starts at the origin in the affine plane. Let

$$W^{\mathbf{o}}_{fu} \subset W^{\mathbf{o}}$$

denote the subset

$$\{\mathbf{a} \in W^{\mathbf{o}} : \mathcal{R}_{\mathbf{o}} \textit{ does not contain a stable point of } W^{\mathbf{o}}\}.$$

†For a family of sets S_ϵ, the existence of the set-theoretic limit means

$$\bigcap_{\epsilon_1 \le 1} \bigcup_{\epsilon_2 \le \epsilon_1} S_{\epsilon_2} = \bigcup_{\epsilon_1 \le 1} \bigcap_{\epsilon_2 \le \epsilon_1} S_{\epsilon_2}$$

We divide W to three disjoint parts.
1) $W_{fu} = \underset{\mathbf{o}\in L}{\cup} W^{\mathbf{o}}_{fu}$, called the set of fully unstable points,

2) $W_s = \underset{\mathbf{o}\in L}{\cup} W^{\mathbf{o}}_{s}$, called the set of stable points,

3) W_{pu} is $W\backslash(W_{fu}\cup W_s)$, called the set of partially unstable points.
Next we blow-up each part by the scalar multiplication $D_{\frac{1}{\epsilon}}$ with $\epsilon \to 0$.

For the fully unstable points W_{fu}, we would like to show they are necessarily on the "boundary" which gives the measure 0. The following is the detail. The boundary of the polyhedron Π_p is defined by multiple hyperplanes. Hence the boundary of C is also defined by multiple hyperplanes H_j. On the other hand in the its target space, we let

$$\begin{array}{cccc} \nu : \mathbb{R}^r\backslash\{\mathbf{0}\} \oplus \mathbb{R}^{p-r} & \to & \mathbb{P}^{r-1} \times \mathbb{R}^{p-r} \\ (\mathbf{u}, \mathbf{v}_1) & \to & ([\mathbf{u}], \mathbf{v}_1) \end{array} \tag{2.4}$$

be the map that is the product of the projectivization map and the identity map (where $\mathbb{P}^{r-1}$ can be regarded as the real projectivization of $T_{\mathbf{0}}\mathbb{R}^r$, the set of directions). Fix a point $\mathbf{o}\in L$. Let $\mathbf{a}\in W^{\mathbf{o}}_{fu}$ other than $\mathbf{o}$. Since $\mathbf{a}$ is a fully unstable point, there are two sequences of points $\mathbf{p}_n, \mathbf{q}_n$ on the ray $\mathcal{R}_{\mathbf{o}}$ such that

$$\lim_{n\to\infty} \mathbf{p}_n = \mathbf{o} = \lim_{n\to\infty} \mathbf{q}_n$$

and

$$\mathbf{p}_n \notin W^{\mathbf{o}}, \mathbf{q}_n \in W^{\mathbf{o}}.$$

Thus the directions $\overrightarrow{\mathbf{o}\mathbf{p}_n}$ and $\overrightarrow{\mathbf{o}\mathbf{q}_n}$, which are all parallel to the tangent vector $\overrightarrow{\mathbf{o}\mathbf{a}}$ must lie on at least one nontrivial plane $\eta_*(H_j)$. Since a subplane properly contained in an Euclidean space has a measure 0, for each fixed $\mathbf{o}$, $\mathbb{P}(W^{\mathbf{o}}_{fu}\backslash\{\mathbf{o}\})$ has measure 0 in the manifold

$$\mathbb{P}(\mathbb{R}^r\backslash\{\mathbf{0}\}) \times \{\mathbf{o}\} \simeq \mathbb{P}^{r-1}$$

where $\mathbf{o}$ is fixed. Since

$$\mathbb{R}^r\backslash\{\mathbf{0}\} \to \mathbb{P}^{r-1}$$

is a bundle's projection, the inverse $W^{\mathbf{o}}_{fu}$ also has measure 0. To go further, we take the union over L to obtain $\nu(W_{fu}\backslash L) = \underset{\mathbf{o}\in L}{\cup} \mathbb{P}(W^{\mathbf{o}}_{fu}\backslash\{\mathbf{o}\})$ has measure 0 in the manifold

$$\mathbb{P}^{r-1} \times \mathbb{R}^{p-r}.$$

Due to the fibre bundle structure of the projectivization, we conclude W_{fu} in $\mathbb{R}^p$ has measure 0. Notice that $D_{\frac{1}{\epsilon}}$ is a linear transformation, $D_{\frac{1}{\epsilon}}(W_{fu})$ which is equal to W_{fu} also has measure 0. Therefore the limit is of 0. ‡

‡But the set W_{fu} is not on the boundary of W.

For stable points W_s, we consider the set $B_\epsilon = D_{\frac{1}{\epsilon}}(W_s)$. We would like to show B_ϵ as $\epsilon \to 0$ is a decreasing set. So it converges to a measurable set. The following is the detail. Let $\mathcal{R}_{\mathbf{o}}$ be the ray starting at $\mathbf{o} \in L$ and through a stable point $\mathbf{a} \in W_s^{\mathbf{o}}$ of $W^{\mathbf{o}}$ for an $\mathbf{o} \in L$. Since $\mathbf{a}$ is stable, the dilation by the scalar multiplication $D_{\frac{1}{\epsilon}}$ yields

$$D_{\frac{1}{\epsilon}}(\mathcal{R}_{\mathbf{o}} \cap W_s) \subset D_{\frac{1}{\epsilon'}}(\mathcal{R}_{\mathbf{o}} \cap W_s), \quad \text{for } \epsilon' < \epsilon < 1.$$

Now taking the union over all the rays through stables points, we obtain

$$D_{\frac{1}{\epsilon}}(W_s) \subset D_{\frac{1}{\epsilon'}}(W_s), \quad \text{for } \epsilon' < \epsilon.$$

Therefore B_ϵ is a decreasing family of measurable sets. Let

$$B_0 := \cup_{\epsilon \in (0,1]} \left(D_{\frac{1}{\epsilon}}(W_s) \right). \tag{2.5}$$

Then set-theoretically the decreasing family yields

$$\lim_{\epsilon \to 0} B_\epsilon = B_0$$

and B_0 is measurable.

For partially unstable point W_{pu}, we consider the set $A_\epsilon = D_{\frac{1}{\epsilon}}(W_{pu})$. We would like to show A_ϵ as the set multiplied by $\frac{1}{\epsilon}$ will be pushed to ∞ as $\epsilon \to 0$. So it is empty. Here is the detail. If $\bigcap_{\epsilon_1 \leq 1} \bigcup_{\epsilon_2 \leq \epsilon_1} A_{\epsilon_2}$ is non-empty, there is a point

$$\mathbf{x} \in \bigcap_{\epsilon_1 \leq 1} \bigcup_{\epsilon_2 \leq \epsilon_1} A_{\epsilon_2}$$

i.e. $\mathbf{x} \in \bigcup_{\epsilon_2 \leq \epsilon_1} A_{\epsilon_2}$ for any $\epsilon_1 < 1$. So, there is a sequence of numbers ϵ_n such that $\lim_{n \to \infty} \epsilon_n = 0$ and $D_{\epsilon_n}(\mathbf{x})$ lies in W_{pu}. Suppose that N is a number in the sequence such that $D_{\epsilon_N}(\mathbf{x}) \in W_{pu}$. By the definition of W_{pu}, there is a smaller $\epsilon_{N'} \neq 0$ such that $D_{\epsilon_{N'}}(\mathbf{x})$ is a stable point, i.e. $D_{\epsilon_{N'}}(\mathbf{x}) \in W_S$. Then all points $D_{\epsilon_n}(\mathbf{x})$ are stable whenever $\epsilon_n < \epsilon_{N'}$. But this contradicts the assertion above: there is a sequence of partially unstable points $\epsilon_n \mathbf{x}$ with $\epsilon_n \to 0$. Thus

$$\lim_{\epsilon \to 0} sup A_\epsilon = \bigcap_{\epsilon_1 \leq 1} \bigcup_{\epsilon_2 \leq \epsilon_1} A_{\epsilon_2} = \varnothing. \tag{2.6}$$

Therefore

$$\lim_{\epsilon \to 0} inf A_\epsilon \subset \lim_{\epsilon \to 0} sup A_\epsilon$$

is also empty. Hence $\lim_{\epsilon \to 0} A_\epsilon$ exists and is equal to an empty set.

Combining the results for W_{fu}, W_s and W_{pu}, we complete the proof.

$\square$

Proof of Theorem 1.2. We continue with all notations in Lemma 2.2. Let ϕ be a test form of degree $p-r$ in $\mathbb{R}^n$. It amounts to show the convergence of the integral

$$\int_c \omega_\epsilon \wedge \phi \tag{2.7}$$

as $\epsilon \to 0$. Let $\mathbb{R}^r$ be the subspace with coordinates $\mathbf{u}$ such that the blow-up form is written as

$$\omega_\epsilon = \frac{1}{\epsilon^r} g(\frac{\mathbf{u}}{\epsilon}) d\mu_{\mathbf{u}} \tag{2.8}$$

where $g(\mathbf{u})$ is a C^∞ function of $\mathbb{R}^r$. Notice that the form $\omega_\epsilon \wedge \phi$ is the sum of simple forms in the coordinates of $\mathbb{R}^n$ that can be explicitly expressed. So, we'll focus on the integral of a single simple form.

We work with the simple form written as

$$\frac{1}{\epsilon^r} g(\frac{\mathbf{u}}{\epsilon}) \psi(\mathbf{u}, \mathbf{v}_1, \mathbf{v}_2) d\mu_{\mathbf{u}} \wedge d\mu_{\mathbf{v}_1} \tag{2.9}$$

where the volume forms $d\mu_{\mathbf{u}}, d\mu_{\mathbf{v}_1}$ determine two coordinate's planes

$$\mathbb{R}^r, \mathbb{R}^{p-r}$$

with coordinates $\mathbf{u}, \mathbf{v}_1$ respectively, and ψ is a C^∞ function on

$$\mathbb{R}^n = \mathbb{R}^r \oplus \mathbb{R}^{p-r} \oplus \mathbb{R}^{n-p}$$

that is the coefficient of the simple form $\psi d\mu_{\mathbf{v}_1}$ in the test form ϕ. Then the integral of (2.7) over $C := c(\Pi)$ is

$$\int_{D_{\frac{1}{\epsilon}}(C)} g(\mathbf{u}) \psi(\epsilon\mathbf{u}, \mathbf{v}_1, \mathbf{v}_2) d\mu_{\mathbf{u}} \wedge d\mu_{\mathbf{v}_1} \tag{2.10}$$

where $\mathbf{u}$ is the new variable obtained from the old $\mathbf{u}$ divided by ϵ. Let K_1 be the support of $g(\mathbf{u})$, and K_2, K_3 be the bounded sets of $\mathbb{R}^{p-r}, \mathbb{R}^{n-p}$ such that C is contained in $\mathbb{R}^r \oplus K_2 \oplus K_3$. Then $\psi(\epsilon\mathbf{u}, \mathbf{v}_1, \mathbf{v}_2)$ uniformly converges to $\psi(\mathbf{0}, \mathbf{v}_1, \mathbf{v}_2)$ in the bounded $K_1 \oplus K_2 \oplus K_3$. So, for any positive δ, we can find sufficiently small ϵ such that

$$|\psi(\epsilon\mathbf{u}, \mathbf{v}_1, \mathbf{v}_2) - \psi(\mathbf{0}, \mathbf{v}_1, \mathbf{v}_2)| \leq \delta. \tag{2.11}$$

Let c_ϵ be the composition

$$\overline{\Pi}_p \quad \xrightarrow{c} \quad \mathbb{R}^n \quad \xrightarrow{D_{\frac{1}{\epsilon}}} \quad \mathbb{R}^n. \tag{2.12}$$

Notice

$$D_{\frac{1}{\epsilon}}(C) \cap (K_1 \oplus K_2 \oplus K_3)$$

is a bounded set. Thus all coefficients of the form $c_\epsilon^*\big(g(u)d\mu_{\mathbf{u}} \wedge d\mu_{\mathbf{v}_1}\big)$ are bounded uniformly for all sufficiently small ϵ. Hence

$$\begin{aligned}&|\int_{D_{\frac{1}{\epsilon}}(C)} g(\mathbf{u})\psi(\epsilon\mathbf{u},\mathbf{v}_1,\mathbf{v}_2)d\mu_{\mathbf{u}} \wedge d\mu_{\mathbf{v}_1} - \int_{D_{\frac{1}{\epsilon}}(C)} g(\mathbf{u})\psi(\mathbf{0},\mathbf{v}_1,\mathbf{v}_2)d\mu_{\mathbf{u}} \wedge d\mu_{\mathbf{v}_1}|\\ &\leq \delta M\end{aligned} \tag{2.13}$$

where M is a constant. For the integral

$$\int_{D_{\frac{1}{\epsilon}}(C)} g(\mathbf{u})\psi(\mathbf{0},\mathbf{v}_1,\mathbf{v}_2)d\mu_{\mathbf{u}} \wedge d\mu_{\mathbf{v}_1} \tag{2.14}$$

we make a change of variable from $\mathbf{u}$ to $\dfrac{\mathbf{u}}{\epsilon}$ to find (2.14) is equal to

$$\frac{1}{\epsilon^r}\int_C g(\frac{\mathbf{u}}{\epsilon})\psi(\mathbf{0},\mathbf{v}_1,\mathbf{v}_2)d\mu_{\mathbf{u}} \wedge d\mu_{\mathbf{v}_1} \tag{2.15}$$

Now we apply Lemma A.1, there is a compactly supported integrable function $\tilde{\xi}_\epsilon(\mathbf{u},\mathbf{v}_1)$ on $\mathbb{R}^p$ such that

$$\frac{1}{\epsilon^r}\int_C g(\frac{\mathbf{u}}{\epsilon})\psi(\mathbf{0},\mathbf{v}_1,\mathbf{v}_2)d\mu_{\mathbf{u}} \wedge d\mu_{\mathbf{v}_1} = \frac{1}{\epsilon^r}\int_W g(\frac{\mathbf{u}}{\epsilon})\tilde{\xi}_\psi(\mathbf{u},\mathbf{v}_1)d\mu_{\mathbf{u}}d\mu_{\mathbf{v}_1} \tag{2.16}$$

where W is the measurable set defined as in Lemma 2.2, and the right hand side is a Lebesgue integral with the density measure $d\mu_{\mathbf{u}}d\mu_{\mathbf{v}_1}$, and $\tilde{\xi}_\psi(\mathbf{u},\mathbf{v}_1)$ in the integrand is a compactly supported L^1 function on $\mathbb{R}^p$. Furthermore, since $\psi(\mathbf{0},\mathbf{v}_1,\mathbf{v}_2)$ is a pullback function from $\mathbb{R}^{p-r}\oplus\mathbb{R}^{n-p}$, then $\tilde{\xi}_\psi(\mathbf{u},\mathbf{v}_1)$ is also a pullback of function $\xi_\psi(\mathbf{v}_1)$ from $\mathbb{R}^{p-r}$. So, in the following, we express the pullback function $\tilde{\xi}_\psi(\mathbf{u},\mathbf{v}_1)$ as $\xi_\psi(\mathbf{v}_1)$. Now changing the variables from $\dfrac{\mathbf{u}}{\epsilon}$ back to $\mathbf{u}$, we have

$$\begin{aligned}\textit{right hand side of}(2.16) &= \textstyle\int_{\mathbb{R}^p}\chi_{D_{\frac{1}{\epsilon}}(W)}(\mathbf{u},\mathbf{v}_1)g(\mathbf{u})\xi_\psi(\mathbf{v}_1)d\mu_{\mathbf{u}}d\mu_{\mathbf{v}_1}\\ &= \textstyle\int_{\mathbb{R}^p}\chi_{D_{\frac{1}{\epsilon}}(W\setminus W_{fu})}(\mathbf{u},\mathbf{v}_1)g(\mathbf{u})\xi_\psi(\mathbf{v}_1)d\mu_{\mathbf{u}}d\mu_{\mathbf{v}_1}\end{aligned} \tag{2.17}$$

where $\chi_\bullet$ denotes the characteristic function of the set $\bullet$. Next for the Lebesgue integrals, we'll omit the notations for variables for the dominant convergence theorem. We'll see that the integrand in (2.17) satisfies

$$|\chi_{D_{\frac{1}{\epsilon}}(W\setminus W_{fu})}g\xi_\psi| \leq |g\xi_\psi|$$

and $|g\xi_\psi|$ is an L^1 function on $\mathbb{R}^p$. The set-theoretic convergence in Lemma 2.2 implies the $\chi_{D_{\frac{1}{\epsilon}}(W\setminus W_{fu})}g\xi_\psi$ point-wisely converges to the function

$$\chi_{B_0}g\xi_\psi.$$

By the dominant convergence theorem

$$\begin{aligned}\lim_{\epsilon\to 0}\int_{\mathbb{R}^p}\chi_{D_{\frac{1}{\epsilon}}(W\setminus W_{fu})}g\xi_{\psi}d\mu_{\mathbf{u}}d\mu_{\mathbf{v}_1} &= \int_{\mathbb{R}^p}\chi_{B_0}g\xi_{\psi}d\mu_{\mathbf{u}}d\mu_{\mathbf{v}_1} \\ &= \int_{B_0}g\xi_{\psi}d\mu_{\mathbf{u}}d\mu_{\mathbf{v}_1}\end{aligned} \tag{2.18}$$

Finally, combining (2.13) and (2.18), we obtain that

$$\lim_{\epsilon\to 0}\int_C \frac{1}{\epsilon^r}g(\frac{\mathbf{u}}{\epsilon})\psi(\mathbf{u},\mathbf{v}_1,\mathbf{v}_2)d\mu_{\mathbf{u}}\wedge d\mu_{\mathbf{v}_1} = \int_{B_0}g(\mathbf{u})\xi_{\psi}(\mathbf{v}_1)d\mu_{\mathbf{u}}d\mu_{\mathbf{v}_1}$$

(Note the left hand side is an integral of a differential form but the right is a Lebegue integral). We conclude

$$T_c\wedge\omega_\epsilon$$

converges to a functional as $\epsilon\to 0$. For the continuity of the functional, we see that if ϕ varies in a bounded set of forms to any orders, then in particular ϕ varies in the bounded set to the order of 0. Hence the formula (2.14) (as a number) is bounded. Then

$$\int_{B_0}g(\mathbf{u})\xi_{\psi}(\mathbf{u},\mathbf{v}_1)d\mu_{\mathbf{u}}d\mu_{\mathbf{v}_1}$$

as a number is bounded. So, the evaluation

$$\lim_{\epsilon\to 0}(T_c\wedge\omega_\epsilon)[\phi]$$

is also bounded. Hence the functional

$$\phi\to\lim_{\epsilon\to 0}(T_c\wedge\omega_\epsilon)[\phi]$$

defines a current. The proof is completed.

□

Appendix A Orthogonal projection of a cell

The integration of forms (2.7) is impossible in geometric analysis since the manifold structure for cells does not exist at the $\epsilon=0$. Our idea is to convert it to a Lebesgue integral (see the right hand side of (2.16)) for the measure still exists at $\epsilon=0$. The following measure-theoretical lemma provides the basis to this key conversion.

Lemma A.1. *Let $p \leq n$ be two whole numbers. Let $\mathbb{R}^p, \mathbb{R}^{n-p}$ be subspaces of $\mathbb{R}^n$ such that $\mathbb{R}^n = \mathbb{R}^p \oplus \mathbb{R}^{n-p}$. Let $\pi : \mathbb{R}^n \to \mathbb{R}^p$ be the orthogonal projection. Let c be a p-dimensional regular cell in $\mathbb{R}^n$, ψ a smooth function on $\mathbb{R}^n$. Then there is a compactly supported L^1 function ξ_ψ on $\mathbb{R}^p$ such that*

$$\pi(T_c \wedge \psi) = \xi_\psi \tag{A.1}$$

where $\pi(currents)$ denotes the pushforward on compactly supported currents, and ξ_ψ represents a current of degree 0.

Proof. Let μ be the Lebesgue measure on $\mathbb{R}^p$, ϕ a test function. Let $C = c(\Pi_p)$. We should note that since T_c is a current with a compact support, the pushforward $\pi(T_c \wedge \psi)$ is a well-defined 0-current. Hence it is both a distribution and a 0-current. So it can be evaluated in two different ways, and the evaluation of the distribution $\pi(T_c \wedge \psi)$ at ϕ is equal to the current's evaluation at forms,

$$\pi(T_c \wedge \psi)[\phi d\mu] \tag{A.2}$$

which has the integral estimate

$$\begin{aligned} \left|\pi(T_c \wedge \psi)[\phi d\mu]\right| &\leq \left|\int_C \psi \wedge \pi^*(\phi) \wedge \pi^*(d\mu)\right| \\ &\leq M||\phi||_\infty \end{aligned} \tag{A.3}$$

where M is a constant independent of the test function and $||\bullet||_\infty = esssup|\bullet|$. Thus, $\pi(T_c \wedge \psi)$ as a distribution has order 0. Therefore it is a signed measure. Let $A \subset \mathbb{R}^p$ be a set of measure 0. Let $\overline{\pi} = \pi|_C$. So, $\overline{\pi}$ is a differential map between two manifolds of the same dimension p. Let

$$\overline{\pi}^{-1}(A) = E_1 \cup E_2$$

where E_1 is a set of critical points of $\overline{\pi}$, and $E_2 = \overline{\pi}^{-1}(A) \backslash E_1$. By the same estimate (A.3), we have

$$\left|\pi(T_c \wedge \psi)[A]\right| \leq M' |\textstyle\int_{E_1+E_2} d\mu| \tag{A.4}$$

where M' is a constant, the integral is of the differential form $d\mu$. Since E_1 consists of critical points, the Jacobian of $\overline{\pi}$ is 0. Thus $\int_{E_1} d\mu = 0$. We let $E_2 = \cup_{i=1}^{\infty} E_2^i$ such that

$$\overline{\pi}|_{E_2^i} : E_2^i \to \overline{\pi}(E_2^i) \tag{A.5}$$

is diffeomorphic. Then each $\overline{\pi}(E_2^i)$ is contained in A. Thus $\mu(\overline{\pi}(E_2^i)) = 0$. Then

$$|\int_{E_2^i} d\mu| \leq |\int_{\overline{\pi}(E_2^i)} J d\mu| \leq k_i \mu(\overline{\pi}(E_2^i)) = 0$$

where J is the Jacobian of the map $\overline{\pi}|_{E_i}$ and k_i is the upper bound of $|J|$. Hence

$$\left|\pi(T_c \wedge \psi)[A]\right| \leq \sum_{i=1}^{\infty} |\int_{E_2^i} d\mu| = 0.$$

Thus the signed measure $\pi(T_c \wedge \psi)$ is absolutely continuous with respect to the Lebesgue measure of $\mathbb{R}^p$. The Radon-Nikodym theorem ([1]) implies that the density function between the signed measure and the positive measure,

$$\xi_\psi = \frac{\pi(T_c \wedge \psi)}{\mu} \tag{A.6}$$

is an L^1 function. The numerator $\pi(T_c \wedge \psi)$ in the formula (A.6) indicates ξ_ψ has the bounded support $\pi(C)$. We complete the proof.

□

Example A.2. *If* $\pi|_C : C \to \mathbb{R}^n$ *is proper, then* $\xi_1 = deg(\pi)\chi_{\pi(C)}$.

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