New continued fraction approximations for the gamma function based on the Tri-gamma function

Hong SonUng, Ri Kwang*, Kim CholRyong
Faculty of Mathematics, **Kim Il Sung** University, Pyongyang, DPR Korea

ABSTRACT: In this paper, we provide some useful lemmas to construct continued fraction based on a given power series. Then we establish new continued fraction approximations for the gamma function via the Tri-gamma function. Especially, we analytically determine all parameters of the continued fraction by Bernoulli numbers.

Keywords: Gamma function, Tri-gamma function, continued fraction, Bernoulli number

1. Introduction

The classical Euler gamma function Γ defined by

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt, \qquad x > 0, \tag{1.1}$$

was first introduced by the Swiss mathematician Leonhard Euler (1707-1783) with the goal to generalize the factorial to non-integer values.

The logarithmic derivative $\psi(x)$ of the gamma function $\Gamma(x)$ given by

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} \qquad \text{or} \qquad \ln\Gamma(x) = \int_1^x \psi(t)dt$$

is well-known as the psi (or digamma) function.

The derivative $\psi'(x)$ is called the Tri-gamma function, while the derivatives $\psi^{(n)}(x)$ are called the poly-gamma functions,
where

$$\psi^{(n)}(x) = \frac{d^n}{dx^n}\{\psi(x)\} \ (\mathrm{n} \in \mathbb{N}).$$

Today the Stirling's formula

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \tag{1.2}$$

is one of the most well-known formulas for approximation of the factorial function by being widely

∗The corresponding author. Email: MH.CHOE@star-co.net.kp

applied in number theory, combinatorics, statistical physics, probability theory and other branches of science.

The Stirling's formula for $n!$ has a generalization to the gamma function,

$$\Gamma(x+1) \approx \sqrt{2\pi x}\left(\frac{x}{e}\right)^x, \quad x \to \infty. \tag{1.3}$$

Also, the Stirling's series for the gamma function is presented (see [1, p.257, Eq. (7.1.40)]) by

$$\Gamma(x+1) \approx \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \exp\left(\sum_{n=1}^{\infty} \frac{B_{2n}}{2n(2n-1)x^{2n-1}}\right), \qquad x \to \infty, \tag{1.4}$$

where B_n $(n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\})$ denotes the Bernoulli numbers defined by the generating formula

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}, \quad |z| < 2\pi,$$

then the first few terms of B_n are as follows.

$$B_{2n+1} = 0, n \geq 1,$$
$$B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_4 = -\frac{1}{30}, B_6 = \frac{1}{42}, B_8 = -\frac{1}{30}, B_{10} = \frac{5}{66}, \cdots$$

Up to now, many researchers made great efforts in the area of establishing more accurate approximations for the gamma function, and had lots of inspiring results. [2-4], [6-12]

Especially, You [13] proved the asymptotic expansion of $\Gamma(x+1)$ via the Tri-gamma function as follows.

$$\Gamma(x+1) \approx \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \exp\left(\frac{1}{12}\psi'\left(x+\frac{1}{2}\right)\right)\exp\left(\sum_{n=1}^{\infty} \frac{c_n}{x^{2n+1}}\right), \quad x \to \infty, \tag{1.5}$$

where

$$c_n = \frac{B_{2n+2}}{2(n+1)(2n+1)} + \frac{(1-2^{1-2n})B_{2n}}{12}.$$

Then, he provided new asymptotic expansion using continued fraction for the factorial n! and the gamma function via the Tri-gamma function.

$$\Gamma(n+1) \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \exp\left(\frac{1}{12}\psi'\left(n+\frac{1}{2}\right)\right)\exp\left(\frac{1}{n}R_m(n)\right), \tag{1.6}$$

where

$$R_m(n) = \cfrac{t_0}{n^2 + s_0 + \cfrac{t_1}{n^2 + s_1 + \cfrac{t_2}{n^2 + s_2 + \cdots \cfrac{t_{m-1}}{n^2 + s_{m-1}}}}},$$

here $t_0 = \frac{1}{240}, s_0 = \frac{11}{28}$; $t_1 = -\frac{193}{1176}, s_1 = \frac{146617}{89166}$; $t_2 = -\frac{865896794}{5273344093}, s_2 = \frac{24573335208457}{6302739063984}$; $\cdots$.

Motivated by these works, we provide some useful lemmas to construct continued fraction based on a given power series. Then we establish new continued fraction approximations for the gamma function via the Tri-gamma function. Especially, we analytically determine all parameters of the continued fraction by Bernoulli numbers.

The rest of this paper is arranged as follows.

In Sect. 2 some useful lemmas are given. In Sect. 3 new continued fraction approximations for the gamma function are provided. In the last section, the conclusions are given.

2. Lemmas

In this section, some useful lemmas are given. Especially, we provide two lemmas to construct the continued fraction based on a given power series.

Lemma 2.1.(The Euler connection [5, p.19, Eq. (1.7.1, 1.7.2)]) Let $\{c_k\}$ be a sequence in $\mathbb{C} \setminus \{0\}$ and

$$f_n = \sum_{k=0}^{n} c_k, \quad n \in \mathbb{N}_0. \tag{2.1}$$

Since $f_0 \neq \infty$, $f_n \neq f_{n-1}$, n ∈ $\mathbb{N}$, there exists a continued fraction $b_0 + K(a_m/b_m)$ with n[th] approximant f_n for all n. This continued fraction is given by

$$c_0 + \frac{c_1}{1} \, \frac{-c_2/c_1}{+1+c_2/c_1} + \cdots + \frac{-c_m/c_{m-1}}{1+c_m/c_{m-1}} + \cdots. \tag{2.2}$$

Lemma 2.2. Let $\{c_k\}$ be a sequence in $\mathbb{R} \setminus \{0\}$.

$$\sum_{i=1}^{m} \frac{c_i}{n^{2i+1}} = \frac{1}{n^2} \mathop{K}_{i=1}^{m} \frac{a_i}{n + \frac{b_i}{n}} = \frac{1}{n^2} \cfrac{a_1}{n + \mathop{K}\limits_{i=2}^{m} \cfrac{a_i}{n - \frac{a_i}{n}}}, \qquad n, m \in \mathbb{N}, \tag{2.3}$$

where

$$a_1 = c_1,\ b_1 = 0,$$
$$a_i = -\frac{c_i}{c_{i-1}},\ b_i = -a_i,\ i = 2, 3, \cdots, m.$$

Proof. Assume that

$$f_0(n) \neq \infty, \quad f_m(n) = \sum_{i=1}^{m} \frac{c_i}{n^{2i+1}}, \quad \text{n, m} \in \mathbb{N}. \tag{2.4}$$

The left-side of (2.3) is equal to $f_m(n)$.

Since

$$f_0(n) \neq \infty, \quad f_m(n) \neq f_{m-1}(n), \text{ m} \in \mathbb{N},$$

using Lemma 2.1,

$$f_m(n) = \frac{1}{n^2}\sum_{i=1}^{m}\frac{c_i}{n^{2i-1}} = \frac{1}{n^2}\frac{\frac{c_1}{n}}{1} + \frac{-\frac{c_2}{c_1 n^2}}{1+\frac{c_2}{c_1 n^2}} + \frac{-\frac{c_3}{c_2 n^2}}{1+\frac{c_3}{c_2 n^2}} + \cdots + \frac{-\frac{c_i}{c_{i-1} n^2}}{1+\frac{c_i}{c_{i-1} n^2}} + \cdots + \frac{-\frac{c_m}{c_{m-1} n^2}}{1+\frac{c_m}{c_{m-1} n^2}}$$

$$= \frac{1}{n^2}\frac{c_1}{n} + \frac{-\frac{c_2}{c_1 n}}{1+\frac{c_2}{c_1 n^2}} + \frac{-\frac{c_3}{c_2 n^2}}{1+\frac{c_3}{c_2 n^2}} + \cdots + \frac{-\frac{c_i}{c_{i-1} n^2}}{1+\frac{c_i}{c_{i-1} n^2}} + \cdots + \frac{-\frac{c_m}{c_{m-1} n^2}}{1+\frac{c_m}{c_{m-1} n^2}}$$

$$= \frac{1}{n^2}\frac{c_1}{n} + \frac{-\frac{c_2}{c_1}}{n+\frac{c_2}{c_1 n}} + \frac{-\frac{c_3}{c_2 n}}{1+\frac{c_3}{c_2 n^2}} + \cdots + \frac{-\frac{c_i}{c_{i-1} n^2}}{1+\frac{c_i}{c_{i-1} n^2}} + \cdots + \frac{-\frac{c_m}{c_{m-1} n^2}}{1+\frac{c_m}{c_{m-1} n^2}}$$

$$= \cdots \quad \cdots \quad \cdots$$

$$= \frac{1}{n^2}\frac{c_1}{n} + \frac{-\frac{c_2}{c_1}}{n+\frac{c_2}{c_1 n}} + \frac{-\frac{c_3}{c_2}}{n+\frac{c_3}{c_2 n}} + \cdots + \frac{-\frac{c_i}{c_{i-1}}}{n+\frac{c_i}{c_{i-1} n}} + \cdots + \frac{-\frac{c_m}{c_{m-1}}}{n+\frac{c_m}{c_{m-1} n}}$$

$$= \frac{1}{n^2}\frac{c_1}{n + \mathop{K}_{i=2}^{m}\frac{-\frac{c_i}{c_{i-1}}}{n+\frac{c_i}{c_{i-1} n}}} = \frac{1}{n^2}\frac{c_1}{n + \frac{0}{n} + \mathop{K}_{i=2}^{m}\frac{-\frac{c_i}{c_{i-1}}}{n+\frac{c_i}{c_{i-1} n}}}. \tag{2.5}$$

The middle expression of (2.3) is equal to

$$\frac{1}{n^2}\mathop{K}_{i=1}^{m}\frac{a_i}{n+\frac{b_i}{n}} = \frac{1}{n^2}\frac{a_1}{n+\frac{b_1}{n}+\mathop{K}_{i=2}^{m}\frac{a_i}{n+\frac{b_i}{n}}}. \tag{2.6}$$

Thus,

$$a_1 = c_1, \; b_1 = 0,$$

$$a_i = -\frac{c_i}{c_{i-1}}, b_i = \frac{c_i}{c_{i-1}} = -a_i, \quad i = 2, 3, \cdots, m.$$

Then, it is obviously true that

$$\frac{1}{n^2}\mathop{K}_{i=1}^{m}\frac{a_i}{n+\frac{b_i}{n}}=\frac{1}{n^2}\frac{a_1}{n+\mathop{K}\limits_{i=2}^{m}\frac{a_i}{n-\frac{a_i}{n}}}. \tag{2.7}$$

The proof of Lemma 2.2 is complete.

Lemma 2.3. Let $\{c_k\}$ be a sequence in $\mathbb{R}\setminus\{0\}$.

$$\sum_{i=1}^{m}\frac{c_i}{n^{2i+1}}=\frac{1}{n^3}\mathop{K}_{i=1}^{m}\frac{p_i n^2}{n^2+q_i}=\frac{1}{n^3}\frac{p_i n^2}{n^2+\mathop{K}\limits_{i=2}^{m}\frac{p_i}{n^2-p_i}},\qquad \text{n, m} \in \mathbb{N}, \tag{2.8}$$

where

$$p_1=c_1,\ q_1=0,$$
$$p_i=-\frac{c_i}{c_{i-1}},\ q_i=-a_i,\ \ i=2,3,\cdots,m.$$

Proof. From (2.4) and Lemma 2.1,

$$f_m(n)=\frac{1}{n^2}\sum_{i=1}^{m}\frac{c_i}{n^{2i-1}}=\frac{1}{n^2}\frac{\frac{c_1}{n}}{1}+\frac{-\frac{c_2}{c_1n^2}}{1+\frac{c_2}{c_1n^2}}+\frac{-\frac{c_3}{c_2n^2}}{1+\frac{c_3}{c_2n^2}}+\cdots+\frac{-\frac{c_i}{c_{i-1}n^2}}{1+\frac{c_i}{c_{i-1}n^2}}+\cdots+\frac{-\frac{c_m}{c_{m-1}n^2}}{1+\frac{c_m}{c_{m-1}n^2}}$$

$$=\frac{1}{n^3}\frac{c_1}{1}+\frac{-\frac{c_2}{c_1n^2}}{1+\frac{c_2}{c_1n^2}}+\frac{-\frac{c_3}{c_2n^2}}{1+\frac{c_3}{c_2n^2}}+\cdots+\frac{-\frac{c_i}{c_{i-1}n^2}}{1+\frac{c_i}{c_{i-1}n^2}}+\cdots+\frac{-\frac{c_m}{c_{m-1}n^2}}{1+\frac{c_m}{c_{m-1}n^2}}$$

$$=\frac{1}{n^3}\frac{c_1n^2}{n^2}+\frac{-\frac{c_2}{c_1}}{1+\frac{c_2}{c_1n^2}}+\frac{-\frac{c_3}{c_2n^2}}{1+\frac{c_3}{c_2n^2}}+\cdots+\frac{-\frac{c_i}{c_{i-1}n^2}}{1+\frac{c_i}{c_{i-1}n^2}}+\cdots+\frac{-\frac{c_m}{c_{m-1}n^2}}{1+\frac{c_m}{c_{m-1}n^2}}$$

$$=\frac{1}{n^3}\frac{c_1n^2}{n^2}+\frac{-\frac{c_2}{c_1}n^2}{n^2+\frac{c_2}{c_1}}+\frac{-\frac{c_3}{c_2}}{1+\frac{c_3}{c_2n^2}}+\cdots+\frac{-\frac{c_i}{c_{i-1}n^2}}{1+\frac{c_i}{c_{i-1}n^2}}+\cdots+\frac{-\frac{c_m}{c_{m-1}n^2}}{1+\frac{c_m}{c_{m-1}n^2}}$$

$$=\cdots\ \cdots\ \cdots$$

$$=\frac{1}{n^3}\frac{c_1n^2}{n^2}+\frac{-\frac{c_2}{c_1}n^2}{n^2+\frac{c_2}{c_1}}+\frac{-\frac{c_3}{c_2}n^2}{n^2+\frac{c_3}{c_2}}+\cdots+\frac{-\frac{c_i}{c_{i-1}}n^2}{n^2+\frac{c_i}{c_{i-1}}}+\cdots+\frac{-\frac{c_m}{c_{m-1}}n^2}{n^2+\frac{c_m}{c_{m-1}}}$$

$$= \frac{1}{n^3} \frac{c_1 n^2}{n^2 + \underset{i=2}{\overset{m}{K}} \frac{-\frac{c_i}{c_{i-1}} n^2}{n^2 + \frac{c_i}{c_{i-1}}}} = \frac{1}{n^3} \frac{c_1 n^2}{n^2 + 0 + \underset{i=2}{\overset{m}{K}} \frac{-\frac{c_i}{c_{i-1}} n^2}{n^2 + \frac{c_i}{c_{i-1}}}}. \tag{2.9}$$

The middle expression of (2.8) is equal to

$$\frac{1}{n^3} \underset{i=1}{\overset{m}{K}} \frac{p_i n^2}{n^2 + q_i} = \frac{1}{n^3} \frac{p_1 n^2}{n^2 + q_1 + \underset{i=2}{\overset{m}{K}} \frac{p_i n^2}{n^2 + q_i}}. \tag{2.10}$$

Thus,

$$p_1 = c_1, \; q_1 = 0,$$
$$p_i = -\frac{c_i}{c_{i-1}}, q_i = \frac{c_i}{c_{i-1}} = -p_i, \quad i = 2, 3, \cdots, m.$$

Then, it is obviously true that

$$\frac{1}{n^3} \underset{i=1}{\overset{m}{K}} \frac{p_i n^2}{n^2 + q_i} = \frac{1}{n^3} \frac{p_1 n^2}{n^2 + \underset{i=2}{\overset{m}{K}} \frac{p_i n^2}{n^2 - p_i}}. \tag{2.11}$$

The proof of Lemma 2.3 is complete.

3. Main results

In this section, we provide new continued fraction approximations for the gamma function via the Tri-gamma function.

Theorem 3.1. For every integer $n \geq 1$, we have

$$\Gamma(n+1) \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \exp\left(\frac{1}{12} \psi'\left(n + \frac{1}{2}\right)\right) \exp\left(\frac{1}{n^2} \underset{i=1}{\overset{\infty}{K}} \frac{a_i}{n + \frac{b_i}{n}}\right)$$

$$= \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \exp\left(\frac{1}{12} \psi'\left(n + \frac{1}{2}\right)\right) \exp\left(\frac{1}{n^2} \frac{a_1}{n + \frac{b_1}{n} + \frac{a_2}{n + \frac{b_2}{n} + \frac{a_3}{n + \frac{b_3}{n} + \ddots}}}\right), \tag{3.1}$$

where

$$a_1 = \frac{1}{12} B_4 + \frac{1}{24} B_2, \; b_1 = 0,$$
$$a_i = -\frac{i(2i-1)}{(i+1)(2i+1)} \frac{6B_{2i+2} + (i+1)(2i+1)(1 - 2^{1-2i}) B_{2i}}{6B_{2i} + i(2i-1)(1 - 2^{3-2i}) B_{2i-2}}, b_i = -a_i, \quad i = 2, 3, \cdots.$$

Proof. Let

$$c_i = \frac{B_{2i+2}}{2(i+1)(2i+1)} + \frac{(1-2^{1-2i})B_{2i}}{12}, \quad i = 1, 2, 3, \cdots. \tag{3.2}$$

From (3.2) and Lemma 2.2,

$$\sum_{i=1}^{\infty} \frac{c_i}{n^{2i+1}} = \frac{1}{n^2} \mathop{K}_{i=1}^{\infty} \frac{a_i}{n + \frac{b_i}{n}}, \tag{3.3}$$

where

$$a_1 = c_1 = \frac{1}{12} B_4 + \frac{1}{24} B_2, \; b_1 = 0,$$
$$a_i = -\frac{c_i}{c_{i-1}} = -\frac{i(2i-1)}{(i+1)(2i+1)} \frac{6B_{2i+2} + (i+1)(2i+1)(1-2^{1-2i})B_{2i}}{6B_{2i} + i(2i-1)(1-2^{3-2i})B_{2i-2}}, b_i = \frac{c_i}{c_{i-1}} = -a_i, \quad i = 2, 3, \cdots.$$

According to (1.5) and (3.3),

$$\Gamma(n+1) \approx \sqrt{2\pi\, n} \left(\frac{n}{e}\right)^n \exp\left(\frac{1}{12}\psi'\left(n + \frac{1}{2}\right)\right) \exp\left(\frac{1}{n^2} \mathop{K}_{i=1}^{\infty} \frac{a_i}{n + \frac{b_i}{n}}\right). \tag{3.4}$$

Thus, our new continued fraction approximation can be obtained.

Remark 3.1. From (2.3), we have another expression of (3.4) as follows:

$$\begin{aligned} \Gamma(n+1) &\approx \sqrt{2\pi\, n} \left(\frac{n}{e}\right)^n \exp\left(\frac{1}{12}\psi'\left(n + \frac{1}{2}\right)\right) \exp\left(\frac{1}{n^2} \frac{a_1}{n + \mathop{K}_{i=2}^{\infty} \frac{a_i}{n - \frac{a_i}{n}}}\right) \\ &= \sqrt{2\pi\, n} \left(\frac{n}{e}\right)^n \exp\left(\frac{1}{12}\psi'\left(n + \frac{1}{2}\right)\right) \exp\left(\frac{1}{n^2} \cfrac{a_1}{n + \cfrac{a_2}{n - \frac{a_2}{n} + \cfrac{a_3}{n - \frac{a_3}{n} + \ddots}}}\right), \end{aligned} \tag{3.5}$$

where

$$a_1 = \frac{11}{28}, a_2 = \frac{107}{132}, a_3 = \frac{20377}{14124}, a_4 = \frac{2426199}{1059604}, a_5 = \frac{10828367}{3234932}, \cdots.$$

For the convenience of readers, we rewrite.

$$\Gamma(n+1) \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \exp\left(\frac{1}{12}\psi'\left(n+\frac{1}{2}\right)\right)\exp\left(\frac{1}{n^2}\cfrac{\frac{11}{28}}{n+\cfrac{\frac{107}{132}}{n-\cfrac{\frac{107}{132}}{n}+\cfrac{\frac{20377}{14124}}{n-\cfrac{\frac{20377}{14124}}{n}+\ddots}}}\right)$$

(3.6)

Theorem 3.2. For every integer n ≥ 1, we have

$$\Gamma(n+1) \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \exp\left(\frac{1}{12}\psi'\left(n+\frac{1}{2}\right)\right)\exp\left(\frac{1}{n^3}\mathop{K}_{i=1}^{\infty}\frac{p_i n^2}{n^2+q_i}\right)$$

$$= \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \exp\left(\frac{1}{12}\psi'\left(n+\frac{1}{2}\right)\right)\exp\left(\frac{1}{n^3}\cfrac{p_1 n^2}{n^2+q_1+\cfrac{p_2 n^2}{n^2+q_2+\cfrac{p_3 n^2}{n^2+q_3+\ddots}}}\right),$$

(3.7)

where

$$p_1 = \frac{1}{12}B_4 + \frac{1}{24}B_2,\ q_1 = 0,$$

$$p_i = -\frac{i(2i-1)}{(i+1)(2i+1)}\frac{6B_{2i+2} + (i+1)(2i+1)(1-2^{1-2i})B_{2i}}{6B_{2i} + i(2i-1)(1-2^{3-2i})B_{2i-2}},\ q_i = -p_i, \quad i = 2, 3, \cdots.$$

Proof. Using Lemma 2.3 and the same method from (3.2) and (3.3), we have

$$\Gamma(n+1) \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \exp\left(\frac{1}{12}\psi'\left(n+\frac{1}{2}\right)\right)\exp\left(\frac{1}{n^3}\mathop{K}_{i=1}^{\infty}\frac{p_i n^2}{n^2+q_i}\right). \tag{3.8}$$

Thus, our new continued fraction approximation can be obtained.

Remark 3.2. From (2.8), we have another expression of (3.8) as follows:

$$\Gamma(n+1) \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \exp\left(\frac{1}{12}\psi'\left(n+\frac{1}{2}\right)\right)\exp\left(\frac{1}{n^3}\frac{p_1 n^2}{n^2+\underset{i=2}{\overset{\infty}{K}}\frac{p_i n^2}{n^2-p_i}}\right)$$

$$= \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \exp\left(\frac{1}{12}\psi'\left(n+\frac{1}{2}\right)\right)\exp\left(\frac{1}{n^3}\frac{p_1 n^2}{n^2+\frac{p_2 n^2}{n^2-p_2+\frac{p_3 n^2}{n^2-p_3+\ddots}}}\right), \quad (3.9)$$

where

$$p_1 = \frac{11}{28}, p_2 = \frac{107}{132}, p_3 = \frac{20377}{14124}, p_4 = \frac{2426199}{1059604}, p_5 = \frac{10828367}{3234932}, \cdots.$$

For the convenience of readers, we rewrite.

$$\Gamma(n+1) \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \exp\left(\frac{1}{12}\psi'\left(n+\frac{1}{2}\right)\right)\exp\left(\frac{1}{n^3}\frac{\frac{11}{28}n^2}{n^2+\frac{\frac{107}{132}n^2}{n^2-\frac{107}{132}+\frac{\frac{20377}{14124}n^2}{n^2-\frac{20377}{14124}+\ddots}}}\right)$$

(3.10)

4. Conclusion

As mentioned above, in our investigation, we provide some useful lemmas to construct continued fraction based on a given power series. Then we establish new continued fraction approximations for the gamma function, via the Tri-gamma function. Especially, we analytically determine all parameters of the continued fraction by Bernoulli numbers.

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