Analytical Exploration and Extension of the Function $$f(x) = \int_{-\infty}^{+\infty} e^{(-x)^{|u|}} \, du$$

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Abstract

This paper investigates the function $f(x) = \int_{-\infty}^{+\infty} e^{(-x)^{|u|}} \, du$, focusing on its analytical expression and extension over the real number domain. We employ techniques analogous to the analytic continuation of the Gamma function to extend $f(x)$ beyond its initial domain, addressing convergence issues and exploring its properties across the entire real line.

Contents

1 Introduction

The function

$$f(x) = \int_{-\infty}^{+\infty} e^{(-x)^{|u|}} \, du$$

arises in various contexts within mathematical analysis, particularly in the study of special functions and integral transforms. The goal of this paper is to derive an explicit analytical expression for $f(x)$ and extend its definition to the entire real number domain, even where the integral diverges, using methods similar to the analytic continuation of the Gamma function.

2 Preliminary Analysis

2.1 Symmetry of the Integrand

The integrand $e^{(-x)^{|u|}}$ is an even function of u due to the absolute value in the exponent. Thus, we can simplify the integral:

$$f(x) = 2 \int_{0}^{\infty} e^{(-x)^{u}} \, du. \tag{1}$$

2.2 Variable Transformation

To facilitate integration, we perform the substitution:

$$t = x^u, \tag{2}$$

$$\ln t = u \ln x, \tag{3}$$

$$u = \frac{\ln t}{\ln x}, \tag{4}$$

$$du = \frac{dt}{t \ln x}. \tag{5}$$

Substituting back into equation (1):

$$f(x) = \frac{2}{\ln x} \int_{x^0}^{x^\infty} \frac{e^{-t}}{t} \, dt. \tag{6}$$

3 Evaluation of the Integral

3.1 Case: $x > 1$

For $x > 1$, the limits of integration become:

$$x^0 = 1, \quad x^\infty = \infty.$$

Thus, equation (6) simplifies to:

$$f(x) = \frac{2}{\ln x} \int_1^\infty \frac{e^{-t}}{t}\, dt = \frac{2E_1(1)}{\ln x}, \tag{7}$$

where $E_1(1)$ denotes the exponential integral:

$$E_1(1) = \int_1^\infty \frac{e^{-t}}{t}\, dt.$$

3.2 Case: $0 < x < 1$

For $0 < x < 1$, the limits become:

$$x^0 = 1, \quad x^\infty = 0^+.$$

The integral becomes:

$$f(x) = -\frac{2}{|\ln x|} \int_0^1 \frac{e^{-t}}{t}\, dt. \tag{8}$$

However, $\int_0^1 \frac{e^{-t}}{t}\, dt$ diverges due to the singularity at $t = 0$. To address this, we consider analytic continuation.

4 Analytic Continuation

4.1 Extension to $x \in (0, \infty)$

Although the integral diverges for $0 < x < 1$, the expression obtained for $x > 1$ in equation (7) can be analytically continued. We define $f(x)$ for $x \in (0, \infty)$ as:

$$f(x) = \frac{2E_1(1)}{\ln x}, \quad x > 0, \quad x \neq 1. \tag{9}$$

This function is well-defined except at $x = 1$, where $\ln x = 0$.

4.2 Behavior Near $x = 1$

As $x \to 1$, $\ln x \to 0$, and $f(x)$ exhibits a singularity:

$$\lim_{x \to 1^+} f(x) = +\infty, \quad \lim_{x \to 1^-} f(x) = -\infty.$$

This indicates a simple pole at $x = 1$.

5 Extension to Negative Real Numbers

5.1 Complex Logarithm

For $x < 0$, we consider the complex logarithm:

$$\ln x = \ln |x| + i\pi.$$

Thus, we extend $f(x)$ to $x < 0$ by:

$$f(x) = \frac{2E_1(1)}{\ln |x| + i\pi}, \quad x < 0. \tag{10}$$

Here, $f(x)$ is a complex-valued function.

5.2 Properties of $f(x)$ for $x < 0$

We can express $f(x)$ in terms of its real and imaginary parts:

$$f(x) = \frac{2E_1(1)[\ln |x| - i\pi]}{(\ln |x|)^2 + \pi^2}. \tag{11}$$

As $x \to 0^-$ or $x \to -\infty$, $f(x)$ approaches zero.

6 Discussion

6.1 Analogy with the Gamma Function

The method used parallels the analytic continuation of the Gamma function, which extends beyond its integral representation:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt, \quad \mathrm{Re}(z) > 0.$$

Similarly, we have extended $f(x)$ to $x \in \mathbb{R} \setminus \{0, 1\}$.

6.2 Implications and Applications

The analytic continuation of $f(x)$ allows us to explore its properties across the entire real line, providing insights into its behavior and potential applications in mathematical physics and complex analysis.

7 Conclusion

We have derived an explicit expression for $f(x)$ and extended its definition to all real numbers excluding $x = 0$ and $x = 1$. This extension reveals interesting properties, including a simple pole at $x = 1$ and complex values for $x < 0$. The techniques employed demonstrate the power of analytic continuation in extending the domain of functions beyond their initial definitions.

A Derivation of the Exponential Integral $E_1(1)$

The exponential integral $E_1(1)$ is defined as:

$$E_1(1) = \int_1^\infty \frac{e^{-t}}{t}\, dt.$$

This integral converges and can be evaluated numerically:

$$E_1(1) \approx 0.219383934.$$

Acknowledgments

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References

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