Standard Form Metric and Static Charged Sphere

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Abstract

For a system of a static charged sphere we make gauge and coordinate transformations so that the electromagnetic vector potential has a unit time component and zero space components. Beginning with a spherically symmetric metric in standard form and electromagetic vector potential having this special form we solve the Einstein field equations outside the sphere. We show the solution has charge outside the sphere.

1 Electromagnetic potential and field

Let $A_\mu(t,r,\theta,\varphi)$ and $g_{\mu\nu}(t,r,\theta,\varphi)$ be the electromagnetic potential and metric tensor respectively. The electromagnetic field is

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} \tag{1}$$

Let $A^\mu(t,r,\theta,\varphi)$ be the electromagnetic vector potential. For a scalar function $\phi(t,r,\theta,\varphi)$ define a gauge transformation of A_μ to $\hat{A}_\mu$ by

$$\hat{A}_\mu = A_\mu + \phi_{,\mu} \tag{2}$$

hence A^μ to $\hat{A}^\mu$ by

$$\hat{A}^\mu = g^{\mu\alpha}\hat{A}_\alpha = g^{\mu\alpha}A_\alpha + g^{\mu\alpha}\phi_{,\alpha} = A^\mu + g^{\mu\alpha}\phi_{,\alpha} \tag{3}$$

We have by (1)-(3), and requiring $\phi_{,\nu\mu} = \phi_{,\mu\nu}$ that

$$\begin{aligned} F_{\mu\nu} &= A_{\nu,\mu} - A_{\mu,\nu} = A_{\nu,\mu} - A_{\mu,\nu} + \phi_{,\nu\mu} - \phi_{,\mu\nu} = (A_\nu + \phi_{,\nu})_{,\mu} - (A_\mu + \phi_{,\mu})_{,\nu} \\ &= (g_{\nu\alpha}[A^\alpha + g^{\alpha\beta}\phi_{,\beta}])_{,\mu} - (g_{\mu\alpha}[A^\alpha + g^{\alpha\beta}\phi_{,\beta}])_{,\nu} = (g_{\nu\alpha}\hat{A}^\alpha)_{,\mu} - (g_{\mu\alpha}\hat{A}^\alpha)_{,\nu} \end{aligned} \tag{4}$$

2 Static charged sphere and Einstein field equations

Let there be a static charged sphere with spherically symmetric charge and mass densities. Require of the electromagnetic potential that

$$A_0(r) < 0 \qquad A_1(r) = A_2(r) = A_3(r) = 0 \tag{5}$$

and $A_0(r) \to 0$ as $r \to \infty$. For the static charged sphere let the metric $g_{\mu\nu}(r)$ having form

$$-d\tau^2 = -C(r)dt^2 + D(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{6}$$

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$C(r) \to 1$ and $D(r) \to 1$ as $r \to \infty$ satisfy the Einstein field equations [1]

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi(E_{\mu\nu} + M_{\mu\nu}) \tag{7}$$

where the electromagnetic energy-momentum tensor $E^{\mu\nu}$ and perfect fluid energy-momentum tensor $M^{\mu\nu}$ are given by [1]

$$E_{\mu\nu} = g^{\sigma\tau} F_{\mu\sigma} F_{\nu\tau} - \frac{1}{4} g_{\mu\nu} g^{\sigma\alpha} g^{\tau\beta} F_{\sigma\tau} F_{\alpha\beta} \qquad M_{\mu\nu} = p g_{\mu\nu} + (\rho + p) U_\mu U_\nu \tag{8}$$

and p is the pressure, ρ the mass density, and U^μ the fluid velocity four-vector.

3 Gauge and coordinate transformation

Let

$$\phi = -t \tag{9}$$

hence by (3) and (9)

$$\hat{A}^0 = \frac{1 - A_0}{C} \qquad \hat{A}^1 = \hat{A}^2 = \hat{A}'^3 = 0 \tag{10}$$

Since $A_0(r) < 0$ we can construct the coordinate transformation given by

$$t' = \frac{C}{1 - A_0} t \qquad r' = r \qquad \theta' = \theta \qquad \varphi' = \varphi \tag{11}$$

Coordinate transformation (11) transforms $\hat{A}^\mu(r)$ given by (10) to the special form [2]

$$\hat{A}'^0 = 1 \qquad \hat{A}'^1 = \hat{A}'^2 = \hat{A}'^3 = 0 \tag{12}$$

By (4), (11), and (12)

$$F'_{\mu\nu} = (g'_{\nu\alpha} \hat{A}'^\alpha)_{,\mu} - (g'_{\mu\alpha} \hat{A}'^\alpha)_{,\nu} = g'_{\nu 0,\mu} - g'_{\mu 0,\nu} \tag{13}$$

4 Solving the field equations for standard metric

The Einstein field equations (7) become after making a coordinate transformation to coordinates $t', r', \theta', \varphi'$

$$\begin{aligned} R'_{\mu\nu} - \frac{1}{2} g'_{\mu\nu} R' &= -8\pi(E'_{\mu\nu} + M'_{\mu\nu}) \\ &= -8\pi g'^{\sigma\tau} F'_{\mu\sigma} F'_{\nu\tau} + 2\pi g'_{\mu\nu} g'^{\sigma\alpha} g'^{\tau\beta} F'_{\sigma\tau} F'_{\alpha\beta} - 8\pi[p g'_{\mu\nu} + (\rho + p) U'_\mu U'_\nu] \end{aligned} \tag{14}$$

By (13) we have (14) becomes

$$\begin{aligned} R'_{\mu\nu} - \frac{1}{2} g'_{\mu\nu} R' &= -8\pi(E'_{\mu\nu} + M'_{\mu\nu}) \\ &= -8\pi g'^{\sigma\tau} [g'_{\sigma 0,\mu} - g'_{\mu 0,\sigma}][g'_{\tau 0,\nu} - g'_{\nu 0,\tau}] + 2\pi g'_{\mu\nu} g'^{\alpha\sigma} g'^{\beta\tau} [g'_{\tau 0,\sigma} - g'_{\sigma 0,\tau}][g'_{\beta 0,\alpha} - g'_{\alpha 0,\beta}] \\ &- 8\pi[p g'_{\mu\nu} + (\rho + p) U'_\mu U'_\nu] \end{aligned} \tag{15}$$

Dropping primes gives

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R &= -8\pi(E_{\mu\nu} + M_{\mu\nu}) \\ &= -8\pi g^{\sigma\tau} [g_{\sigma 0,\mu} - g_{\mu 0,\sigma}][g_{\tau 0,\nu} - g_{\nu 0,\tau}] + 2\pi g_{\mu\nu} g^{\alpha\sigma} g^{\beta\tau} [g_{\tau 0,\sigma} - g_{\sigma 0,\tau}][g_{\beta 0,\alpha} - g_{\alpha 0,\beta}] \\ &- 8\pi[p g_{\mu\nu} + (\rho + p) U_\mu U_\nu] \end{aligned} \tag{16}$$

We now consider other possible metric solutions of (16). Instead let $g_{\mu\nu}(t,r)$ be any solution of (16) having the standard form [1]

$$-d\tau^2 = -B(r,t)dt^2 + A(t,r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{17}$$

We have by (8), (13), and primes dropped that

$$E_{\mu\nu} = g^{\sigma\tau}[g_{\sigma 0,\mu} - g_{\mu 0,\sigma}][g_{\tau 0,\nu} - g_{\nu 0,\tau}] - \frac{1}{4}g_{\mu\nu}g^{\alpha\sigma}g^{\beta\tau}[g_{\tau 0,\sigma} - g_{\sigma 0,\tau}][g_{\beta 0,\alpha} - g_{\alpha 0,\beta}] \qquad M_{\mu\nu} = 0 \tag{18}$$

outside the sphere so by (17) and (18)

$$E_{00} = \frac{B'^2}{2A} \qquad E_{01} = 0 \qquad E_{11} = -\frac{B'^2}{2B} \qquad E_{22} = \frac{r^2B'^2}{2AB} \qquad E_{33} = \frac{r^2\sin^2\theta B'^2}{2AB} \tag{19}$$

where the prime now means derivative with respect to r. From (16)-(19) we have outside the sphere

$$R - 2R = -8\pi g^{\alpha\beta}E_{\alpha\beta} = 0 \tag{20}$$

hence $R = 0$. For the standard form metric (17) and [1]

$$\begin{aligned}
R_{00} &= -\frac{B''}{2A} + \frac{A'B'}{4A^2} - \frac{B'}{Ar} + \frac{B'^2}{4AB} + \frac{\ddot{A}}{2A} - \frac{A'^2}{4A^2} - \frac{\dot{A}\dot{B}}{4AB} && (21)\\
R_{01} &= -\frac{\dot{A}}{Ar} && (22)\\
R_{11} &= \frac{B''}{2B} - \frac{B'^2}{4B^2} - \frac{A'B'}{4AB} - \frac{A'}{Ar} - \frac{\ddot{A}}{2B} + \frac{\dot{A}\dot{B}}{4B^2} + \frac{\dot{A}^2}{4AB} && (23)\\
& && (24)\\
R_{22} &= -1 + \frac{1}{A} - \frac{rA'}{2A^2} + \frac{rB'}{2AB} && (25)\\
R_{33} &= \sin^2\theta R_{22} && (26)\\
R_{02} &= R_{03} = R_{12} = R_{13} = R_{23} = 0 && (27)
\end{aligned}$$

where the dot denotes derivative with respect to time. We have outside the sphere from (16)

$$R_{01} - \frac{1}{2}g_{01}R = -8\pi E_{01} \tag{28}$$

hence by (19), (22), and $R = 0$ we have $A(t,r)$ does not depend on time. Now outside the sphere from the field equations

$$R_{00} + 8\pi E_{00} = 0 \qquad R_{11} + 8\pi E_{11} = 0 \tag{29}$$

hence

$$\frac{R_{00} + 8\pi E_{00}}{B} + \frac{R_{11} + 8\pi E_{11}}{A} = -\frac{1}{rA}\left(\frac{A'}{A} + \frac{B'}{B}\right) = 0 \tag{30}$$

so AB does not depend on r. We have outside the sphere from

$$R_{22} + 8\pi E_{22} = 0 \tag{31}$$

that

$$-1 + \frac{1}{A} - \frac{rA'}{2A^2} - \frac{rA'}{2A^2} + 4\pi\frac{r^2A'^2B}{A^3} = 0 \tag{32}$$

hence

$$B = \frac{1}{4\pi r^2 A'^2}[A^3 - A^2 + rAA'] \tag{33}$$

Since A does not depend on t we then have by (33) that B does not depend on t. We can conclude the spherically symmetric metric in standard form does not depend on t outside the sphere. Consequently AB is constant. Require $A \to 1$ and $B \to 1$ as $r \to \infty$ hence outside the sphere $A = B^{-1}$. From (33) and $A = B^{-1}$ we have

$$-1 + B + rB' + 4\pi r^2 B'^2 = 0 \tag{34}$$

A series solution to this equation gives

$$B(r) = 1 + \frac{a}{r} + 4\pi\frac{a^2}{r^2} + 32\pi^2\frac{a^3}{r^3} + \cdots \tag{35}$$

where a is a parameter.

5 Conclusions

We have outside the sphere that

$$-d\tau^2 = -B(r)dt^2 + \frac{1}{B(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{36}$$

Maxwell equations are

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} \qquad -\frac{1}{\sqrt{-g}}[\sqrt{-g}g^{\gamma\alpha}g^{\mu\beta}F_{\alpha\beta}]_{,\gamma} = J^\mu \tag{37}$$

where $J^\nu = \sigma U^\mu$ and σ is the charge density. Using (4) and transforming to coordinates (11) and then dropping primes so (37) becomes

$$F_{\mu\nu} = g_{\nu 0,\mu} - g_{\mu 0,\nu} \qquad -\frac{1}{\sqrt{-g}}\left[\sqrt{-g}g^{\gamma\alpha}g^{\mu\beta}(g_{\beta 0,\alpha} - g_{\alpha 0,\beta})\right]_{,\gamma} = J^\mu \tag{38}$$

We have

$$\hat{A}_0(r) = g_{0\alpha}\hat{A}^\alpha = g_{00} = -B(r) \qquad \hat{A}_1(r) = \hat{A}_2(r) = \hat{A}_3(r) = 0 \tag{39}$$

$$F_{01} = B'(r) \qquad F_{02} = F_{03} = F_{12} = F_{13} = F_{23} = 0 \tag{40}$$

$$J^0(r) = B''(r) + \frac{2}{r}B'(r) = \frac{a}{r^3} + 8\pi\frac{a^2}{r^4} + \cdots \qquad J^1(r) = J^2(r) = J^3(r) = 0 \tag{41}$$

By (41) we can conclude for metric in standard form there is charge outside the sphere. We however expect whether there is charge outside the sphere to be independent of coordinates.

References

[1] S. Weinberg, *Gravitation and Cosmology*

[2] K. De Paepe, *Physics Essays*, September 2007