# Light trajectories near a point mass

A.J.Owen

v1 September 16th 2024

Author's contact email address:
ajowen@physics.org

## Abstract

Light trajectories near a static point mass have been calculated for two cases: the first is the usual black-hole solution in general relativity that results from using Schwarzschild coordinates to determine the curvature of spacetime, and the second is for a model described in a previous paper, *viXra:2409.0030*, in which spacetime is completely regular with no event horizon.

## 1 Introduction

The first and probably most important test of Einstein's theory of general relativity *(GR)* [1] was to predict correctly the bending of starlight as it passed near the Sun on its way to being observed on Earth during the solar eclipse of 1919 [2]. Since then, numerous examples of light bending have been observed and catalogued, and the term gravitational lensing used to describe the effect generally [3].

Using *GR*, an approximate expression for the angle of bending $\delta$ is given by $\delta = 2\alpha/R$, where $\alpha = 2GM/c^2$ and $R$ is the perihelion or closest distance of approach of a ray of light to the mass $M$ causing gravitational bending [4]. This applies when the bending is small and the light ray continues its travels. However, when a light ray passes very close to a point-like mass, below a certain distance, it will be captured and spiral inwards towards the central mass. There is also a theoretical distance where a light ray will do neither of these things and stay in a circular orbit, called the photon sphere.

These trajectories can be calculated numerically using the well known *GR* solution for a point mass in terms of Schwarzschild coordinates. However, there is a potential problem here, in my opinion, in that the Schwarzschild radial coordinate (denoted in this paper by $\tilde{r}$) is not strictly identical to the true radial coordinate distance $r$ from the point mass when distances are small. Whether this discrepancy leads to a false impression of the trajectories has, to my knowledge, not been discussed in the scientific literature. In this paper I therefore want to show how light ray trajectories might differ from what one expects from the standard view, by adopting a spacetime metric that quantifies the difference between $\tilde{r}$ and $r$.

## 2 Background theory

The spacetime solution for a point mass in terms of Schwarzschild coordinates leads to it being a black hole with the defining feature of an event horizon (see, for example, [5]). If the black hole is spinning, instead of being static, the metric for this is called the Kerr metric [6]. However, in this paper it will suffice to consider just the well-known solution for a static point mass, for which a metric line element in a curved spacetime $(t, \tilde{r}, \theta, \phi)$ with spherical spatial symmetry about the origin may be written:

$$d\tilde{s}^2 = c^2 dt'^2 = A(\tilde{r})\, c^2 dt^2 - B(\tilde{r})\, d\tilde{r}^2 - \tilde{r}^2\, d\Omega^2 \tag{1}$$

Here, $d\tilde{s}$ is a spacetime increment, $c$ the speed of light, $dt'$ an increment of proper time, $dt$ an increment of coordinate time, $d\tilde{r}$ an increment of the radial coordinate $\tilde{r}$, and $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$; $A$ and $B$ are radially dependent functions describing the curvature of the time and radial metric coefficients, respectively, while $\tilde{r}$ is the Schwarzschild radial coordinate. The solution satisfying Einstein's field equations of general relativity for the vacuum outside the point mass is well known, and given by:

$$A(\tilde{r}) = \frac{1}{B(\tilde{r})} = 1 - \frac{\alpha}{\tilde{r}} \tag{2}$$

where $\alpha$ is a constant of integration equal to $2GM/c^2$. This leads to the idea of a black hole and event horizon, since the functions $A$ and $B$ become negative if $\tilde{r} < \alpha$.

In order to calculate how a ray of light is bent in the gravitational field - or curved spacetime - of a static point mass, we use the fact that for light the proper time is zero, so the metric in Equation 1 may be rearranged to read:

$$dt^2 = \frac{B}{Ac^2}\, d\tilde{r}^2 + \frac{\tilde{r}^2}{Ac^2}\, d\Omega^2 \tag{3}$$

Now consider a ray of light travelling in a plane given by $\theta = \pi/2$, in which case $d\Omega^2 = d\phi^2$, and we have

$$dt^2 = \frac{B}{Ac^2}\,d\tilde{r}^2 + \frac{\tilde{r}^2}{Ac^2}\,d\phi^2 \tag{4}$$

This can be rewritten as:

$$\frac{B}{A}\frac{\dot{r}^2}{c^2} + \frac{\tilde{r}^2}{A}\frac{\dot{\phi}^2}{c^2} = 1 \tag{5}$$

where $\dot{r} = d\tilde{r}/dt$ and $\dot{\phi} = d\phi/dt$. Using the coordinate time $t$ as Lagrangian parameter in Equation 4 it is then straightforward to show that the geodesic equation in $\phi$ leads to

$$\frac{\tilde{r}^2}{A}\,\dot{\phi} = k \tag{6}$$

where $k$ is a constant.

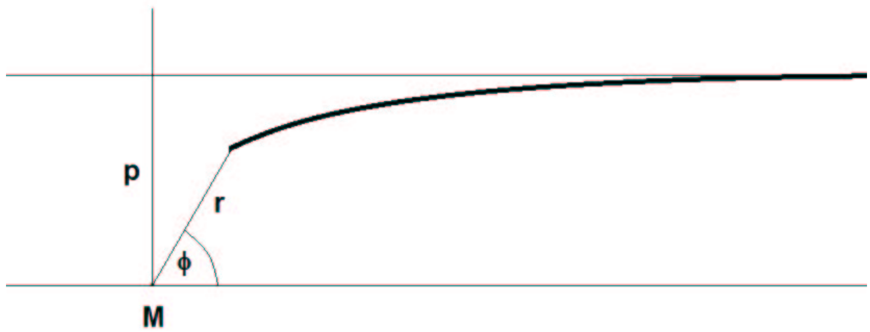


Figure 1: A light ray approaches the mass $M$ with an impact parameter $p$. The ray deviates from a straight line near $M$ as a result of the curvature of spacetime.

We now need to find the constant $k$ for the particular "experiment" envisaged. Consider a ray of light directed from a large distance away towards a point mass $M$, where the "impact parameter" is $p$ (Figure 1). When the ray approaches $M$ (from the right) it will be bent towards it. For large $p$ it would pass $M$, being deflected by a small angle $\delta \approx 2\alpha/R$, as mentioned previously. Instead, we essentially want to know here how the path of the ray would be affected when $p$ is small or of the same order of magnitude as $\alpha$.

From the initial condition, i.e. when the ray is a long way from the mass $M$, it can be deduced that the constant $k$ in Equation 6 is given by:

$$k = p\,c \tag{7}$$

giving

$$\frac{\dot{\phi}^2}{c^2} = \frac{A^2 p^2}{\tilde{r}^4} \tag{8}$$

Inserting this into Equation 5 with $B = 1/A$ then gives:

$$\frac{\dot{r}^2}{c^2} = A^2 \left(1 - \frac{A p^2}{\tilde{r}^2}\right) \tag{9}$$

Dividing Equation 8 by Equation 9 gives

$$\frac{d\phi}{d\tilde{r}} = \frac{p}{\tilde{r}^2 \sqrt{1 - \frac{A p^2}{\tilde{r}^2}}} \tag{10}$$

which is the fundamental equation I shall be using in this paper. This now enables us in principle to find the trajectory of a light ray in terms of the parameters $\alpha$ and $p$.

In addition to this, it is interesting to derive the equation for a circular orbit for the light trajectory, the so-called photon sphere. In that case we write $dr = 0$ in Equation 4 to give

$$dt^2 = \frac{\tilde{r}^2}{A\,c^2} d\phi^2$$

or

$$\dot{\phi} = \frac{c\sqrt{A}}{\tilde{r}} \tag{11}$$

To complete the solution, we need to use the radial geodesic equation, which is given by :

$$\ddot{r} + \frac{A' c^2}{2B} \dot{t}^2 + \frac{B'}{2B} \dot{r}^2 - \frac{\tilde{r}}{B} (\dot{\theta}^2 + \sin^2\theta\, \dot{\phi}^2) = 0 \tag{12}$$

and set $dr = 0, d\theta = 0, \theta = \pi/2$, which gives

$$\dot{\phi}^2 = \frac{A' c^2}{2\tilde{r}} \tag{13}$$

Combining Equations 11 and 13 gives

$$\frac{A'}{A} = \frac{2}{\tilde{r}}$$

and with $A = 1 - \alpha/\tilde{r}$, we finally obtain

$$\tilde{r} = \frac{3}{2}\alpha \tag{14}$$

for the radius of the photon sphere.

## 3 Conventional results

Integrating Equation 10 to obtain $\phi$ directly as a function of $\tilde{r}$ does not seem to be analytically possible. However, numerical solutions can be obtained using online software (see, e.g. [7]), or by writing one's own programmes, which I did here. The following diagrams give examples of numerically calculated solutions using the above equation.

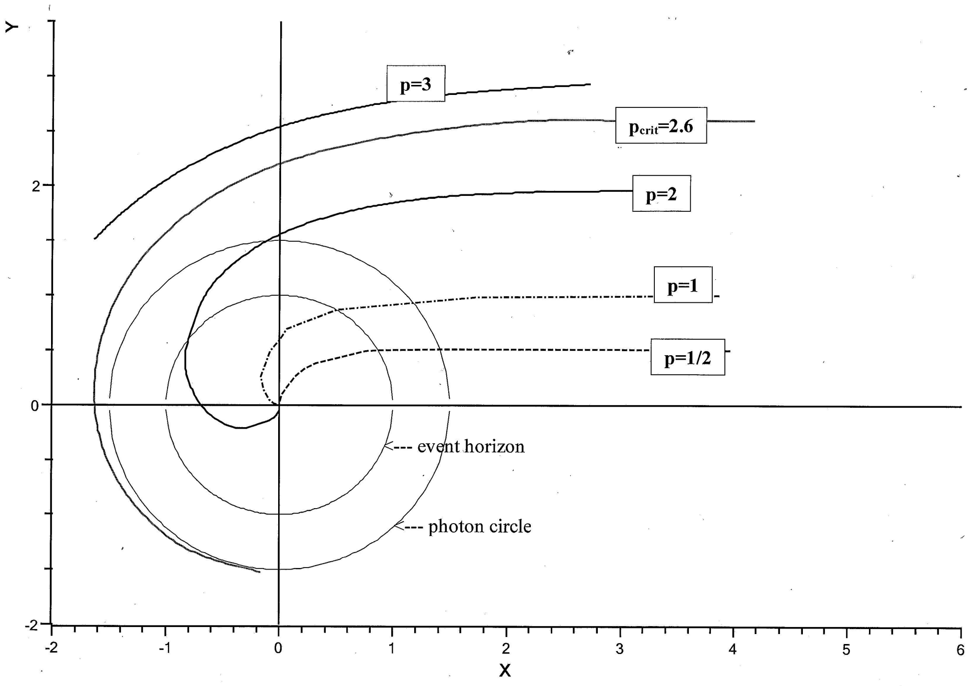


Figure 2: Light trajectories near a point mass (black-hole model). The impact parameter $p$ is in units of $\alpha = 2GM/c^2$

For the usual model - which I shall call the black-hole model - using the Schwarzschild radial coordinate $\tilde{r}$, we have $A = 1 - \alpha/\tilde{r}$. The ray trajectory is shown in terms of $\tilde{r}$ in Figure 2 for various values of the impact parameter $p$. Distances are plotted in units of $\alpha$, by setting $\alpha = 1$. For large values of $p$ the light ray is bent slightly towards the black hole, but passes by and is not captured. This occurs for $p > 3\sqrt{3}/2 \approx 2.6$, and the trajectory is then symmetrical about a straight line joining the origin to the perihelion. When $p$ is less than 2.6, the light ray overshoots the point mass, but is nevertheless captured by it. It bends around and spirals in towards the origin at $\tilde{r} = 0$. If $p$ is less than about 0.5, the ray is simply captured by the point mass without overshooting it. The event horizon is indicated as a circle a distance $\alpha$ from the origin and the photon sphere is at a distance $3\alpha/2$ from the origin. This condition can also be established by setting $dr/d\phi = 0$ in Equation 10. These characteristics are well known, and do not represent anything new, so far.

## 4 Additional theory and results

Adapting Newton's law of gravitation to a four-dimensional spacetime, i.e. with a time dimension (instead of absolute time), I showed in previous papers [8], [9] that the true radial coordinate $r$ is related to the Schwarzschild coordinate $\tilde{r}$ by the linear expression

$$\tilde{r} = r + \alpha \tag{15}$$

This difference is of no consequence when $r >> \alpha$. Then, all the usual predictions of *GR*, such as gravitational time dilation, the perihelion rotation of Mercury and the weak bending of light past a gravitational mass are quantitatively valid, as they stand. However, when distances are of the order of $\alpha$, which is crucial for the black-hole prediction, the situation is entirely different. The time curvature metric coefficient $A(r)$ is then given by:

$$A(r) = \left(1 + \frac{\alpha}{r}\right)^{-1} \tag{16}$$

and by applying *GR* to satisfy Einstein's vacuum field equations, since the functions $A$ and $B$ are reciprocal to each other, we have $B = (1 + \alpha/r)$, and in the metric we have to replace $\tilde{r}^2$ by $(r + \alpha)^2$. All these functions then remain regular for all values of $r$ ($\infty > r > 0$), $A$ and $B$ only becoming zero or infinity, respectively, at the position of the point mass itself. There is therefore no event horizon in this model, and spacetime is completely regular.

With $\tilde{r} = (r + \alpha)$ and $A = r/(r + \alpha)$, we then replace Equation 10 with the following expression:

$$\frac{d\phi}{dr} = \frac{1}{\sqrt{\frac{(r+\alpha)^4}{p^2} - r(r + \alpha)}} \tag{17}$$

It is fortunate that $\tilde{r}$ and $r$ are linearly related; this means we have $d\tilde{r} = dr$, which simplifies the analysis considerably.

Trajectories are plotted in Fig.3 using the above equation for various values of the parameter $p$. For $p > 2.6$ a ray again escapes to infinity,

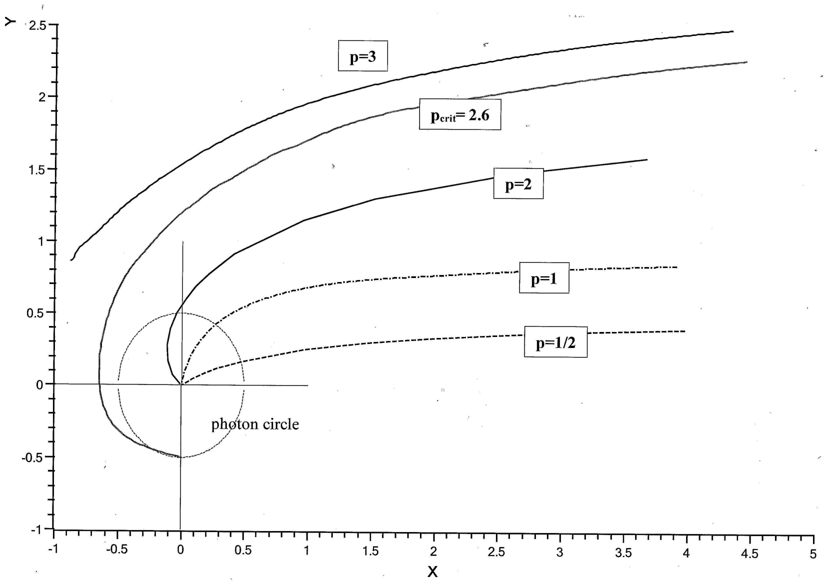


Figure 3: Light trajectories near a point mass (this solution - no event horizon) for different values of the impact parameter $p$.

and below this value it is captured, but the overshoot and spiralling is less pronounced than for the black-hole solution. The critical impact parameter for a photon sphere remains at $p = 2.6$, but now, the radius of the photon sphere is at a distance $\alpha/2$ from the point mass and there is no event horizon.

# 5 Discussion

The differences between the two scenarios examined here may not seem particularly important, but there is a significant difference in principle between them. As I have already explained, the Schwarzschild radial coordinate $\tilde{r}$ used to depict the trajectories in Figure 2 is not a true representation of the radial coordinate distance when $r$ is small and of the order of $\alpha = 2GM/c^2$. In my interpretation, the region $\tilde{r} < \alpha$ is physically prohibited, i.e. the region inside a sphere of radius $\tilde{r} = \alpha$ does not exist as part of the spacetime manifold. The trajectories in Fig.3 thus give a truer representation to a distant observer of how light would be curved near a point mass.

The photon sphere is an important concept, because it is known (see, e.g. Koga *et al*) as the geometrical structure that shapes a black hole shadow; the radius is a threshold for photons coming from distant light sources to escape to infinity or fall into the black hole; the photon sphere accumulates photons along this radius, and the observer sees a very bright shadow edge corresponding to the photon sphere as actually observed by the EHT [12].

In my work, there is no black hole and event horizon, but there is a photon sphere. To obtain an image similar to that published by the EHT does not necessarily imply a black hole is present at the centre. It just means there is a compact mass whose photon sphere lies outside its physical radius. The point mass in my calculations would indeed be essentially invisible due to complete gravitational redshift of any light it emitted itself. The bright ring in the image would then be due to light emitted from particles localised in the photon sphere that have been energised by the influx of radiation there. i.e. photons and relativistic particles accumulating in the photon sphere will be scattered and absorbed by gaseous matter there, which then re-emits radiation in all directions.

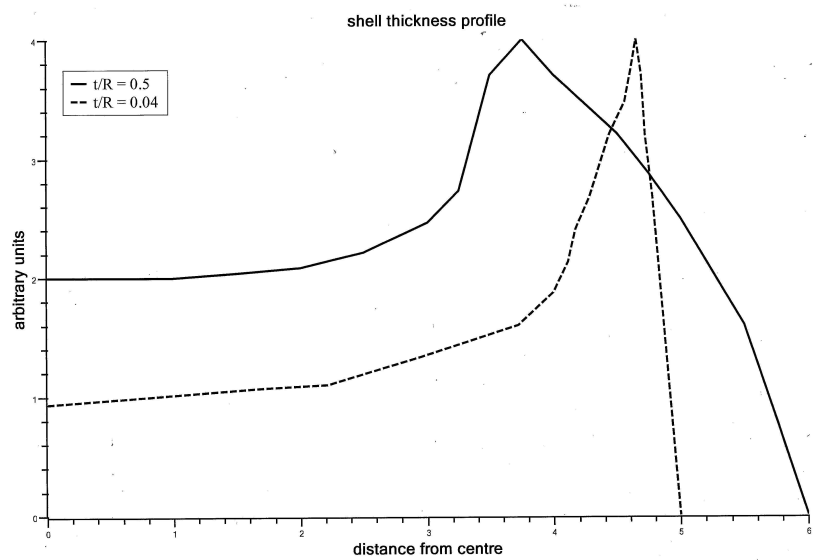


Figure 4: Schematic of light intensity from a spherical shell light source for two different thickness to radius ratios.

The next diagram shows schematically how a spherical shell as a light source might appear to a distant observer. The thinner the shell, the sharper is the contrast between the peak intensity and the central intensity; however the intensity is overall lower for a thin shell. Thus, it is quite easy to see that a thin spherical shell acting as a source of light would appear to an observer as a circular image, rather than a uniform disk of light. Obviously, this is just a simple suggestion to explain a very interesting and complex phenomenon.

## References


[1] A.Einstein, Sitzungsberichte der Preussischen Akademie der Wissenschaften zu Berlin: 778, (11 November 1915)

[2] en.wikipedia.org *Eddington experiment*

[3] en.wikipedia.org *Gravitational lensing*

[4] C.Schiller, *Motion Mountain, The Adventure of Physics, Relativity, p.148*

[5] C.W.Misner, K.S.Thorne, J.A.Wheeler, *Gravitation*, W.H.Freeman (1973)

[6] R.P.Kerr, Phys.Rev.Lett. 11,237(1963)

[7] wolframalpha.com

[8] A.J.Owen, *The black hole fallacy*, viXra: 2211.0007, October 2022

[9] A.J.Owen, *Newton's Law of Gravitation, GR, SR and the Correspondence Principle*, viXra: 2409.0030, September 2024

[10] https://en.wikipedia.org/wiki/*Photon sphere*

[11] Y.Koga, N.Asaki, M.kimura, K.Okabayashi, *Dynamical photon sphere and time evolving shadow around black holes with temporal accretion*, arXiv:2202.00201, March 2022

[12] *First M87 Event Horizon Telescope Results, I, The Shadow of the Supermassive Black Hole*, The Astronomical Journal Letters, 875,Li (April 2019)