A Generalisation of Sommerfeld's *On the Composition of Velocities in the Theory of Relativity*

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Abstract

The Michelson-Morley (MM) experiment and its resolution by the special theory of relativity form a foundational truth in modern physics. In this paper we examine and generalise the geometry of the sequence of events within a standard MM interferometer to arrive at a geometry that merges the perspectives of the rest and moving frames within a common stationary circle in space. Further we show that this theoretical approach leads us into spherical trigonometry that supplies a simple solution of the Michelson-Morley problem.

1 Introduction

Special relativity [1], formulated by Albert Einstein in 1905, fundamentally altered our understanding of space, time, and motion. It provided a framework where the speed of light remains constant in all inertial frames, leading to counter-intuitive but experimentally verified concepts like time dilation and length contraction. A contemporary work by Sommerfeld, *On the Composition of Velocities in the Theory of Relativity* [2] was published just four years after Einstein. In his short paper Sommerfeld examines the problem of relative velocity composition from the perspective of spherical trigonometry. Restricting himself at first to two congruent right spherical triangles (shown later to be equivalent to fig. 1 $\triangle ABQ$ and $\triangle CBQ$), Sommerfeld arrives at the spherical equivalent of Einstein's addition theorem [3] for velocities. Proceeding further, they invoke the cosine rule of spherical trigonometry to present a general solution to all triangles of the form $AB'C$ in fig. 1. Sommerfeld summarizes, "For the composition of velocities in the theory of relativity, not the formulas of the plane, but the formulas of the spherical trigonometry (with imaginary sides) are valid. By this remark the complicated transformation calculus becomes dispensable, and can be replaced by a lucid construction on a sphere" [2].

The aim of this paper is to conduct an in-depth theoretical re-visitation of the paradigm shifting Michelson-Morley (MM) experiment, its famous null result [4] and the resulting paradox of space and time whose solution [1] forms the foundational basis of modern physics. We will examine arguments that show that the event sequence within an MM interferometer may be theorised by the rest frame in an unconventional fashion. This approach will demonstrate that under inertial conditions and independent of its orientation or its relative velocity with respect to the rest frame, the locus of all points in space where a reflection event can occur within an MM interferometer is a stationary circle in space. Restricting the discussion to inertial conditions, we attempt to reconcile the MM paradox through a generalisation of Sommerfeld [2] that retains this circular geometry.

2 Euclidean Geometry

On a flat surface [1], we draw any angle θ at origin Q bounded by two equal length line segments $QB = QB' = h$. We join points B and B' to points A and C such that the line segment AC is perpendicular to QB and centred at Q. We will restrict our arguments to the domain $x < h$. Fig. 1 illustrates.

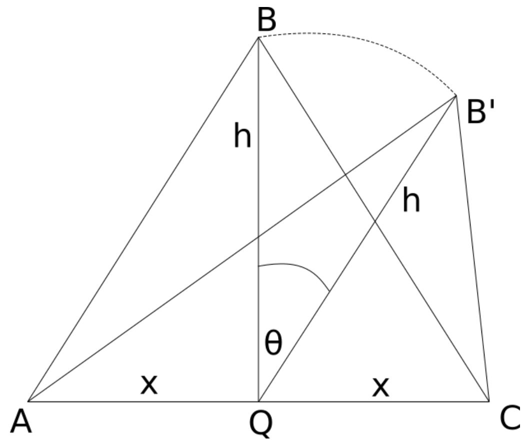

Figure 1: Triangles ABC and $AB'C$ rendered on a flat surface.

From fig. 1, we posit the following:

1. If $x > 0$, physical measurements will verify the theoretical statement $AB + BC \neq AB' + B'C$ remains true for all $\theta \neq 0, \pi, 2\pi...$
2. Since h is constant, curve BB' will take the form of a circle as $0 \leq \theta \leq 2\pi$ independent of x.
3. If $x > 0$, physical measurements will verify the theoretical statement $\angle AB'Q \neq \angle QB'C$ remains true over all $\theta \neq 0, \pi/2, \pi...$

3 A Template of the MM Experiment

Now we turn to theoretical aspects of relativistic optical interferometry to demonstrate that the geometry and sequence of events within an MM interferometer always templates to that of fig. 1.

3.1 Frames of Reference

Consider two imaginary euclidean reference frames that are in relative motion with respect to each other. Let us arbitrarily assume one of these frames is at rest and the other moves with some velocity v with respect to the rest frame. Accordingly we refer to fig. 1 and declare,

1. A rest frame I_0 centered at point Q.
2. A moving frame I_1 that translates from point A to point C with some velocity v relative to rest frame I_0.

3.2 Geometry and Sequence of Events

Now let us consider the structure of an MM interferometer [4](see fig. 2). By fixing $\angle B_1'QB_2' = \pi/2$, line segments QB_1' and QB_2' form the arms of the interferometer. Mirrors B_1 and B_2 are aligned perpendicular to their respective arms. The apparatus may be rotated about its source and consequently each arm subtends its own angle θ_i measured from a perpendicular to line segment AC. Let us affix moving frame I_1 to the source of the interferometer. Now let us imagine this interferometer moving through space under inertial rules such that,

1. v remains constant ($AQ = QC$).
2. The interferometer orientation (θ_i) with respect to line segment AC remains constant.

Reference frame I_1 (affixed to the source) translates with constant velocity v from point A to point C. From the perspective of the rest frame I_0, a discrete event cycle begins with the source at point A marking the simultaneous emission of a pair of photons (wavelength=λ). As the entire apparatus moves with some constant ($AQ = QC$) velocity v relative to origin Q along line segment AC, the photons are emitted at point A, reflect from mirrors B_1 and B_2 to finally arrive simultaneously (in phase with each other) at point C. This geometry and sequence of events remains true over all possible orientations θ of an MM interferometer [5] and over all $0 \leq v < c$ where c represents the velocity of light in free space [6].

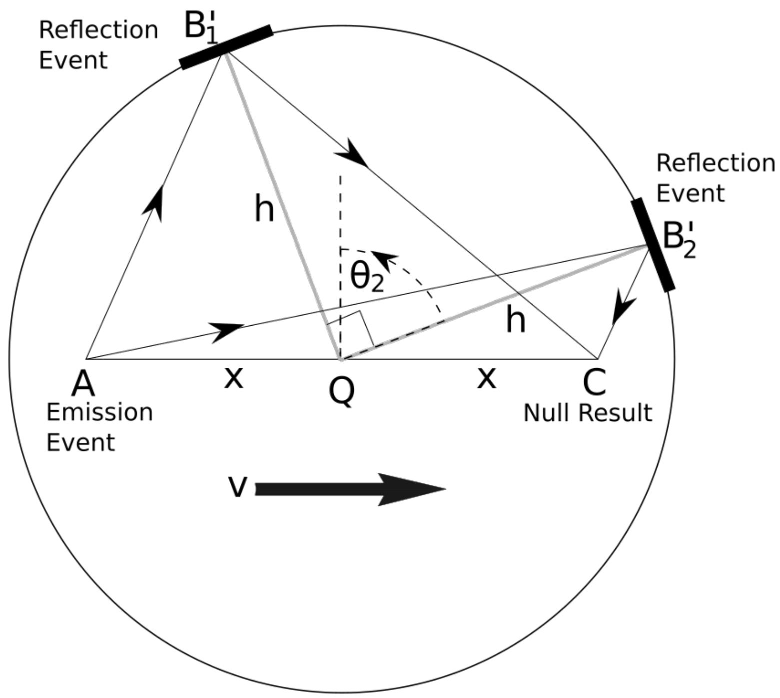

Figure 2: Geometry of the Michelson-Morley experiment depicting the general case $v \neq 0$ and $\theta_i \neq 0, \pi/2, \pi....$ Point Q is chosen as the origin. Only the events within the interferometer that are relevant to relativistic discussion are shown. Independent of the orientation of the interferometer, rest frame I_0 will find triangle $AB_i'C$ is a generalisation of triangle $AB'C$ in fig. 1. Identical to fig.1, physical measurements of the geometry of events will confirm that $AB_i'+B_i'C \neq AB_j'+B_j'C$ for all $\sin\theta_i \neq \sin\theta_j$ (inequality in path lengths) and $\angle AB'Q \neq \angle QB'C$ (inequality in angles of incidence and reflection) for all $\theta_i \neq 0, \pi/2, \pi...$ By setting $v = 0$ ($x = 0$), the figure represents the observational perspective of moving frame I_1. By setting $v > 0$ ($x > 0$), the figure represents the observational perspective of rest frame I_0. It is evident from fig. 1 that curve BB' will take the form of a stationary circle of radius h about point Q independent of θ_i (i.e. orientation) and v (i.e. frame of reference).

4 Special Relativity

At this stage of investigation, rest frame I_0 recognises the inequalities in path lengths depicted in fig. 1 coupled with the experimental null result of the MM experiment to arrive at a well understood paradox of space and time that is traditionally reconciled by selecting point A as the origin followed by the application of special relativity [7]. Let us examine the process by which this paradox and its resolution occur:

1. Rest frame I_0 observes a single cycle of an MM interferometer moving from point A (the origin) to point C under inertial rules and draws a diagram of the events on a flat surface (see fig. 2).
2. The rest frame uses a stopwatch to count the observed time interval t between emission and null result events.
3. The rest frame uses a measuring rod to measure the distance AC to determine x.
4. From x and t, the rest frame determines the relative velocity $v = 2x/t$.
5. Rest frame employs a measuring rod to determine $AB'_i + B'_iC$ and $AB'_j + B'_jC$ and finds them unequal in all cases where $v > 0$ and $\sin\theta_i \neq \sin\theta_j$.
6. For subsequent computation purposes and rest frame I_0 determines the cartesian coordinates of points $B'_i = (b_1, b_2)$ and $B'_j = (b_3, b_4)$ in fig. 2 by analytical means or by measuring rod.

From the above steps rest frame I_0 concludes that if $x > 0$ (equivalent to $v > 0$), the observed null result conflicts with the application of the wave equation [8] to the geometry under consideration. This leads rest frame I_0 into a paradox of space and time that is resolved by special relativity [1] in the following manner:

1. Rest frame I_0 invokes the lorentz factor:
$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \tag{1}$$
2. From their physical observations of x and t, rest frame determines γ:
$$\gamma = \frac{1}{\sqrt{1 - (2x/t)^2/c^2}} \tag{2}$$
3. The rest frame now determines b'_1 and b'_3 according to the rule:
$$b'_i = \gamma(b_i - vt) \tag{3}$$
4. Leaving b_2 and b_4 unchanged, the rest frame redraws fig. 2, on this occasion setting (i) mirrors B_i and B_j in positions (b'_1, b_2) and (b'_3, b_4) and (ii) $AC = \gamma(2x - vt)$. If $x > 0$ (equivalently $v > 0$), this procedure results in a "shortening" [1] of the observed distance AC and of the physical dimensions of the interferometer along the AC axis.
5. The rest frame also determines $t' = \gamma(t - 2vx/c^2)$ to observe a lengthening of the time interval between emission and null result events.

Importantly, it may be noted here that in order to apply special relativity to the MM interferometer geometry, rest frame I_0 must determine the co-ordinates of mirror B_i and B_j in fig. 2 by some means i.e. the orientation of the interferometer *must* be known to the rest frame.

5 Selecting a New Origin

By imagining afresh the sequence of events within an MM interferometer, we may also posit that by selecting instead point Q as a common origin (refer fig. 2), rest frame I_0 and *any* moving frame I_i moving with velocity v_i along the AC or CA directions are all assured that over all $0 \leq \theta_i \leq 2\pi$ and $0 \leq v_i < c$, the locus of all points in space where a reflection event can occur is a common stationary circle of radius h about point Q. Invoking the symmetry of the circle, a rest frame I_0 may also rotate fig. 2 in entirety about point Q by any angle $0 \leq \phi \leq 2\pi$ and may incorporate any number of moving frames $I_1, I_2, I_3, ...I_i$, each moving in any possible direction ϕ_i and each set an any orientation $0 \leq \theta_i \leq 2\pi$ for all $0 \leq v_i < c$ within a single common stationary circle i.e. curve BB'. Further, invoking the superposition property of waves [8], we may posit [9] that this single circle BB' is capable of hosting an infinite number of MM null result cycles moving at all possible velocities, in all possible directions, simultaneously [10]. To this end, let us theorise a model of space upon which rest frame I_0 is able reconcile the paradox of unequal path lengths presented above in a manner that retains the circular geometry of curve BB'. Since special relativity leads to distortions in curve BB' (refer sec. 4), it is unsuitable for this manner of theoretical exploration.

6 Spherical Trigonometry

Before we dive into spherical trigonometry, recall Sommerfeld, "it apparently better corresponds to the meaning of the theory of relativity, to calculate and (by consideration of the reality relations) to construct by rotation angles, instead of only using its tangents, the velocities" [2]. With this in mind, let rest frame I_0 project fig. 1 onto the surface of an imaginary sphere of arbitrary radius R such that the shortest distance path between any two points are described by great circles on the sphere [11]. Thus the magnitude of physical distances $x, h, AB', B'C$ in fig. 1 are measured analytically as rotations in radians subtended at the centre of this sphere. The angles depicted in fig. 1 are measured on the surface of the sphere and curve BB' takes the form of a small circle on the surface of this sphere having radius h radians and centred at point Q. Since Sommerfeld has already provided the cosine rule as a solution to all triangles of the form $AB'C$, let us generalise further and invoke instead the sine rule of spherical trigonometry to see where it leads us.

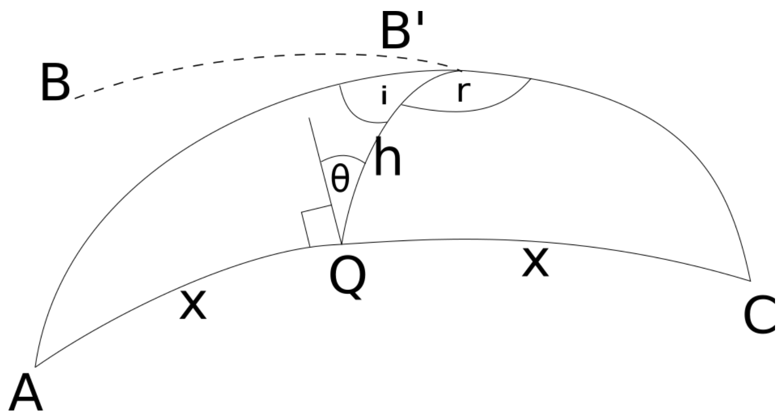

Figure 3: Spherical Trigonometry. Angles $h, x, AB', B'C$ are measured analytically at the centre of an imaginary sphere. Angles i, r, A, C, θ are measured analytically on the surface of the sphere.

6.1 Analysis of Spherical Model

From fig. 3 and the rule of sines for spherical triangles [12], rest frame I_0 finds in $\triangle AB'Q$:

$$\frac{\sin AB'}{\sin\left(\pi/2+\theta\right)}=\frac{\sin h}{\sin A}=\frac{\sin x}{\sin i} \tag{4}$$

where $i = \angle AB'Q$. Similarly for $\triangle CB'Q$:

$$\frac{\sin CB'}{\sin\left(\pi/2-\theta\right)}=\frac{\sin h}{\sin C}=\frac{\sin x}{\sin r} \tag{5}$$

where $r = \angle CB'Q$.

From equations, 4 and 5 rest frame I_0 finds in all spherical triangles of the form $AB'C$:

$$\frac{\sin(AB')}{\sin(CB')}=1 \tag{6}$$

From eq. 6, we find AB' and $B'C$ are supplementary angles i.e. rendering the same result when subjected to the sine function. Referring now to fig. 2, eq. 6 guarantees that by interpreting the MM null result geometry with this analytical approach, rest frame I_0 is assured the theoretical statement $AB'_i + B'_iC = AB'_j + B'_jC$ remains true independent of v, h, θ. Thus the paradox of unequal path lengths presented by physical measurements of fig. 1 vanishes independent of frame of reference v_i or orientation of the interferometer θ_i, and the solution to the null result retains the MM geometry *verbatim.* Further by selecting point Q as a common origin, every frame of reference $I_0, I_1, I_2 ... I_\infty$, whether at rest or moving are all assured that the commonality and the circularity of curve BB' (refer sec. 4) remain unaffected if projected onto this theoretical model of space.

Equations 4 and 5 also show that:

$$\frac{\sin i}{\sin r}=\frac{\sin A}{\sin C}=\rho \tag{7}$$

From eq. 7 and by physical measurements of $\angle ABQ$ and $\angle CBQ$ in fig. 1, rest frame I_0 finds that under inertial conditions, the constant $\rho = 1$ and recognises that in Sommerfeld's analytical space, independent of x, h, θ, surface angles $\angle i$ and $\angle r$ (also $\angle A$ and $\angle C$) are (i) equal if $\theta = 0$ or (ii) supplementary angles if $\theta \neq 0$.

6.2 Duality in Space

Consider a function *Polar()* that takes an element-wise spherical triangle as its argument and returns the corresponding polar spherical triangle [13]. Under Sommerfeld's model, rest frame I_0 will find in all analytical triangles of the form $AB'C$:

$$Polar(AB'Q) = [\pi - A, \pi - AB', \pi - i, \pi - h, \pi/2 + \theta, \pi - x] \tag{8}$$

and,

$$CB'Q = [C, B'C, r, h, \pi/2 - \theta, x] \tag{9}$$

Recalling the relationships from eq. 6 and 7 rest frame I_0 finds that spherical triangles $Polar(AB'Q)$ and $CB'Q$ are analytical duals of each other and their corresponding elements of $\triangle AB'C$ form pairs of supplementary angles. By invoking the reversibility [14] of the *Polar()* function, we may conclude that under this model of space, triangles $AB'Q$ and $CB'Q$ are also analytical duals of each other and render identical results when subjected element-wise to the sine function.

7 Conclusion

Equation 6, demonstrates that in a spherical model of space and independent of frame of reference $I_1, I_2, I_3 ... I_i$, the total analytical light path in an MM cycle, $AB' + B'C$ is always equal to the maximal value i.e. π radians and this remains true independent of h, θ and over all $0 \leq x/h < \infty$. Further, unlike special relativity, this solution of the MM problem does not mandate distortions in the structure of curve BB'. Thus Sommerfeld's model of analytical space respects the circularity and commonality of curve BB' and is valid over all $0 \leq v/c < \infty$ as compared to special relativity, which Einstein himself recognizes is "meaningless" [1] except within the domain $v < c$. Thus the continued exploration of Sommerfeld's model would allow physics discussions in the domain $1 < v/c < \infty$ to become *meaningful*. Also, the property of duality in space presented above is intriguing given its symmetry with the widely accepted concept of duality in particle-wave theory. Further, it is noteworthy that this generalisation of Sommerfeld, being independent of θ, is also applicable in the discipline of quantum mechanics, an identical problem of space and time where the orientation of the "interferometer" is hidden from rest frame I_0 [15].

8 Statements and Declarations

The author has no competing interests to declare that are relevant to the content of this article. There are no data associated with this article.

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