

Relativity predicts a variable G

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It is demonstrated that General Relativity strictly predicts an environment-dependent, variable gravitational constant G . Starting from the spacetime structure of a single particle in an otherwise empty universe—which yields pervasive singularities—we extend the formulation to multi-particle systems and eventually to arbitrary, discrete sets of dimensionless particles representing realistic energy distributions. By applying the interchange of limits theorem, some tailor-made sequences of energy distributions are rigorously linked to these physical energy distributions, while the pervasive singularities naturally vanish for systems of three or more particles. This construction reveals that the dynamical evolution of local rest frames is driven by an inverted Green's function acting on particle four-momenta. While the foundational principles of Einstein's field equations remain fully intact, the standard solar-system constant G must be replaced by a local effective coupling that is suppressed in high-density environments and enhanced in low-density regimes—a phenomenon designated as the surrounding effect [1, 2]. From straightforward physical reasoning, an explicit dynamical equation for G is derived, naturally reproducing galactic rotation curves, MOND dynamics, and the Tully–Fisher relation without invoking ad hoc dark matter. This revised formulation of General Relativity establishes the foundation of Surrounding Matter Theory, offering a unified framework that resolves major observational anomalies across astrophysics and cosmology. Furthermore, applying this mechanism to quantum gauge theories provides the key mechanism to suppress field divergences and resolve the Yang-Mills Mass Gap problem [29].

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1. Introduction

The primary objective of this work is to establish that General Relativity strictly predicts an environment-dependent, variable gravitational constant G , and to demonstrate how this foundational revision provides the mathematical key to resolving the Yang-Mills Mass Gap Millennium problem [3,4].

This study extends and completes previous foundational works [1, 2]. While earlier formulations demonstrated that specific physical conditions—such as the relativistic dynamics of subatomic constituents—imply a non-rigid gravitational coupling, the present work proves that the spacetime variation of G —governed by the past light cone centered at each spacetime point—is an inescapable, first-principles theoretical prediction of relativity itself, independent of any external phenomenological inputs or ad hoc assumptions regarding energy distributions.

The mathematical proof proceeds constructively, starting from the spacetime geometry of a single dimensionless particle embedded in an otherwise empty universe—a configuration that generates pervasive singularities. We then extend the derivation to multi-particle systems and ultimately to arbitrary, discrete sets of dimensionless particles representing realistic physical energy distributions. By applying the interchange of limits theorem, some tailor-made sequences of energy distributions are rigorously linked to these physical energy distributions. Crucially, this construction demonstrates that while a single-particle universe remains everywhere singular, these singularities naturally vanish for systems containing three or more particles.

This foundational derivation is driven by a hierarchy of physical and mathematical motivations:

- 1) Resolving the Yang-Mills Mass Gap problem and curing quantum field divergences.
- 2) Addressing major observational and conceptual challenges in astrophysics and cosmology.
- 3) Uncovering the fundamental link between local energy distributions and the first derivatives of the metric tensor.
- 4) Resolving long-standing conceptual caveats within standard General Relativity [2].
- 5) Establishing a rigorous, first-principles theoretical reconstruction of Newton's gravitational law and Einstein's field equations.
- 6) Replacing the anthropocentric solar-system constant G with an environment-driven field quantity.
- 7) Retrieving the microscopic physical information lost during the spatial coarse-graining of the stress-energy tensor $T_{\mu\nu}$.
- 8) Formulating the dynamical evolution of local rest frames across spacetime.

Central to this approach is the systematic generalization of local rest frames into the Generalized Rest Frame (GRF) framework. This reveals that Special Relativity is not merely an asymptotic algebraic constraint on General Relativity, but an active description

of how local energy directly determines spacetime deformation. Mathematically, the dynamical evolution of these rest frames is driven by an inverted Green's function acting on particle four-momenta, establishing that local physics is intrinsically coupled to the global distribution of surrounding matter.

This mechanism formalizes the surrounding effect [1]: gravitational coupling is systematically suppressed in high energy-density environments and enhanced in low-density regimes. From straightforward physical reasoning, an explicit simplified equation for G naturally emerges, yielding the $1/r$ spatial behavior required for galactic velocity profiles. A slight refinement of this relation directly recovers empirical MOND dynamics and the a_0 acceleration threshold from first principles. Building upon this surrounding effect, Surrounding Matter Theory establishes a unified framework that addresses and resolves the major observational and conceptual challenges across astrophysics and cosmology without requiring ad hoc dark matter or dark energy, while simultaneously providing the mass-generation mechanism and the suppression of quantum field divergences in pure gauge theories [29].

The remainder of this document is structured as a rigorous mathematical proof. To maintain clarity and readability, extensive technical proofs are compiled in the Appendix, and key conceptual definitions are provided in the preliminary glossary.

2. Glossary

Energy distribution: In this work, energy distributions are defined on a shared bounded domain and vanish everywhere except within a bounded subset of spacetime. They are modeled as finite sets of non-zero mass, dimensionless particles. No spacetime event may contain more than one dimensionless particle. The physical and mathematical justifications for these restrictions are addressed throughout the document.

Frame in which time elapses the most: A didactic concept originating from the resolution of the twin paradox (wherein the stay-at-home twin experiences maximal proper time elapsed relative to the traveling twin). In this document, this notion serves as an intuitive transition toward defining the Generalized Rest Frame (GRF).

GRF: Generalized Rest Frame. The central concept introduced in this extension of relativity theory. Initially introduced via the frame of maximal elapsed proper time, it is formally defined as the generalization of the standard rest frame.

Matter at rest: Matter that remains stationary relative to a specified inertial frame.

Matter in motion: Matter moving at the speed of light c . This restriction ensures that such motion remains non-zero across all inertial frames.

Rest frame: A reference frame in which a given particle remains at rest. Within this framework, which considers solely gravitational interactions among sets of dimensionless particles (neglecting non-gravitational forces or treating them as gravitational manifestations), a standard relativistic rest frame can always be associated with a co-moving test particle.

S: The functional mapping an energy distribution to its corresponding spacetime geometric structure (see Appendix A for further mathematical details).

Test particle: An idealized, dimensionless particle of negligible mass in free fall that exerts no back-reaction on the ambient spacetime geometry.

Thought experiment: Terminology retained primarily for historical continuity. Within the rigorous and self-consistent framework of general relativity, a thought experiment corresponds directly to a formal mathematical demonstration.

3. Demonstration

3.1. Principle of the demonstration

The principle of the demonstration is based on a remark about a dimensionless particle P . For such a particle, two extremely different assumptions can be made about the nature of its energy:

- Assumption (1): Its energy is made of matter at rest. This means there is an inertial frame in which the content of this particle is a solid block of matter at rest.
- Assumption (2): Its energy is made of matter in motion. This means the internal matter of P is always in motion, regardless of the chosen inertial frame. Hence, the speed of this matter is the speed of light. The simplest model is to consider P made of a set of dimensionless bunches of matter in a Brownian distribution, each moving at the speed of light.

The first assumption appears natural and obvious. The second assumption might seem unusual because P is dimensionless and composed of a set of bunches of matter. However, this assumption is as legitimate as the first, where P is dimensionless and composed of a solid block of matter at rest. Therefore, one can state that the second assumption is always true. Hence, one can state that matter is always internally made of infinitely small bunches of matter always moving at the speed of light along infinitely small closed trajectories in a Brownian motion and distribution.

The bunches of matter in the second assumption are unrelated to the particles of particle physics, even though reality in this field has proven to be more compliant with this assumption.

Regardless of the assumption chosen, the result is always a dimensionless particle P . Thus, the generated spacetime structure will be the same. However, as will be seen later, the calculation of the spacetime structure will be entirely different.

From this remark, the principle is to calculate this deformation under the second assumption.

3.2. S function and Context

Let S be the function giving spacetime structure from energy distribution. The first constraint for the domain of S is that it is assumed to be the set of energy distributions following the rule of energy conservation. Therefore, for each of them, the following is true:

$$\nabla_\mu T^{\mu 0} = 0 \tag{1}$$

$T^{\mu\nu}$ is the stress energy tensor. ∇_μ is the partial covariant derivative with respect to the μ dimension. This equation is simpler for energy distributions composed of a set of dimensionless particles: it represents the conservation of the number of particles and their energy at rest.

For ease of demonstration, a second constraint is added: the energy distributions are restricted to sets of dimensionless particles. Apart from these two constraints, there are no further constraints for the domain of the S function.

In addition, both energy distributions and their associated spacetime deformations are assumed to be free of pathological mathematical irregularities (such as non-differentiable discontinuities or unmanageable singularities).

The context described in this paragraph applies only to the calculation of the spacetime deformation generated by P under the second assumption. This will be done from paragraph 3.5 to 3.8. Outside of these paragraphs, this context will be withdrawn. The new context will be indicated when needed.

3.3. Mathematical reminder: interchange of limits theorems

Norms are equivalent in finite dimensions, hence this is true for the four-dimensional spacetime of GR. By other means, spacetime structure is a function of energy distribution. In the present document, the considered energy distributions will be assumed to share the same bounded domain. This will allow the demonstration of the continuity of S . This restriction is allowed since the size of this bounded domain can be huge, for example, it can contain the observable universe.

Under the previous restriction, S is a continuous function using the uniform norm for the two involved spaces. Indeed, if any amount of energy at any spacetime location is decreased to 0, then the effect of this amount of energy on spacetime structure decreases also to 0. This continuity is the result of the conservation of energy principle. The detailed proof is given in Appendix A.

From this continuity of S , the interchange of limits theorem is valid. If any f_n sequence of energy distributions over spacetime tends uniformly to some limit energy distribution, then the following equation arises.

$$S\left(\lim_{n \rightarrow +\infty} (f_n)\right) = \lim_{n \rightarrow +\infty} (S(f_n)) \quad (2)$$

In equation (2), $\lim_u$ means the uniform limit, for energy distributions, and for spacetime structures.

The continuity of spacetime structure as a function of energy distribution leads to another notable consequence. It is possible to formulate thought experiments in which the energy is progressively increased or decreased, producing a continuous modification of the spacetime structure—all while strictly preserving the principle of conservation of energy. Indeed, General Relativity constitutes a mathematical framework that can, and at times must, be evaluated independently of physical reality. Within this framework, one can consistently model alternative or hypothetical universes.

Regarding the term "thought experiment", its usage here is maintained primarily for historical continuity. Within the established and self-consistent mathematical structure of relativity, such an experiment corresponds simply to a rigorous mathematical demonstration.

3.4. A generalized rest frame in relativity

3.4.1. Definition

In relativity, rest frames exist (see glossary, paragraph 2). This concept can be generalized and is then referred to as a generalized rest frame (GRF). This generalization is achieved by evaluating the metric between rest frames. In vacuum, GRFs can be constructed by iteratively applying a process involving small displacements. This construction is detailed in Appendix B.

3.4.2. Old and new GRFs

Thus, for any particle located at a given event x , a GRF exists at x . Let us denote the "former GRF" as the GRF "just before the particle," which then transforms into the "new GRF," i.e., the one "just after the particle." Alternatively, the former GRF locally describes spacetime without the presence of the particle, while the new GRF describes it just after the particle's appearance. In other words, the identification of the new GRF from the old one can be carried out progressively, using the continuity of the function that relates the spacetime structure to the energy distribution. To illustrate this, consider the following scenario — not realistic, but valid as a thought experiment as previously stated: the energy of the particle is increased gradually from 0 to its actual value.

3.4.3. Behavior

The nature of these GRFs depends on the existing energy distribution. However, transitioning from one energy distribution to another by introducing a dimensionless particle can be analyzed. Suppose a dimensionless particle is inserted into the universe. The former GRF refers to the frame before this insertion, and the new GRF refers to the frame after it. The boost associated with the motion of this new matter in the former GRF describes the evolution from the former GRF to the new one and allows the calculation of the new GRF. More details and demonstrations are provided in Appendix B.

3.4.4. Rescaling

Additionally, a rescaling of time and space units occurs after the boost. This will be discussed in more detail later. However, in the present document, we will, somewhat abusively, refer to the local deformation simply as a boost. The context will determine whether a rescaling also occurs. Further discussion on this can be found in Appendix B and Appendix C.

3.4.5. Conclusion

This GRF concept and its behavior are local and involve only the first derivative of the metric. From this, the second derivative can be inferred, enabling a comparison with Einstein's equation.

3.4.6. Examples

Consider a simple example: a particle P moving along a straight line in a static universe filled with uniform matter density. In the absence of P , this universe is flat and Minkowskian. The old system of GRFs, denoted $R_{0 \text{ system}}$, corresponds to a constant R_0 frame in which the universe is at rest. This is traditionally referred to as the "frame of fixed stars." The new system, $R'_{0 \text{ system}}$, is the one that actually exists and results from the presence of P .

This scenario can be refined by assuming that P appears progressively in space and time. In this case, a continuous set of GRFs is constructed progressively from $R_{0 \text{ system}}$ to $R'_{0 \text{ system}}$.

In this example, the spacetime deformation caused by P — that is, by its very existence — is locally described by the boost associated with the velocity of P in its former GRF. This boost yields the new GRF from the old one.

For a GW, the same definition applies: the "former GRF" is the one that would exist if the GW and the matter generating it were absent. The "new GRF" accounts for their presence.

3.5. Fundamental assumption

From now on in the present document, it will be assumed the following assumption.

Assumption (I): a GW propagates at the speed of light.

The relevance of this assumption (I) is well described in the literature [5]. This is for the experimental data. Also a theoretical argument is made by the usual study of a GW using the linearized Einstein equation.

3.6. Spacetime structure around a particle moving at the speed of light

We now aim to describe the local spacetime deformation of a GW generated by a particle moving at the speed of light. We adopt the following setup: a particle P propagates at speed c along a straight trajectory D in an otherwise empty Minkowski spacetime, a property that remains invariant in any inertial frame R .

The spacetime deformations of the GW will be studied at the locations where the amplitude of the deformation is maximal.

Since the deformation under consideration is purely local, it transforms the former GRF into the new GRF. Consequently, this local deformation is described by a linear transformation.

Let us analyze the envelope of this GW. Owing to the spatial rotational symmetry of the energy distribution around D , the generated spacetime geometry inherits the same ax-

isymmetry. Therefore, in a frame R at any given instant, the region of maximal deformation forms a spatial right circular cone centered on D , whose apex angle may vary depending on R . Indeed, changing the reference frame modifies the apparent energy of P , resulting in a different cone geometry (wider or narrower).

Appendix H provides full details of this deformation for the special case where the envelope forms an equilateral cone.

3.7. *Spacetime structure around a particle moving along a circle at the speed of light*

3.7.1. *Objective and Model*

The next step is to calculate the spacetime deformation generated by a particle at rest, under assumption (2) of Section 3.1. For modeling this particle, the best model would be a Brownian motion of a huge set of bunches of matter moving at the speed of light. But a simpler model will be used. It's a bunch of matter moving along a circle at the speed of light. The relevance of those models is discussed in Appendix K.

3.7.2. *Spacetime deformation generated by the particle moving along a circle*

Therefore, it is assumed that the previous P particle is forced to move along a circle. Of course, using the word "particle" here is an abuse of language. Indeed, this identifies a bunch of matter whose existence is only a hypothesis and this is assumed only for theoretical purposes. But this abuse allows for easier reading.

Then the previous cone transforms into a more complicated geometric figure. But infinitely far from the circle, therefore asymptotically, the envelope of the GW propagation at constant time is a sequence of spheres centered at O , the center of the circle. Asymptotically, these spheres inflate themselves around O at the speed of light. The radius of these spheres are separated by the same constant value d , which is the circle's circumference. Now the deformation that is propagated asymptotically is described by a boost associated with the speed of light. This speed is the vector that is normal to the propagation sphere, in the direction that goes out of the sphere. Also, this deformation propagates with this speed. This is proven in Appendix J.

3.7.3. *When the circle's radius tends to 0*

In the mathematical framework of relativity, if the radius of the circle decreases progressively and tends to zero, it means in reality that the circle tends to its limit, which is a dimensionless particle located in O . Then the envelope of the GW transforms itself into the previous sequence of spheres, but also d tends to O . The final result is a sequence of spheres which are infinitely close to each other, all centered on O .

The interchange of limits theorem can be applied to this Dirac distribution of matter, namely here, the dimensionless particle located in O . Let's denote P' as this particle. This can be applied because the circle-like trajectories of microscopic bunches of matter

represent energy distributions that tend toward this Dirac distribution. If any doubt exists because P is in a rotating motion around O , then it is possible to add another bunch of matter sharing the same energy and moving along the same circle-like trajectory in the opposite direction. This would cancel the whole rotating motion without changing the final result, which is the following. Finally, this distribution tends to the Dirac distribution centered in O , when d tends to 0.

$$\lim_{n \rightarrow +\infty} (S_n) = m\delta \quad (3)$$

S_n is this energy distribution made of one circle-like trajectory of one bunch of matter like P , in an empty universe, this circle being centered in O . m is the mass of P' , which is assumed to be located in O , the center of the δ Dirac distribution. Equation (3) tells that S_n tends to the Dirac distribution of P' since the radius of the S_n circle tends to O .

3.7.4. Spacetime deformation generated by a particle at rest

Therefore, let's apply the interchange of limit theorem. The spacetime structure generated by a 3D Dirac distribution centered in O is the limit of the previously described sequence of spheres, limit when d tends to 0, as written in the following equation.

$$S(m\delta) = S\left(\lim_{n \rightarrow +\infty} (S_n)\right) = \lim_{n \rightarrow +\infty} S(S_n) \quad (4)$$

Those limit deformations are described in any M space point by the boost associated with the speed of light which is oriented along OM . Let's write R , a rest frame attached to P' (in which P' is at rest). M belongs to a sphere centered in O which contains all the deformations arriving at the same time in R . Moreover, these spheres are now infinitely close to each other. It means that each spacetime event of the universe is modified by this singular boost. This is the spacetime structure generated by the 3D Dirac distribution.

Hence, this result differs radically from the current literature. Let's remind that with today's literature the spacetime deformation here is the one which corresponds exactly to Newton's law occurring in the solar system. But what has been proven here is that this deformation is a singularity everywhere.

This huge difference will explain why G is not a constant but a variable.

3.7.5. Avoiding the unknown forces

P was forced to follow the circle-like trajectory which is not a geodesic. Therefore, there is an unknown force which allows this trajectory not to follow a geodesic. This force is not gravity. Therefore, at first glance, modeling the Dirac distribution under assumption (2) of Section 3.1 would not be possible in a coherent way and implying only gravitation. But this apparent result is wrong; the final result is quite the opposite, as shown in Appendix K. Interestingly, the result here is a very important result. The result is that in relativity, assumption (1) is impossible and must be replaced.

3.8. Partial resolution of the issue of Mach's principle

This resulting spacetime structure might be argued to be wrong because it is not realistic. But the correct argument is the opposite. This description appears to be more accurate than the one given in the literature because the distribution was initially assumed to be unrealistic. Indeed, a Dirac distribution of matter is inherently unrealistic since it assumes an empty universe outside of the center of the Dirac distribution. For that reason, only an unrealistic result should be expected.

Moreover, this GR prediction is in perfect agreement with Mach's principle [6]. Let's briefly recall one aspect of the problem with Mach's principle in GR. For example, the issue arises in the case of a static spherically symmetric universe. A particle is located at the center of this spherical symmetry. If ρ is the matter density filling the universe, then one can distinguish two assumptions. The first is $\rho > 0$, and the second is $\rho = 0$. Close to the particle, ρ appears insignificant in both cases. Therefore, in this region, the spacetime deformations will be approximately the same for the two assumptions. However, in the first assumption, it is possible to find an inertial frame, R , at rest with the particle, which is not in rotation with respect to the universe. In R , there are no fictitious forces such as centrifugal forces. But in the second assumption, it is not possible to find such a frame. Assuming that R is at rest with respect to the object is not enough. It is not possible to determine if R remains inertial or not. One cannot say if fictitious forces will appear in R .

The new GR prediction solves this problem. Now, the spacetime structure becomes singular everywhere for the second case. An answer is provided: in each frame at rest with the particle, no fictitious force could ever appear, since those singularities would dissolve them completely. Therefore, with this new, correctly unrealistic prediction, GR becomes more Machian.

3.9. Two particles

Now let's add another particle, apart from the first one. So there are two particles, at different locations, and the universe outside them is still empty. Everything is still assumed to be static, meaning that the two particles are at rest in some given inertial frame.

There are still singularities, but they are located only along the straight line containing the particles' locations, and only at the points that are not between the particles' locations. They are described by the boost associated with the speed of light in the direction moving away from the particles.

Outside of the particles and those singularities, the spacetime structure is determined by the conservation of the GR Lagrangian in vacuum. It is simpler to say that the Ricci tensor is null in vacuum, as shown by the following equation.

$$R_{\mu}^{\nu}(g) = 0 \tag{5}$$

where $R_{\mu}^{\nu}(g)$ represents the Ricci tensor operator applied to the metric tensor g . This is a second-degree differential equation. An integration constant is still required at the end of the calculation. Under today's version of GR, the G solar system constant value and the

mass of each particle are used for this. Now, the determination of this integration constant remains to be given. Hints and clues for this determination are provided in Appendix D. Although this determination would be an improvement, it is not mandatory for the study of the present document.

Also, asymptotically, the spacetime deformation is singular. Indeed, this asymptotic deformation is the same as the previous asymptotic deformation of the distribution with only one particle. And of course, each particle still generates locally a spacetime singularity. If the masses of the particles are not equal, a difficult calculation is required. Let's suppose that they are equal. Then, the calculation is simpler since there is perpendicular symmetry with respect to the mediating plane between the particles. This picture is still vastly different from what is written in today's literature.

3.10. Three or more particles

Adding a third particle to the system dissolves the singularities that were previously located along the straight line connecting the two initial particles. Here, we explicitly assume that the positions of the three particles are not strictly collinear (a standard non-degeneracy condition). As usual, the third particle is assumed to be at rest with respect to the others.

With three or more particles, the spacetime structure remains determined by Equation (R.1). Under the assumption that the rest masses of the particles are equal, symmetry considerations can be exploited to compute the resulting spacetime geometry. The integration constants are determined using the same reasoning as in Section 3.9.

3.11. Spacetime structure for any distribution of energy

As stated in Section 3.2, the studied distribution of energy is a realistic one: a set of dimensionless particles. Indeed, modeling reality in this way is often a good approximation in astrophysics, from planets to cosmology scales, and in particle physics, because of the sparse nature of matter. Here the wave nature of particles of particle's physics is simply ignored since only gravity is studied at first. Of course, a more general distribution of energy might be studied. Notably, a uniformly continuous distribution of energy could still allow the use of the interchange of limits theorem. But this is outside the scope of the present document.

Hence, let's apply the interchange of limits theorem to this set of Dirac distributions of matter, namely here, the dimensionless particles of the universe. Let's recall that for each P_i particle, where i is the particle's number, i from 0 to infinity, there is a S_n^i sequence of energy distributions. For each i and n , S_n^i is a distribution made of a circle-like trajectory of a virtual microscopic bunch of matter moving at the speed of light in an empty universe, this circle being centered on P_i . For each i , i from 0 to infinity, the circle's radius of S_n^i tends to 0, and S_n^i tends to the Dirac distribution of the P_i particle, as n tends to infinity.

$$\lim_{n \rightarrow +\infty} (S_n^i) = m_i \delta_i \quad (6)$$

In equation (6), δ_i is the Dirac space distribution centered on the P_i particle, and m_i is the mass of P_i . Although $\lim_u$ represents uniform convergence, from the perspective of the set of P_i , this is simple convergence. This simple convergence for each P_i can be transformed into uniform convergence for the entire set of P_i . To achieve this, it suffices to choose, for each n , the same S_n^i circle's radius for any i . For example, the value of $1/(n+2)$ can be chosen for the circle's radius of S_{n+1}^i , in the system of GRFs of spacetime structure generated by S_n^i .

$$S\left(\sum_{i=1}^N m_i \delta_i\right) = S\left(\sum_{i=1}^N \lim_{n \rightarrow +\infty} (S_n^i)\right) = \lim_{n \rightarrow +\infty} S\left(\sum_{i=1}^N (S_n^i)\right) \quad (7)$$

Equation (6) has been used, as well as the interchange of limits theorem. N is the number of particles in the universe. Of course an infinite number of such particles is not required in the study. The finite N number of the visible particles of the universe is enough. Hence we are left with the spacetime structure given by the final term $\lim_{n \rightarrow +\infty} S\left(\sum_{i=1}^N (S_n^i)\right)$.

Each S_n^i amount of matter is moving at the speed of light. Asymptotically it has been shown that each of them propagates a GW which is described locally by the boost associated with the motion of the envelope of this GW, which propagates at the speed of light.

Furthermore, for each particle, asymptotically close to it, the gravitational effect of the particle is infinitely greater than the effects of the other particles. Therefore asymptotically close to the particle, the arising context is exactly the same as studied in paragraph 3.7.4. Consequently, each of the previous result applies: the shape of the GW envelope, which is generated by the bunches of matter which are in motion inside the particle, is a sphere and its ray increases at the speed of light. And the GW spacetime deformation is described by the boost associated with the speed of light of the ray of this envelope. Asymptotically when the ray of the circle of the trajectory of the bunch of matter tends to zero, and, after that, asymptotically close to the resulting particle, the whole spacetime close to it receives the GW. In other words, in the infinitely small solid sphere centered in the location of the particle, every space location receives the GW, at any time.

Now far from the particle, this description remains valid. Indeed, the previous demonstration still holds if the old spacetime structure (the spacetime structure before the particle) is not assumed to be flat Minkowskian. The reasoning was working because of the vacuum around the particle and because the particle is dimensionless. But whatever is the nature of former GRFs of the particle, it is possible to consider a system of frames in which the GW propagates along (geodesics which are) straight lines containing the location of the particle. By the way the former GRFs is such a system of frames (as stated in Section 3.2 the possible mathematical global irregularities has been assumed not to happen here, for example such as closed loops, or singularities. It means that the context is assumed to be regular enough in order to allow the following sentences). Even if the Riemann tensor does not describe a Minkowskian flat spacetime, those straight lines implies that the envelope is a sphere in this system of frames, and then the previous reasoning remains correct. Hence all the previous results hold, asymptotically, the GW propagates at the speed of light, is described by the boost associated with this speed, and each event of space-time receives

the GW.

$$\left\{ B\left(\sum_{i=1}^N 1_w(x, y_i) f(x, y_i) C^\mu(y_i)\right), x \in U \right\} \rightarrow S\left(\sum_{i=1}^N m_i \delta_i\right) \quad (8)$$

Relation (8) describes exactly that. Here, B is a function giving the boost associated with a four-momentum, and y_i is the position of the P_i particle. $C^\mu(y_i)$ is the drastic boost (associated with a null four-velocity) describing the local spacetime deformation of the GW generated by the virtual particle modeling P_i . The variable x denotes the generic event describing the set on the LHS of Eq. (8). The $1_w(x, y_i)$ and $f(x, y_i)$ terms are related to the propagation of this deformation and will be detailed further below during the explanation of the subsequent, similar equation. Here, U denotes the entire universe, highlighting the purely formal nature of this relation.

An interesting ambiguity regarding the identification of the former system of GRFs arises here. In particular, this stems from the non-Abelian nature of the S function: $S(a, b)$ (meaning inserting a first, then b) is generally not equal to $S(b, a)$, even though static distributions allow this commutativity. Nevertheless, once a system of frames is given, Relation (8) describes a transformation between the $C^\mu(y_i)$ four-momenta of the i particles at y_i and the resulting spacetime structure. In other words, formally, the relation holds for any choice of former GRFs. The result depends on the choice of this system of frame, which is the system of frames which represent space-time structure in the vacuum before the appearance of the particles of the universe. It is interesting to see the role of vacuum arising here since vacuum is a source of particle physics mysteries. Therefore let us formulate this more rigorously. Relation (8) becomes the following.

$$\left(S(\text{vacuum}), \left\{ B\left(\sum_{i=1}^N 1_w(x, y_i) f(x, y_i) C^\mu(y_i)\right), x \in U \right\} \right) \rightarrow S\left(\sum_{i=1}^N m_i \delta_i\right) \quad (9)$$

Now the set of those boosts over spacetime identifies uniquely the new spacetime structure resulting from the existence of the P_i particles.

Of course this relation is purely mathematical. The description of the universe which is given here can avoid the problem of this choice since the result of the present demonstration is only the one which follows.

The final picture is as follows. The spacetime structure is the result of all the microscopic GWs. These microscopic GWs are generated by the virtual microscopic bunches of matter moving at the speed of light and representing the real P_i particles in the above study. From now on in this document, these GWs generated by the microscopic virtual bunches of matter will be called "virtual gravitational waves" (VGW). Considering only the limit distributions ($n \rightarrow +\infty$), the set of S_n^i transforms into the set of $m_i \delta_i$. Thus, each spacetime event receives a VGW from each real particle in the universe. This provides a clue for calculating the spacetime structure in a different way.

3.12. Calculation of spacetime structure

3.12.1. Four-momentum equation

Let us analyze how arbitrary gravitational waves (GWs) superpose at a given spacetime event x . We first derive this property for general GWs, incorporating specific features of VGWs only as needed.

At x , the total four-momentum is the sum of the four-momenta of all GWs passing through that event. This relation directly follows from the principle of conservation of energy (see Appendix M).

The resulting equation, previously introduced in this work and in [2], is expressed as follows:

$$D^\mu(x) = \sum_{n=0}^{\infty} 1_w(x, y_n) f(x, y_n) C^\mu(y_n) \quad (10)$$

Equation (10) shows the calculation of the resulting four-momentum at x . For n ranging from 0 to infinity, each event y_n represents a spacetime location from which a particle P_n potentially emits a GW reaching x . The indicator function $1_w(x, y_n)$ equals 1 if events x and y_n are connected by a future-directed null geodesic from y_n to x , meaning that the GW generated at y_n is received at x , and 0 otherwise. $f(x, y_n)$ is a scalar positive function. It is assumed to be equal to 1 if y_n is equal to x . This allows to retrieve the rule of local spacetime deformation generated by a particle. It expresses the attenuation of the GW energy which is emitted from y_n . The four-vector $C^\mu(y_n)$ encodes the GW energy information at y_n . As established in Appendix B.3, treating particles as dimensionless point sources implies that each energy component $C^0(y_n)$ diverges at $x = y_n$. Alternatively, Eq. (10) can be formulated by assigning a finite physical dimension to particles—for instance, modeling them as uniform spheres of fixed radius, such as the Planck length—which ensures a finite local energy density throughout space.

Although it will be shown later that $C^0(y_n)$ is proportional to the square root of the effective GW energy rather than the energy itself, the terms "four-momentum" and "energy" will be used for $C^\mu(y_n)$ and $C^0(y_n)$ to simplify the narrative. The precise meaning (effective quantity vs. algebraic contribution) will remain clear from the context. Here, "contribution" specifically refers to the $1_w(x, y_n) f(x, y_n) C^\mu(y_n)$ terms on the right-hand side of Eq. (10).

3.12.2. Formal divergence of the four-momentum equation

A key question concerning Eq. (10) is whether the spatial summation can yield a divergent result. For instance, in an infinite universe with a uniform spatial distribution of particles, the sum diverges if the attenuation function $f(x, y_n)$ decays more slowly than $1/r^3$. This behavior is directly analogous to Olbers' paradox [7].

However, several straightforward resolutions present themselves. A first approach is to assume that the number of contributing particles is finite, either because the observable

universe itself is spatially bounded or due to physical mechanisms such as the progressive cancellation of emitted wave energies over distance. Alternatively, a natural resolution arises when applying Eq. (10) within specific gravitational frameworks, such as Surrounding Matter Theory [1], where fitting the model to observational data naturally constrains the summation to a finite value. Consequently, a practical approach at this stage is to set aside generic distributional divergences in Eq. (10) and address them directly within the context of specific gravitational models.

3.12.3. *Other remarks on the four-momentum equation*

Naturally, while selecting a different reference frame alters the coordinate components of the four-momenta in Eq. (10), these quantities transform covariantly as tensors. For instance, any relative motion of a source location y_n within the chosen frame modifies the corresponding component $C^0(y_n)$ via the associated relativistic Doppler factor.

Other remarks can be made regarding the structure of this equation. Although it resembles an energy conservation law, it also acts as a weighted barycentric combination, relating the final four-velocity vector of spacetime at x to the GW four-velocity vectors. The weights in this barycentric combination are given by the total energy of each contribution. This resolves the initial coherence problem of spacetime deformation: a P bunch of matter deforms spacetime globally according to its motion, but if it is composed of A and B components moving differently, then the spacetime deformations due to A and B differ from the one generated by P . The associative property of the barycentric operator solves this inconsistency. This also addresses the question raised by the existence of Willer rotations [8, 9] in the composition of boosts. Furthermore, the equation can be viewed as a Green's function integral representation involving four-vectors. However, the underlying mechanism is non-linear. In addition, the attenuation function f is unknown — yet this apparent obstacle can be overcome. Indeed, in the most common realistic distributions of energy, this function is approximately the same for each y_n event, allowing for both reasoning and simple calculations.

3.12.4. *Calculating the deformation four-momentum via virtual gravitational waves*

We now resume the derivation previously interrupted by the generalization of the fundamental equation, picking up directly from the end of Section 3.11.

This section presents the primary result of this work. Specifically, Eq. (10)—formulated above in a broader mathematical context—is applied directly to virtual gravitational waves (VGWs), as motivated by the conclusions of Section 3.11.

When considering the limiting distributions in Eq. (6), the propagation factor $1_w(x, y_n)$ in Eq. (10) reduces identically to unity for all VGWs and can therefore be omitted. This reduction follows from restricting the framework exclusively to VGWs: each particle situated at y_n emits a unique VGW that intersects any given spacetime event x .

To illustrate the mechanism of calculating the local deformation, let us examine two fundamental configurations and how Eq. (10) applies to each:

- **A single dimensionless particle Q located at x :** $C^0(y_n)$ diverges at $x = y_n$. Consequently, the local spacetime deformation at x is fully determined by Q , in accordance with the rigidity property detailed in Appendix B.3.
- **Multiple co-located dimensionless particles at x :** When several dimensionless particles coincide at the exact same location x , Eq. (10) reduces to the local energy conservation principle at x , as the aforementioned rigidity constraint applies collectively to all co-located particles.

3.12.5. From the four-momentum to the boost

Following the established framework, we maintain the energy distribution defined in Section 3.11. The resulting four-momentum from Eq. (10) takes the form:

$$D^\mu(x) = \gamma \frac{E}{c} \left(1, \frac{v}{c}, 0, 0 \right), \quad (11)$$

where E and v represent the rest energy and the velocity associated with the four-momentum, respectively, and $\gamma = 1/\sqrt{1 - v^2/c^2}$ is the Lorentz factor. Equation (11) is expressed in a global rest frame $R_0(O; ct, x, y, z)$ anchored to the background universe, where the spatial orientation is chosen such that v lies along the x -axis. From $D^\mu(x)$, the generated local spacetime deformation is determined following the prescription in [2], represented in R_0 by the boost matrix:

$$B^\mu{}_\nu(x) = \begin{pmatrix} \gamma & -\gamma \frac{v}{c} & 0 & 0 \\ -\gamma \frac{v}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (12)$$

This boost transformation is directly specified by the velocity vector of the four-momentum in Eq. (11).

From a physical standpoint, in a static universe, the boost reduces identically to the identity matrix in any comoving reference frame. However, as soon as relative motion is present, the dynamic transition between local reference frames generates a non-trivial boost. Under the continuum description of spacetime, this Lorentz boost remains infinitesimally close to the identity transformation, inducing a smooth and continuous modification of the local GRFs across all spacetime events x .

3.12.6. From the boost to the metric

We now derive the spacetime metric from the local boost matrix $B^\mu{}_\nu(x)$.

We maintain the energy distribution assumptions defined in Section 3.11, further assuming a static universe with a constant spatial matter density. In this continuum limit, the discrete particle grid is taken to be sufficiently fine-grained to justify a smooth density approximation.

Let R_0 denote a rest frame tied to the background universe. Owing to the uniform matter distribution, the baseline spacetime structure is globally flat Minkowski space, and the GRFs system of frames is such that each frame is R_0 .

Now, consider the introduction of a particle P situated at origin O at time $t = 0$ and at rest in R_0 . As established, this configuration does not imply a violation of energy conservation; rather, it represents a formal comparison between two distinct spacetime configurations: one without P , and one with P present.

Let x denote a spacetime event corresponding to a spatial point $M \neq O$ at $t = 0$ in R_0 . Prior to the introduction of P , R_0 serves as the local GRF at x . Upon introducing P , the local GRF at M transitions to R'_0 . Thus, relative to the presence of P at point M , R_0 represents the "former GRF" and R'_0 the "new GRF". Let x' designate the event at M (along the timeline of R_0) where R_0 transforms into R'_0 . The new frame R'_0 is obtained by applying the Lorentz boost $B^\mu{}_\nu(x)$ to R_0 at event x .

Furthermore, a continuous sequence of intermediate GRFs connecting R_0 to R'_0 can be constructed by smoothly increasing the energy of particle P from 0 to its full physical value (a valid construction within a relativistic thought experiment).

Finally, a rescaling of the boosted spacetime coordinates is required. The boosted temporal and spatial axes—modified locally by the action of $B^\mu{}_\nu(x)$ —are rescaled such that the continuous timeline traced out by these successive infinitesimal steps forms a spacetime geodesic. This transformation chain is expressed mathematically by mapping the original coordinates X^μ in R_0 to the boosted coordinates X'^ν , and subsequently to the fully rescaled coordinates X''^ρ in R'_0 :

$$X'^\nu(x') = B^\nu{}_\mu(x)X^\mu(x), \quad (13)$$

$$X''^\rho(x') = S^\rho{}_\nu(x')X'^\nu(x), \quad (14)$$

$$g_{\alpha\beta}(x) = B^\rho{}_\alpha(x)S^\mu{}_\rho(x')g_{\mu\nu}(x')S^\nu{}_\kappa(x')B^\kappa{}_\beta(x). \quad (15)$$

Here, $S^\mu{}_\rho(x')$ represents a symmetric linear transformation diagonalizable in R'_0 , whose explicit form is uniquely determined by the geodesic constraint imposed on the sequence of local GRFs. Equations (13)–(15) demonstrate how the updated metric $g_{\mu\nu}(x')$ is derived from the initial metric $g_{\alpha\beta}(x)$ through the action of $B^\nu{}_\mu(x)$, generated by the added energy-momentum density $D^\mu(x)$ of particle P . Inverting Eq. (15) using the inverse transformation tensors $(B^{-1})^\beta{}_\kappa(x)$ and $(S^{-1})^\rho{}_\mu(x')$ yields:

$$g_{\mu\nu}(x') = (S^{-1})^\rho{}_\mu(x')(B^{-1})^\alpha{}_\rho(x)g_{\alpha\beta}(x)(B^{-1})^\beta{}_\kappa(x)(S^{-1})^\kappa{}_\nu(x'). \quad (16)$$

Equations (13)–(15) were initially derived within the context of spherically symmetric static configurations. In the Schwarzschild metric, a test particle P_1 (as defined in the

Glossary, Section 2) that is initially at rest at spatial infinity traces out a timeline transformed precisely by these relations [10]. These equations remain valid under more general conditions, as demonstrated in Appendix F.3. However, for the scope of the present work, focusing on the spherically symmetric static configuration is sufficient.

It is established that $g_{\mu\nu}(x)$ is diagonal in the R_0 frame and $g_{\mu\nu}(x')$ remains diagonal in the R'_0 frame. According to Eq. (11), since the boost is directed along the x -axis, only the temporal and x -spatial components evolve under this metric deformation. If the rescaling transformation $(S^{-1})^\rho{}_\nu(x')$ is parametrized by a temporal rescaling factor α and a spatial rescaling factor β , applying Eq. (16) alongside the standard condition $\alpha\beta = 1$ yields the metric components $g_{00}(x') = \alpha^2$ and $g_{11}(x') = -\alpha^{-2}$. This result clarifies the underlying mathematical mechanism governing Eqs. (13)–(15).

3.12.7. Overview of the spacetime metric derivation

In summary, the determination of the spacetime structure proceeds from the VGW framework established in Section 3.11. This enables the computation of the net local four-momentum at event x , which encodes the local geometric deformation induced by particle P . Relative to the former GRF at x , the local spatial velocity associated with this four-momentum determines a corresponding Lorentz boost, which encapsulates the local spacetime deformation at x . The local metric tensor is then evaluated pointwise from this boost across all spacetime events, yielding the global spacetime geometry.

The stability of this self-induced geometry naturally arises as a fundamental question. In a static universe, this iterative mechanism converges toward a stable spacetime configuration. Indeed, one can formulate a thought experiment in which the energy distribution is assembled incrementally by introducing particles sequentially and ramping their rest energies continuously from zero to their physical values. Each intermediate stage represents a static energy configuration. As is standard in General Relativity, the local spacetime structure at each step can be represented within Riemann normal coordinates [11], where spacetime appears locally Minkowski flat. In this coordinate system, the net vector sum of VGW momentum passing through any event x vanishes identically. This property holds continuously in the limit of the sequence of energy distributions and their associated reference frames.

4. Predicted surrounding effect

4.1. Emergence of the surrounding mechanism from relativity

The preceding analysis demonstrates that relativity predicts a spatially variable G , whose variation obeys the "surrounding matter" rule introduced in [1]. Equation (10) directly encodes this surrounding effect. To make this dependence explicit, we rewrite Eq. (10) by transferring the total energy to the right-hand side and isolating the resulting spatial velocity fraction:

$$\frac{v}{c} = \frac{f(x, y_0)C^0(y_0)}{\sum_{n=0}^{\infty} f(x, y_n)C^0(y_n)}. \quad (17)$$

Equation (17) reveals the underlying mechanism: the relative spatial velocity ratio v/c is determined by a weighted barycentric combination of the spatial velocities of all VGWs propagating through event x , where each VGW contribution is weighted by its total energy. The spatial attenuation of these propagating VGWs conforms to the vacuum Einstein field equations ($R_{\mu\nu} = 0$).

We maintain the assumption of a static baseline universe filled with a uniform matter density, perturbed only by a single particle P situated at rest at origin O . Let y_0 denote a spacetime event located spatially at O , with $C^\mu(y_0)$ representing the four-momentum of the unique VGW generated by P .

Due to the assumption of a homogeneous background universe, the spatial velocity evaluated at x in the absence of particle P would vanish ($v = 0$). The introduction of P induces a non-zero velocity ($v \neq 0$), whose magnitude remains constrained by the denominator in Eq. (17). Since the intrinsic energy of particle P is fixed, any increase or decrease in the total surrounding energy near event x leads to a corresponding increase or decrease in the denominator, thereby suppressing or enhancing the resulting velocity v , respectively.

Directly derived from Eq. (10), Eq. (17) thus demonstrates that the magnitude of the induced spatial velocity is inversely proportional to its denominator, which grows monotonically with the total ambient energy surrounding event x . This phenomenon was termed the "surrounding effect" in [1].

4.2. Some consequences of the surrounding effect

Furthermore, the denominator on the right-hand side of Eq. (17) represents an isotropic summation of positive scalars. This mathematical structure naturally motivates the formulation of a modified gravitational model in which this denominator is identified with the total local surrounding energy near the point where the gravitational interaction is evaluated. Consequently, the surrounding matter model—or a closely related gravitational framework—emerges directly as a regular prediction of general relativity. Within this context, contemporary gravitational anomalies may not represent breakdowns of standard physics, but rather natural consequences of relativistic dynamics.

Modern modified gravity frameworks must broadly agree with MOND phenomenological predictions [15] across the vast majority of observational data, a requirement established by decades of astrophysics research. This compatibility is inherently realized within the surrounding matter model: while MOND enhances gravitational acceleration in the low-acceleration regime, low local acceleration typically correlates with a low ambient density of surrounding matter and energy, which similarly enhances effective acceleration under the surrounding effect. Notably, Appendix P demonstrates that relativity directly recovers MOND's characteristic $1/r$ gravitational acceleration law in galactic environments.

Another mechanism emerging from this effect is the scale-invariant nature of spacetime deformations under global rescalings. Multiplying every contribution on the right-hand side

of Eq. (17) by a positive scaling factor $\lambda > 0$ leaves the ratio v/c on the left-hand side invariant. This scale independence is particularly relevant at particle physics scales, where matter is far sparser than in astrophysical environments, offering a physical mechanism for mass gap phenomena [29].

Furthermore, if the background matter density of the universe approaches zero while the energy of particle P remains fixed, the denominator on the right-hand side of Eq. (17) tends to $f(x, y_0)C^0(y_0)$, driving the fraction to unity and $v \rightarrow c$ on the left-hand side. This recovers precisely the limiting behavior described at the outset of this derivation in Section 3.7.4, providing a natural regularization mechanism for potential divergences in high-energy field calculations [29].

The physical implications of the surrounding effect extend well beyond these immediate applications, pointing toward a broader reinterpretation of scale-dependent phenomena in fundamental physics.

5. Revisiting the foundations of gravitational laws

The theoretical derivation of a variable G naturally prompts a critical reassessment of how gravitational field equations were historically constructed—from Newtonian mechanics to General Relativity. A fundamental critique reveals that fixing G as a universal constant in Einstein’s field equations imposes an empirical constraint not required by first principles.

The formal foundational analyses are detailed in two dedicated appendices:

- Appendix Q revisits Newton’s law and Poisson’s formulation from the perspective of vacuum gravitational wave (VGW) flux conservation and environmental energy exchange.
- Appendix R analyzes how these Newtonian considerations carry over to the Einstein tensor, the stress-energy tensor, and the broader arguments supporting a surrounding-dependent coupling constant G .

Ultimately, these foundational considerations demonstrate that General Relativity naturally accommodates—and arguably favors—a spatially variable G governed by the local surrounding energy density.

6. Conclusion about G

6.1. General statement

Therefore, in addition to the whole mathematical demonstration of the present document, further arguments support a variable G governed by the surrounding effect. These considerations show that the presented results of this demonstration are also grounded in established principles:

- 1) Mach’s principle.
- 2) Correct theoretical construction of Newton’s law.
- 3) Sophisticating the construction of Einstein equation.
- 4) Loss of information in the construction of the stress-energy tensor.

5) Implicit assumption of GR.

Items 2) and 4) were listed as motivations in the introduction. Item 1) is part of the motivation 4) of the introduction. Those items were transformed into arguments by the present study. Items 1) 2) and 3) have been described above. Item 4) was used implicitly in the present study. Indeed, there were no such loss of information in the descriptions of the energy distributions of the demonstrations of the present document. This item, as well as item 5), is described in [2]. Item 5) is related to the remark that the quarks are moving at a speed which is close to the speed of light.

Consequently, the outcome of the mathematical derivation presented in this work is entirely consistent with general relativity. Beyond the solar system, the surrounding effect must absolutely be taken into account to re-evaluate the local effective value of G , rather than relying on the solar system constant. Since this surrounding mechanism and the resulting variation of G emerge strictly from mathematical derivation, it was imperative to verify that such a feature remains fully compatible with established relativistic principles. Remarkably, beyond simple compatibility, this mechanism resolves several long-standing conceptual challenges in theoretical physics. In short, adopting a variable G governed by the surrounding effect provides a far more accurate, comprehensive, and physically sound representation.

6.2. *Derivation of the dynamical equation*

An explicit, first-approximation formula for G can be derived directly from Eq. (10); complete mathematical details are provided in Appendix N.

It should be noted that these calculations do not explicitly invoke the Ricci scalar curvature R . While a full variational treatment involving the Ricci scalar would represent the most rigorous formulation, it remains mathematically far more complex. Instead, a simpler effective approach is adopted here, which directly yields an explicit dynamical equation for G .

As demonstrated in Appendix P, this relation directly recovers the characteristic $1/r$ acceleration law in galactic environments. Furthermore, a refined formulation of this relation naturally reproduces the MOND equation, yielding the following key results:

- (1) The characteristic flat rotation curves (constant asymptotic velocity) of spiral galaxies are naturally recovered.
- (2) The empirical Tully–Fisher relation is successfully reproduced [30, 31].

These findings strongly support the physical validity and observational consistency of this refined formulation.

7. Discussion

In this work, a fundamental determination of spacetime structure by energy distribution has been presented. By extending the relativistic concept of local rest frames into the generalized rest frame (GRF) framework, this study uncovers a crucial property previously

overlooked in the literature: the dynamical evolution of these rest frames is directly driven by energy density gradients. Specifically, these gradients are mathematically expressed through an inverted Green’s function that incorporates the energy density and explicit four-momenta of the constituent particles of the universe. Mathematically, this behavior leads directly to Eq. (10), which links energy distribution to the first derivatives of the metric. By bridging first-order metric variations to the standard second-order Einstein field equations, this formulation opens an enriched, deeply Machian chapter of General Relativity.

A primary outcome of this framework is that physics intrinsically obeys a surrounding effect: increasing the energy density in the vicinity of an interaction suppresses the effective gravitational coupling compared to the standard Newtonian prediction. Consequently, Newton’s constant G is no longer a rigid universal constant, but an environment-dependent field quantity. Beyond the mathematical derivation itself, these results are strongly supported by foundational physical principles—including Mach’s principle, a rigorous reconstruction of Newton’s law and Einstein’s equations, the elimination of information loss inherent in coarse-grained stress-energy tensors $T_{\mu\nu}$, and the relativistic dynamics of subatomic constituents such as quarks moving near light speed [2]. Furthermore, a simplified expression for G derived from straightforward physical reasoning establishes the characteristic $1/r$ spatial variation required for galactic velocity profiles; fine-tuning this relation naturally recovers the empirical MOND acceleration threshold a_0 , thereby reproducing flat rotation curves and the baryonic Tully–Fisher law from first principles.

While evaluating the full attenuation function presents computational complexity, utilizing an environment-driven G offers a vastly superior physical description compared to enforcing an anthropocentric solar-system constant across galactic and cosmological scales. Based on the surrounding effect proven in this document, Surrounding Matter Theory explicitly addresses and resolves the major observational and conceptual challenges in astrophysics and cosmology—such as galaxy rotation curves, cosmological expansion, and the cosmological constant problem—without requiring ad hoc dark matter or dark energy [1]. Furthermore, this framework demonstrates remarkable universality: applying Eq. (10) and the resulting surrounding effect to quantum field theory provides the key mechanism to resolve the Yang-Mills Mass Gap problem, successfully demonstrating the emergence of a non-zero mass gap from pure gauge theories while naturally suppressing quantum field divergences [3, 4, 29].

Since this surrounding mechanism and the resulting variation of G emerge strictly from a mathematical derivation, verifying its compatibility with established relativistic principles was paramount. Far from being restricted to standard gravitational phenomenology, this mechanism produces intense local spacetime deformations under extreme conditions—such as the microscopic vacuum eventually surrounding an isolated particle, which strongly amplifies the spacetime geometry mediating the interaction—making its consequences vital across all domains of physics. Remarkably, beyond simple compatibility, this framework resolves major foundational puzzles across both macro- and micro-physics—from galactic dynamics and cosmology to quantum gauge theories and the Yang-Mills Mass Gap. Should gravitational interactions ultimately serve as the foundation for physical unification, such extreme surrounding effects would naturally extend further the

scope of resolved problems, offering promising pathways to address long-standing puzzles such as nuclear saturation and the proton charge radius anomaly [2]. The successful alignment of Surrounding Matter Theory with observational data strongly suggests that these "anomalies" are not breakdowns of relativity, but direct, unexploited predictions of General Relativity itself, establishing a far more accurate representation where effective gravitational coupling is intimately bound to the global energy distribution of the universe.

8. Conclusion

In conclusion, this work demonstrates that the spacetime variation of Newton's constant G is an inescapable prediction of General Relativity when spacetime structure is constructed directly from local and surrounding energy distributions. By moving beyond single-particle singularities to discrete multi-particle systems via the Generalized Rest Frame framework, the dynamical evolution of local rest frames—mediated by an inverted Green's function—naturally resolves long-standing mathematical pathologies while preserving the fundamental principles of Einstein's field equations. From this first-principles construction, a simplified equation for G naturally emerges; upon minor refinement, it directly recovers empirical MOND phenomena, flat galactic rotation curves, and the baryonic Tully–Fisher law. Building upon this surrounding effect, Surrounding Matter Theory establishes a unified framework that explicitly addresses and resolves the remaining gravitational and cosmological anomalies without relying on ad hoc dark matter or dark energy.

Furthermore, the physical principles unveiled in this gravitational context exhibit remarkable universality across scale. When applied to quantum gauge theories, the exact same surrounding mechanism provides a natural regulator that suppresses field divergences and generates a non-zero mass gap from first principles. By demonstrating that key astrophysical anomalies and quantum field challenges stem from the same overlooked predictions of General Relativity, this framework bridges the historical divide between local couplings and global energy geometry, opening clear avenues for future developments in high-energy phenomenology, particle physics, and quantum gravity.

Appendix

Appendix A. Continuity of the S function

The S function gives spacetime structure from energy distribution. S starts from the set of energy distributions, which is a linear space of infinite dimension. It reaches another linear space of infinite dimension, which is the space of spacetime structures. The latter can be defined using the matrices of the metric in some given system of coordinates. Then, this space is the linear space of the distributions over spacetime of 4×4 matrices. This definition depends on the choice of the system of coordinates. Both linear spaces are equipped with the uniform norm. Let's remind that the considered energy distributions are assumed to share the same bounded domain.

With the uniform norm, and for this restricted set of energy distributions, the claim is that S is continuous.

To prove this, let's show that if any amount of energy at any spacetime location is decreased to 0, then the effect of this amount of energy on spacetime structure also decreases to 0. For this demonstration, let's assume that this is wrong (assumption "w"). Then there exists an f_n sequence of energy distributions tending to 0 and such that their deformations are greater than some given non-null value. This can be written as follows:

$$\begin{aligned} \lim_{n \rightarrow +\infty} (f_n) &= 0 \\ \forall n \text{ in } \mathbb{N}, \|S(E + f_n) - S(E)\| &\geq M \end{aligned} \tag{A.1}$$

In these equations, E is the energy distribution of the universe, to which the f_n energy is added. $\|\cdot\|$ is the uniform norm on the set of spacetime structures. M is some strictly positive real number. Therefore, an infinitely small amount of energy provokes a non-null modification of spacetime structure. Let's remind that this word "small" means that the maximum density of energy which is added is small (uniform continuity), and also that the whole quantity of added energy is "small" (the considered energy distributions share the same bounded domain). Since this added amount of energy is infinitely small, it is possible to imagine a thought experiment in which this energy is added instantaneously. But the result on spacetime structure is noticeable. This generates gravitational waves. If the universe is empty, then no contradiction arises. But this would be unrealistic. If the universe is not empty, then those gravitational waves cause the motion of matter in the universe. But with this infinitely small and sudden "addition" of energy, of course, nothing happens since the added amount is infinitely small. Therefore, a sudden modification of energy distribution happens without anything to provoke it. This violates the conservation of energy principle. Therefore, assumption "w" was wrong, and the claim is proven.

Then, writing this continuity property with the limit of sequences, equation (2) is given.

Appendix B. Generalized rest frame

Appendix B.1. Definition of a GRF

The construction of GRFs of the present appendix is done for an energy distribution following the principle of energy conservation and following the principle stating that a free

fall particle follows the trajectory of a geodesic.

In relativity, rest frames exist. This concept can be generalized into what is called a generalized rest frame (GRF). Let us now perform this generalization. The metric is evaluated between rest frames using the GR Lagrangian in vacuum. Based on this, GRFs can be defined in the following way.

A rest frame R is chosen. This means there is a particle P at rest in R , where P is a dimensionless particle with nonzero mass. A displacement is then considered: that is, a spacetime event E is chosen such that P is located at E , and $E' = E + dl$ is any other event infinitesimally close to E , where dl is an infinitesimal displacement. From E' , a test particle P' is considered, meaning that P' is at rest relative to R . However, since P' is not exactly at E , its "potential" rest frame is not R . The term "potential" is used because P' has no mass. Therefore, the rest frame of P' is another GRF passing through E' . By repeating this process infinitely, a GRF can be constructed at every spacetime event in vacuum.

It may be proven that this generalization ensures the existence of a time line for the generalized frame, which is a congruent geodesic (part of the same congruence of geodesics [11]) with respect to the time lines of the rest frames. (A "congruent geodesic" is the generalization of the concept of a "parallel straight line" in curved spacetime.) However, such a demonstration is not necessary for the present document.

Appendix B.2. *Fundamental behavior of a rest frame*

The fundamental behavior of a generalized rest frame can be illustrated through a thought experiment. Consider gradually increasing the rest energy of a particle P , while simultaneously decreasing the total energy of the universe. At the end of the experiment, P and the universe have exchanged roles: P now contains the energy that was previously in the universe, and the universe contains the energy that was previously in the particle.

The first result is that the frame in which time elapses the most is no longer the one attached to the universe. That frame is now attached to P . This implies that the spacetime structure is now the symmetric result of a frame exchange between the two. It also means that the spacetime structure has been gradually modified from the initial state to the final one. This process effectively reverses time dilation. For example, in a twin paradox configuration, at the end of the traveling twin's journey, the older twin would become the younger after this thought experiment. Therefore, the spacetime modification is described simply by the boost that transforms one frame into the other. This reasoning is based on the well-established assumption that general relativity is internally consistent.

At this point, it becomes useful to assign names to the frames. Let R_u be a frame attached to the universe. Assume the universe is filled with a constant, homogeneous distribution of matter; hence, this matter is considered to be at rest in R_u . Let R_p be a frame attached to P . The result of the thought experiment is that P locally generates a spacetime deformation described by the boost from R_u to R_p . Of course, this deformation is local to P , but the greater the rest energy of P , the more valid this deformation becomes around P . A "more valid deformation" means that the spacetime deformation has significant influence over a larger spacetime region.

The spacetime deformation arising in the experiment is described by the boost that progressively transforms the frame from R_u to R_p . This frame remains a rest frame throughout the process, even if it is no longer the frame in which time elapses the most. It is, nonetheless, the frame in which P is at rest.

This implies that the "absolute" spatial lines used in explaining the twin paradox have been exchanged between R_u and R_p . Accordingly, their time lines have also been exchanged (and the paradox can likewise be interpreted using absolute time lines). Furthermore, it follows that only the relative values of the respective energies determine which frame experiences the fastest passage of time. Consequently, any dimensionless particle P with nonzero mass deforms spacetime locally in such a way that, sufficiently close to P , its deformation dominates over that of any other energy source. Thus, in the vicinity of P , the absolute space and time lines are the space and time axes of P 's rest frame.

The result is as follows: for any particle located at a given event x , an former GRF exists at x and is transformed into a new GRF by the presence of the particle, using the boost associated with the particle's four-momentum. Equation 12 describes the formulation of this boost in the former GRF, as derived from the expression of the four-momentum in the former GRF, given by equation 11.

Appendix B.3. *Rigidity of the local spacetime deformation*

The local spacetime deformation in the immediate vicinity of a point-like particle P exhibits a fundamental rigidity: the associated local boost remains invariant regardless of the surrounding energy distribution outside P . While increasing the total energy of P expands the spatial region over which this deformation dominates, it leaves the field structure at the exact location of P unaltered. Furthermore, the presence of surrounding particles near P does not modify this local deformation.

This behavior stems directly from the standard continuum-to-discrete transition, which can be formally demonstrated by broadening the domain of acceptable energy distributions. Consider, for instance, a sphere of radius r containing a uniform mass-energy density associated with a total energy E . Taking the limit as $r \rightarrow 0^+$ while keeping E constant, the spatial deformation evaluated asymptotically far from the origin remains strictly invariant at every stage of the contraction. Realistically, a physical source is represented by a distribution with $r > 0$. However, adopting the point-particle idealization ($r = 0$) necessarily yields an infinite local energy density at the location of the particle.

Consequently, Eq. (10) admits two physically equivalent formulations:

- (1) **Dimensionless formulation:** Particles are treated as zero-dimensional point sources ($r = 0$), in which case the amplitude factor $C^0(y_n)$ diverges at each source location y_n .
- (2) **Finite-extent formulation:** Particles are modeled with a non-zero physical dimension—for instance, represented by uniform spheres of a fixed conventional radius, such as the Planck length—thereby ensuring a finite local energy density throughout space.

In the present document, convention (1) is adopted. Due to this local divergence, all external contributions in Eq. (10) become negligible at $x = x_P$, leaving the term corresponding to $y_n = x_P$ as the sole dominant factor governing the local field. Surrounding particles exert no additional influence directly at x_P . As a result, the local deformation generated by P is fully determined by P itself and remains invariant under changes in its global environment.

From a physical standpoint, this result holds to a high degree of approximation even for extended bodies, provided their matter density is sufficiently high. Given the extreme densities characteristic of fundamental particles, this model remains robust when applied to the virtual microscopic matter clusters analyzed in the thought experiments throughout this work.

Appendix B.4. Conclusion

The conclusions of this appendix are as follows:

- 1) For each spacetime event, a frame called a GRF has been constructed.
- 2) This GRF concept generalizes the notion of a rest frame.
- 3) Each particle P located at a given spacetime event x is associated with an former GRF and a new GRF. The former GRF is defined in x under the assumption that P is absent; the new GRF is defined in x under the assumption that P is present.
- 4) At x , the new GRF is obtained from the former GRF by applying the boost associated with the motion of P relative to the former GRF.

Appendix C. Rescaling after the boost

Let's consider P , a particle moving uniformly at speed v , where $v < c$, along the D space line in a flat Minkowskian spacetime. The $R(O; ct, x, y, z)$ frame is chosen such that O lies on D , and Ox is in the direction of P 's motion. The local deformation around P is first described by the b_v boost associated with the v speed, as shown previously.

But equations (13), (14), and (15) show that a rescaling of ct time and x space axis is required. If no rescaling occurs, it means that the deformation generated by P is entirely described by a boost. In that case, it is simply the usual change of coordinates following SR rules, and no deformation is observed. Therefore, a rescaling occurs, and this is consistent with time dilation between R and R' . From equations (11), (13), (14), and (15), if α and β are the s rescalings of ct and x , respectively, we have $\alpha^2 = \beta^{-2} = 1 - v^2/c^2$.

Hence, the complete deformation is described by $b_v s$. This results from equations (13), (14), and (15).

This is true for the Schwarzschild metric and for a test particle at rest infinitely far from the center of symmetry. But the calculation of this rescaling can be done for any energy distribution, with respect to any R_0 frame, at any t given time in R_0 . For this calculation a test particle being at rest with respect to R_0 at t is used for calculating the boost. And then the rescaling is the time dilatation from the time of R_0 to the proper time of the particle.

Appendix D. Integration constant

This appendix deals with the integration constant in the spacetime structure calculation for the energy distribution of two particles in an empty universe.

This constant can potentially be determined from the following information. Singularities occur along the straight line containing the particles' locations, but only at those points that are not between the particles' locations. These are described by the boost associated with the speed of light in the direction moving away from the particles.

The rigorous approach is to calculate the spacetime structure using symmetry considerations and equation (R.1). Then, the asymptotic values of the deformation should allow the calculation of the integration constant.

Let's recall that this procedure has already been done for the energy distribution with one particle in an empty universe. Symmetry considerations produce the Schwarzschild metric. Equation (R.1) provides the information that the metric takes the form $g_{00}(x) = 1 - M/r$. Here, $g_{00}(x)$ is the time-time component of the metric at the x spacetime event, r is the spatial distance from x to the particle located at the center of symmetry, and M is the unknown integration constant. Then, the asymptotic value of the local spacetime deformation, as given by paragraph 3.7, allows the calculation of $M = \infty$.

Therefore, it should be possible to calculate this constant for the energy distribution of two particles in an empty universe.

Appendix E. Particle moving at the speed of light: local deformation

Now, assume $v = c$. Let's prove that the deformation in C is the b_C boost associated with the speed of light along the D line in the direction of the GW propagation.

Starting from the fundamental behavior of a rest frame which is described in Appendix B, taking the limit as v approaches c , the result is obtained. As usual, the interchange of limits theorem is used. The simple convergence is sufficient for it to work since the studied deformation is only local.

Appendix F. Particle moving at the speed of light: propagation of the deformation along the trajectory

Appendix F.1. Aim

The main aim of the present appendix is to study simply the GW generated by a given particle P in motion at the speed of light. This study is "simple" because it is restricted to the trajectory of the particle and also to the geodesics which share the same tangent as the P trajectory (if this trajectory deviates from being a geodesic at some point).

The complete description of this GW is done in Appendix H, but no demonstration is really given for this description, only clues. That's why here the aim is to have a rigorous demonstration of the nature of the GW along those particular geodesics. This will be enough for the whole demonstration of the present document.

The study will start by the description of important properties of test particles.

Appendix F.2. Context and notations

A given E energy distribution is in the universe. Let's denote $GRF(E)$ the system of GRFs associated to E . In other words, $GRF(E)$ is the set of the new GRFs after insertion of E . An e spacetime event is chosen. Let's denote, for any such e , $GRF(E, e)$ the GRF which is a part of the $GRF(E)$ system, and which is located in e .

Then a Q test particle is considered, and added to E . Q is such that it is moving at the V speed with respect to $R_0 = GRF(E, e)$ at e . Hence after the insertion of Q , the total energy distribution of the universe is $E + E_Q$, where E_Q is the Dirac distribution of Q . Let's always denote E_M the Dirac distribution of any given M dimensionless particle. Let's denote $v(t)$ the speed of Q with respect to R_0 at any t time.

Appendix F.3. Deformation generated by a test particle at rest in R_0

First, let's assume $V = 0$. Therefore Q is assumed to be at rest with respect to R_0 at e .

In e , a geodesic, let's call it $g(e)$, has been constructed at the construction of the $GRF(E)$ system. $g(e)$ is congruent to the nearby particle trajectories which are geodesics. This has been done at the construction of the $GRF(E, e)$ system, in Appendix B. Let's remind that this construction is done for an energy distribution following the principle of energy conservation and following the principle stating that a free fall particle follows the trajectory of a geodesic. $g(e)$ as well as the Q trajectory are geodesics. But they share the same tangent in e because $V = 0$. Hence those geodesics are equal. Therefore, the speed of Q , with respect locally to each frame of the $GRF(E)$ system of frames, remains equal to 0.

Therefore, along $g(e)$, the new GRF, $GRF(E + E_Q)$, is equal to its old, that is, $GRF(E)$. But by other means, along the Q trajectory, $GRF(E + E_Q)$ is given by transforming $R_0 = GRF(E, e)$ by the boost associated with $v(t)$ in R_0 at any t . Therefore since $GRF(E) = GRF(E + E_Q)$, it is also $GRF(E)$ which is given by transforming $R_0 = GRF(E, e)$ by the boost associated with $v(t)$ in R_0 at any t . Therefore, a test particle allows to calculate the spacetime structure generated by E , along its trajectory.

Therefore a test particle being at rest with respect to $GRF(E)$ at a given e event allows to calculate spacetime structure along its trajectory.

The final result is that the test particles being at rest with respect to $GRF(E)$ at any given event allow to calculate spacetime structure everywhere. This is an important result which will be used directly throughout the document.

Appendix F.4. Deformation generated by a test particle moving at V speed in R_0

Second, $V = 0$ is no longer assumed. The Q test particle being in free fall, its motion with respect to the local GRF which is part of $GRF(E)$, the system of former GRFs, is always equal to V . This is ensured, for instance, by conservation of energy of motion. Therefore, along Q 's trajectory, $GRF(E + E_Q)$, is given by transforming $GRF(E)$ by the boost associated with V with respect to $GRF(E)$ at any t time.

Therefore $GRF(E)$, is given by transforming R_0 by the boost associated with the rel-

ativistic subtraction of $v(t)$ by V in R_0 at any t .

This allows to calculate spacetime structure in another way in which V is not null. This way might appear wired because it is simpler to choose R_0 such that $V = 0$ at t_0 , which is always possible. But this wired result helps for the following paragraph.

Appendix F.5. Deformation generated by a particle moving at the speed of light along its trajectory

Now let's add another particle, P . Let's assume that P is moving at the speed of light along a geodesic before a given e spacetime event, at time t in a given R frame, and that after t , P deviates from this geodesic. Let's call g this geodesic. This deviation may result from the existence of some unknown force. The question about the existence of such a force is discussed in Appendix K. The aim here is only to study the deformation generated by the GW produced by P .

g is a null geodesic before e therefore g is always a null geodesic. Let's add another particle, Q . Q is a test particle "surfing" on the GW generated by P along g . It means that Q is located in e , infinitely close to P , and in e the two particles share the same speed of light. Then after e , P and Q get apart from each other.

Since Q is in free fall and since it moves at the speed of light in e with respect to $GRF(E)$, its speed remains the speed of light after e , with respect to $GRF(E)$, along g .

Therefore, after e , $GRF(E + E_P + E_Q)$ is given by transforming $GRF(E)$, by the boost associated with this particular speed of light. A first demonstration is given by considering that this is the rule valid for any test particle not moving at the speed of light. It remains valid for a particle moving at the speed of light this can be proven by taking the limit for a test particle moving at the $V < c$ speed, limit when V tend to c , using also the interchange of limit theorem. Another demonstration uses the Doppler effect. It is that if the deformation generated by this particle was not equal to this one then it would mean that its energy would become null. Indeed with respect to the system of $GRF(E + E_P + E_Q)$, in e , Q

Q is at rest with respect to $GRF(E + E_P)$ at e .

Therefore Q which is a test particle, which remains at rest with respect, locally, to $GRF(E + E_P)$, along g .

Therefore, along g , $GRF(E + E_P) = GRF(E + E_P + E_Q)$ (the transforming boost is associated to the null speed).

Therefore, along g , $GRF(E + E_P)$ is given by transforming $GRF(E)$ by the boost associated with the speed of light, the speed of Q .

But $GRF(E + E_P)$ is the system of new GRFs given by the existence of E and P . Therefore this system describes the GW generated by P .

Appendix F.6. Conclusion

Therefore the GW propagates at the speed of light along g and the local deformation which is propagated is the drastic deformation described by the boost which is associated with the speed of light of this GW.

Using the notations of Figure (2), the final picture is the following. The deformation of P in C propagates along the D line in the same direction as the motion of P and is described by the b_C boost associated with the speed of light in the same direction. This can't be really observed when P moves along D , because the deformation generated by P can't be distinguished from the deformation generated by its GW propagating in the same direction. But this can be observed as soon as P deviates from this D line trajectory. The propagation speed is the speed of light (this has been demonstrated without using assumption (I).) The important result here is that this propagated local deformation is described by b_C .

Appendix G. Simple description of the local spacetime deformation

Let's describe the local spacetime deformation generated by a particle or a GW, using the simplest possible approach. For that, the Poincaré-Einstein synchronization [28] will be used. The Minguzzi synchronization is not strictly required since the transitivity of the procedure is not mandatory here. This procedure is used here for the construction of an inertial frame $R'(C, ct', x')$, moving at speed v with respect to another frame, denoted $R(O, ct, x)$. No Lorentz transform or boost is needed for that, only the Einstein synchronization rule and geometry in two dimensions are used. Figure (1) represents this construction. After this construction, x' is the symmetric of ct' through the orthogonal symmetry with respect to the light trajectory in the increasing x direction. Indeed, the circle passing through O and M , and having C as its center, also contains N because the OMN triangle is a right triangle at N . Hence, $CM = CN$. Then, if H is the midpoint of M and N , then CH is parallel to ON by Thales' theorem applied to the OMN triangle. And, of course, CN is the direction of the light trajectory in the increasing x direction.

Moreover, there is a natural coherence in the calculation of the space axis in four dimensions using the synchronization procedure. This is demonstrated by using different light beam trajectories. For example, the procedure can be executed with a light beam in

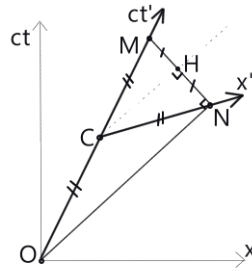


Fig. 1. Two inertial frames are represented in two dimensions, $R(O; ct, x)$, and $R'(C; ct', x')$. The latter is constructed with respect to the first using Einstein's synchronization rule. ON and NM together represent the trajectory of a light beam emitted from O in the direction of increasing x , and bouncing back at N . C is the midpoint of OM . The x' axis is supported by CN following the synchronization rule. H is the midpoint of MN . A simple geometric reasoning in two dimensions shows that x' is symmetric to ct' through the orthogonal symmetry with respect to CH , which is parallel to ON .

the decreasing x direction, and then, after the turning point, in the increasing x direction. The result is, of course, the same x' axis. But also, using a trajectory along the y or z directions will show that along these space axes no deformation is noticed, which is the expected result. Any other light beam back-and-forth trajectory in space will show a deformed axis, which is also coherent since the procedure yields linear results.

The causal link proceeds as follows. The starting point is the relative velocity v of R' with respect to R , meaning that C moves at speed v in R . From this parameter, the ct' axis of R' is constructed in the frame R . Subsequently, the synchronization rule determines the orientation of the x' axis. Although ct' and x' are mathematically equivalent and mutually retrievable, this specific causal chain is particularly simple and direct, even if other constructions of the transformation remain possible.

A remark can be added about the rescaling studied in Appendix C. The boost transforming R into R' is deduced and, therefore, implicitly identified by the direction of the new ct' time axis with respect to R . The new GRF and the new metric valid after the insertion of the particle or the new GW are deduced from this ct' axis only, without the need for the details of the boost itself. Even the usual rescaling of the new time and space units can be done based on the ct' axis alone.

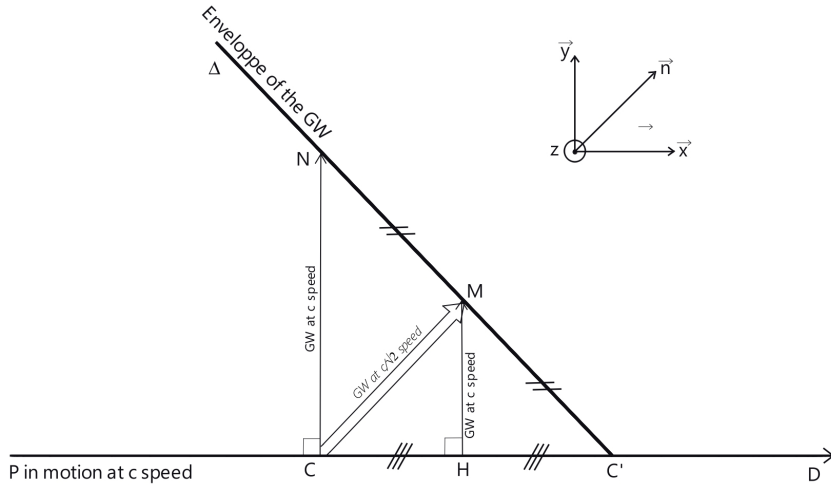


Fig. 2. The GW generated by a particle P traveling at the speed of light in two spatial dimensions. D is the trajectory of P . The spacetime deformation produced by P propagates perpendicularly to D from C to N at speed c , as well as from H to M at speed c . H and M are the midpoints of CC' and NC' , respectively. Δ is the GW envelope, propagating from C to M at speed $c/\sqrt{2}$.

Appendix H. Particle moving at the speed of light: final picture of the deformation everywhere

Appendix H.1. Demonstration of the GW structure

We study the envelope of the GW generated by a particle moving at the speed of light along the trajectory D , using the notation introduced in Figure 2. The universe is assumed to be empty outside of P .

This envelope forms a cone coaxial with D . Owing to the spatial rotational symmetry of the energy distribution about D , the generated spacetime geometry inherits the same axisymmetry.

The following analysis focuses on the special case where the envelope is an equilateral cone, corresponding to the configuration shown in Figure 2.

Appendix H.2. Description of the GW envelope in the case of an equilateral cone

Figure (2) shows the GW in two dimensions. In $R(C; ct, x, y, z)$ the P particle moves in the direction of the x axis, and along the D line at the c speed. Let's focus on the GW which is generated by P from the E_C event which is a given C space location pertaining to D , and a given t_C time. This GW propagates from C at the c speed along the space directions of the (y, z) plane which is perpendicular to x and therefore to D , starting from C . Let's choose an E_N spacetime event composed of the N space point and the t_N time in R , such that CN is along the y axis perpendicular to D and E_N receives the GW propagation starting from E_C . During the same $t_N - t_C$ time interval, P moves from C to some C' space point pertaining to D . Also, when P is located in H , which is the middle of C and C' , the GW starts another propagation from H in the direction of CN . This propagation is located in M at t_N , such that M is the middle of N and C' . The propagations along CN and HM are done at the c speed. Therefore, this GW propagation is also along the CM line at the $c/\sqrt{2}$ speed. This is the speed of the envelope of the GW along its trajectory. The spatial vector of this propagation speed is normal to the envelope. The envelope is a cone centered on D . This cone is "isosceles". In other words, there is $CN = CC'$ and $HM = HC'$.

Now let's study the deformation generated by the GW. First of all, in C the local deformation which is generated by P is the b_C boost associated with the speed of light in the same direction as P . This is proven in Appendix E.

Along the D line, exactly the same boost propagates itself, at the speed of light. This is proven in Appendix F. Here, since the P trajectory is the same D line, of course, this can't be noticed. But it will be noticed as soon as P will deviate from this D trajectory.

Outside of D , the local GW deformation is still described by the b_C boost. Indeed, this is the initial deformation in C . As such it is the deformation which is propagated. Also, this is coherent with the propagation speed of the envelope. More details are available in Appendix H and Appendix I.

Let's study the space submanifold at some given time. It is assumed that a slicing of spacetime along time is possible. The spacetime deformation at any X space location depends on the position of X with respect to the cone of the maximum GW deformation.

Outside of the cone (external part of the cone), the spacetime structure is unchanged for causality reasons. Indeed, here P has no possible causal link. In other words, it means that here the old and new GRFs are equal. Inside the cone, each event has been reached in the past by the GW generated by P . Indeed, nothing more has ever happened there (no more GW or any effect due to any other matter than P has ever happened there, since there is a vacuum instead of P in the universe). Therefore, here the local spacetime deformation is described by b_C .

When P moves from C to C' , this is done at the speed of light with respect to any given inertial R frame. During the same delay of time in R , the generated GW moves from C to N . This is the definition of N , along with CN being perpendicular to D . Let's take an R' inertial frame in motion with respect to R along CN .

Appendix H.3. Details about the resulting GW in the case of an equilateral cone

Figure (2) will illustrate the events described in the present appendix. Let's argue the following.

Assumption (j): The spacetime local deformation described by the b_C boost, generated by the particle P moving at the speed of light in the x direction, also generates a GW that propagates at the speed of light along the directions perpendicular to x .

This assumption is the assumption that the envelope is an equilateral cone.

Moreover, let's show in the present appendix that this is coherent with the fact that the envelope of the GW propagates globally at the speed $c/\sqrt{2}$ along the CM direction. This will be an additional argument for assumption (j).

Let's start by studying the motion of the GW because this is the starting point, as shown in Appendix G. The GW propagates at the speed of light, as stated by assumption (I) of paragraph 3.5. This propagation at speed c occurs along directions perpendicular to the P trajectory, as stated by assumption (j) above. Therefore, this propagation occurs from C to N at speed c . Let's study the Δ half-line containing N and C' , starting at C' and going in the direction of N . The events along Δ at time t_N correspond to the propagation of events at the locations of P along the CC' segment. This means that Δ is the envelope of the GW along the (x, y) plane. It is noted that this Δ envelope propagates along the CN direction at speed $c/\sqrt{2}$.

From this motion, the new time axis is deduced, let's call it ct'' , starting the causal link presented in Appendix G. This ct'' time axis in M is located in the (ct, n) plane, where n is the axis aligned with the CM direction. Due to the $c/\sqrt{2}$ motion of the GW, the equation of the trajectory of the GW along the n axis is as follows.

$$r = \frac{c}{\sqrt{2}}t \quad (\text{H.1})$$

In equation (H.1), r is the coordinate along the n direction starting from M , with respect to R . It can be written as $ct = \sqrt{2}r$. Therefore, the angle β between ct'' and the (x, y) plane is given by the following equations.

$$\tan(\beta) = \sqrt{2} \quad (\text{H.2})$$

$$\cos(\beta) = \frac{1}{\sqrt{3}} \quad (\text{H.3})$$

Now let's show that this result is coherent with the local spacetime deformation generated by P and its local propagation along the x axis. At time t_N , along Δ , the ct time axis has been rotated around the (y, z) plane by an angle of $\pi/4$ in the direction of x , resulting in the ct' axis. Indeed, this is the effect of the b_C boost along the space points of Δ : the b_C boost transforms R into the $R'(ct', x', y, z)$ frame in this way. There is $ct' = x'$. This creates a Π plane containing all these lines starting from the points of Δ in the direction of ct' . In other words, Π is the new plane of time after the action of the b_C boost executed along Δ at $t = t_N$ in R . Let's calculate the angle α between Π and the (x, y) plane, with respect to R . This is also the angle between the n direction and Π , and therefore the angle between CM and Π . The calculation follows.

$$\cos(\alpha) = \frac{\overrightarrow{CM} \cdot \overrightarrow{CM'}}{|CM|_3 |CM'|_3} = \frac{\left(\frac{L}{2}\right)^2 + \left(\frac{L}{2}\right)^2}{\sqrt{\left(\frac{L}{2}\right)^2 + \left(\frac{L}{2}\right)^2} \sqrt{\left(\frac{L}{2}\right)^2 + \left(\frac{L}{2}\right)^2 + L^2}} = \frac{1}{\sqrt{3}} \quad (\text{H.4})$$

L is the $CN = CC'$ distance. Let's denote Π' as the plane parallel to Π and containing C . M' is the spacetime point that is the projection of M onto Π' along the ct direction. Equation (H.4) is coherent with equation (H.3). The result is the following.

$$\alpha = \beta \quad (\text{H.5})$$

This means that the GW propagation along its envelope creates a spacetime deformation that is coherent with the propagation of the b_C boost perpendicularly to the P trajectory. It should also be noted that the Π plane makes a $\pi/4$ angle with respect to the (x, y) plane, but in the direction of y . In other words, the intersection of Π with the (y, ct) plane is a line that makes such an angle with respect to the (x, y) plane. This is coherent with the local GW propagation along the y axis at the speed of light.

The final result is a coherence between the following GW features:

- 1) Locally, the GW propagates the b_C boost at the speed of light along the P trajectory,
- 2) It also propagates this boost at the speed of light, perpendicularly to the P trajectory. This describes the entire GW spacetime deformation (in the whole space at a given time).
- 3) The GW propagation along any space direction perpendicular to the P trajectory occurs at the speed of light. This is a spacetime deformation also described by the boost associated with the speed of light in this direction. It describes exactly the same entire GW spacetime deformation as described above.

- 4) Globally, the envelope of the GW propagates at the speed $c/\sqrt{2}$. The boost associated with this speed and direction also describes exactly the same entire GW spacetime deformation as described above.

Then, still following the causal link presented in Appendix G, the new space axis of the GW deformation is deduced from the location of the new time axis. This new n'' space axis is part of the (ct, n) plane because the GW global motion is along the n axis. n'' is the symmetric of ct'' with respect to the line supported by the $\vec{ct} + \vec{n}$ vector, where $\vec{ct}$ and $\vec{n}$ are the unit vectors of the ct and n axes, respectively. Of course, the same coherence arises for the new space axes along the n , x , and y directions, that is, the same coherence as shown above for the new time axis (items 1 to 4).

The conclusion of this appendix is that under assumptions (I) and (j), a coherent and symmetrical picture of this GW is presented. Since assumption (j) has not been proven, the picture cannot be considered definitively correct. Hopefully, the full picture is not required for the demonstration in the current document. What is required is the result from Appendix F, namely, the fact that the b_C boost propagates itself along x and the increasing x direction.

Appendix I. Final picture for a particle moving at the speed of light

This follows from Appendix H. The final picture for a particle moving at the speed of light is that, in R , the deformation occurring at event E_C in C propagates globally along the Cn spatial direction, at a speed of $c/\sqrt{2}$. This deformation is described by the boost associated with the speed of light in the x direction, which propagates locally at the speed of light along the Cx direction and also at the speed of light along the Cy direction. However, along the Cn direction, the deformation is described by the boost associated with the $c/\sqrt{2}$ speed. This picture is Lorentz invariant.

Appendix J. Asymptotic spacetime deformation around a particle moving at the speed of light along a circle

The asymptotic local spacetime deformation generated by a particle moving at the speed of light along a circle is deduced from Appendix I. The GW propagates in the direction perpendicular to the particle's trajectory. This is what is assumed by assumption (j) of Appendix H, and this assumption provides a coherent picture, as described in this appendix.

Moreover, asymptotically, this direction is also the line which is tangent to the particle's trajectory. In Appendix F, it has been proven that this propagation happens at the speed of light, and that this GW is the propagation of the b_C boost along D .

Therefore, it has been demonstrated the present starting paragraph. Asymptotically, the envelope is represented by a sequence of spheres (the sequence of spheres described in section 3.7), all centered on O , the center of the circle. Asymptotically, these spheres expand around O at the speed of light, and thus this envelope propagates at the speed of light. The boost which describes the local deformation of the GW on those spheres is associated with the speed of light which is normal to these spheres, in the direction that goes out of the spheres.

Appendix K. Avoiding the Unknown Forces

P was forced to follow a circular trajectory, which is not a geodesic. Therefore, there is an unknown force that prevents this trajectory from following a geodesic. This force is not gravity. At first glance, this suggests that modeling the Dirac distribution under assumption (2) of Section 3.1 would not be possible in a coherent way which would only involve gravitation. However, this apparent result is incorrect. The final result is actually quite the opposite, as will be shown.

Let us assume that P is at rest in a given inertial frame R , at the point O in space. The goal is to model P using a set of energy distributions that converge to it.

Let us start by reformulating the constraint above, which is indeed necessary for modeling a dimensionless particle P . Another constraint is added, and the result is the following two conditions, which must be satisfied by each energy distribution in the sequence converging to P :

- The energy distribution has a non-zero size.
- The spacetime structure generated by the energy distribution allows it to remain static.

As usual, the definition of the word "static" is that, as time passes, the energy distribution remains unchanged with respect to R , but matter may change. This means that matter can still be in motion in R , but in such a way that the matter density remains the same at any given space point.

The first constraint simply prevents trivial sequences. The second constraint is mandatory because the goal is to model the final Dirac distribution of matter for P by studying a sequence of energy distributions that converge to this Dirac distribution. The Dirac distribution is static because the aim is to model the gravity of this dimensionless particle P , which is at rest in R . Strictly speaking, this constraint may not be strictly mandatory. Indeed, one could imagine a cumbersome sequence of complex energy distributions, each of which is non-static, but still avoids any force other than gravity, allowing their existence, and ultimately converging to the energy distribution of P . However, the goal here is simply to examine the remark made at the beginning of the present appendix regarding the required unknown force for the static distribution of a circular trajectory. Moreover, the ultimate goal here in the present document is to construct a static sequence of energy distributions converging to the energy distribution of P without the need of any other force than gravity and following the rule given by assumption (2) of Section 3.1.

However, under assumption (1), such a distribution does not exist. Indeed, for a static distribution with a non-zero size, the spacetime structure would generate a radial acceleration toward some space point I , located somewhere within this energy distribution. For example, if the distribution is symmetric with respect to O , then this point is $I = O$. Therefore, matter would be accelerated toward I . Any infinitesimal chunk of matter located at M , different from I , would be accelerated along and in the direction of the MI vector. The result is that this is not a static energy distribution.

On the other hand, under assumption (2) of Section 3.1, such static distributions may be

possible because geodesic trajectories can be circles around O . It is more difficult to prove that assumption (2) of Section 3.1 does not allow such a sequence of static distributions converging to P . And it remains challenging to prove that such a sequence exists under this assumption. In Appendix L, this problem will be studied. In this appendix, it will be shown that, under assumption (2) of Section 3.1, there exists a static energy distribution that is extremely close to the one of P (it is important to recall that this is not possible under assumption (1), as shown above).

Therefore, the result is exactly the opposite of what was claimed at the beginning of the present paragraph. It is straightforwardly modeling the Dirac distribution under assumption (1), which is, obviously, not possible. Under assumption (2) of Section 3.1, no such obvious contradiction arises, and some arguments suggest that it might be possible.

Interestingly, this result is very important. It shows that in relativity, assumption (1) is, to say the least, not straightforward. For this replacement, assumption (2) of Section 3.1 might be a viable alternative.

Appendix L. Existence of a static spherically symmetric distribution

Appendix L.1. *Existence of a sequence of static distributions converging to P*

Instead of proving the existence of such a sequence, let us list some specifications for such energy distributions that might converge to P in a static manner.

- A set of particles moving at the speed of light,
- This set is distributed homogeneously around O , following the spherical symmetry around O ,
- At each point M different from O , an infinite set of such particles exists, each moving perpendicularly with respect to OM . This means that the v speed of light of each particle is perpendicular to OM ,
- The distribution of these v vectors is homogeneous in M .

There are many energy distributions that satisfy these constraints. The goal is to find a sequence of such distributions converging to P , while also fulfilling the following constraint.

- The spacetime structure generated by each energy distribution must allow it to remain static.

This work is still in progress. However, the following example will show that assumption (2) of Section 3.1 is better than assumption (1) for modeling P .

Appendix L.2. *Existence of a static distribution close to P under assumption (2)*

Let us try to roughly calculate the radius of the circle in the trajectory of a particle moving at the speed of light, in such a way that the resulting energy distribution would remain static. This radius is roughly the r Schwarzschild radius of the mass M of the particle at rest which is being modeled.

$$r = \frac{2MG}{c^2} \quad (\text{L.1})$$

Let us calculate this using the gravitational constant G given by Surrounding. (Using the usual gravitational value valid in the solar system would yield a very similar result.)

$$G = G_0 \frac{\alpha_0 \rho_0 + \rho_{u0}}{\rho_{15kpc} + \rho_u} \quad (\text{L.2})$$

G_0 is the gravitational constant valid in the solar system. ρ_{15kpc} is the matter density in the sphere centered on the measurement location, with a 15 *kpc* radius. ρ_{u0} is the universe's matter density today, and ρ_u is the universe's matter density at the time of measurement. The result is the following:

$$r \approx 10^{-18} \text{ Planck Length} \quad (\text{L.3})$$

We used $G_0 = 6.67 \times 10^{-11} \text{ m}^3 \text{kg}^{-1} \text{s}^{-2}$, $\alpha_0 = 1.6 \times 10^{-5}$, $\rho_0 = 0.98 \times 10^{-21} \text{ kg m}^{-3}$, $\rho_{15} = \rho_{u0} = \rho_u = 0.05 \rho_c$ with $\rho_c = 8.5 \times 10^{-27} \text{ kg m}^{-3}$, and $PlanckLength = 10^{-35} \text{ m}$. For the mass M , we used the proton mass of $M = 1.67 \times 10^{-27} \text{ kg}$.

The result shows that the following modeling of a proton is possible:

- Modeling a proton as being made of an internal bunch of matter moving at the speed of light along a circular trajectory,
- Using the gravitational constant given by Surrounding in the particular situation of a proton located outside any galaxy at the current time,
- In a static manner, that is, without collapse,
- Without the help of any force except gravitation.

Of course, it has not been demonstrated that the resulting spacetime structure allows this energy distribution to remain static. Nevertheless, since the size of the circle is extremely small, this energy distribution is close to another one, which consists of a circle of matter rotating at the speed of light around its center and in the plane of the circle. This energy distribution generates a symmetric spacetime structure with respect to the straight line passing through the center of the circle and perpendicular to the plane of the circle. Also there is a symmetry with respect to the plane of the circle. At the locations of the circle, the attraction of matter towards O is exactly the one that allows the orbital motion along the circle at the speed of light because the distance to O is r , the Schwarzschild radius. If any doubt exists about the fact that this orbital distance is r , then it is possible to use this orbital distance as the radius of the circle. This orbital distance will be close to r .

Also, this model does not account for the fact that, in reality, a proton is made of quarks. Nevertheless, this modeling is not far from reality. The goal here is purely theoretical: it models a dimensionless particle P at rest in R . The mass of P has simply been assumed to be equal to the mass of a proton.

Therefore, under the above conditions, it is realistic and possible to model a particle with the mass of a proton, using only static distributions, and under assumption (2) of

Section 3.1. This result must be compared with the fact that such a modeling is not possible under assumption (1).

Appendix M. The gravitational wave four-momenta add themselves

Let's suppose that n gravitational waves (GWs) are propagating and encountering each other at a given spacetime event x . Let GW_i represent the i -th gravitational wave, for i ranging from 0 to n . For each i , a P_i particle can be created such that its momentum exactly counteracts the effect of GW_i at x . This means that the P_i speed is the speed of light and that its total energy in any reference frame R is the energy of GW_i in R . It also implies that the direction of P_i 's motion is opposite to that of GW_i . The result is that P_i and GW_i together generate no local spacetime deformation. Therefore, the overall effect of all these P_i particles is such that they cancel the spacetime deformation generated by all the GW_i . But the sum of all the P_i particles forms a large compound object, which we will call P . Thus, the spacetime deformation of all the GW_i is exactly counteracted by the spacetime deformation generated by P . According to the principle of conservation of energy, the four-momentum of P is the sum of the four-momenta of all the P_i . Therefore, the four-momentum of the spacetime deformation of the entire set of GW_i is the sum of the four-momenta of the GW_i .

$$\begin{aligned}
 F_m(\{GW_i, i = 0..n\}) &= -F_m(P) \\
 &= -\sum_{i=0}^n F_m(P_i) \\
 &= -\sum_{i=0}^n (-F_m(GW_i)) \\
 &= \sum_{i=0}^n F_m(GW_i)
 \end{aligned} \tag{M.1}$$

$F_m()$ is a function that gives the four-momentum of a particle or a GW. This concludes the proof.

Appendix N. An equation for G

We adopt the context and notation of 3.12.6. The reference frame $R_0(O; ct, x, y, z)$ is maintained, with particle P situated at origin O , while x represents an arbitrary spacetime event. To formulate an analytical expression for G , we introduce the following four assumptions:

Assumption (i): Newton's law of gravitation remains structurally valid, though G may deviate from its solar-system value.

This assumption is empirically grounded: Newtonian gravity is verified with high precision within the solar system [14]. Theoretical arguments suggest that this functional form persists beyond the solar system, albeit with a spatially varying effective coupling constant G .

Assumption (ii): Spherically propagating gravitational wave (GW) energy obeys a universal attenuation law (governed solely by initial source energy and propagation distance), independent of location or emission energy. This represents a strong assumption.

Assumption (iii): In Eq. (10), the energy contribution generated by P is far smaller than the background sum of all other energy contributions.

This assumption is consistent with the surrounding matter model, where a characteristic radius of $R_{\text{cut}} = 15$ kpc is fitted to experimental data. The energy enclosed within this sphere corresponds to the temporal component on the right-hand side of Eq. (10).

Assumption (iv): The terms in Eq. (10) can be replaced by their asymptotic forms without significantly altering the global result. This assumption is likewise strong.

This approximation holds when particles in the universe are sufficiently isolated from one another—a realistic condition given the extreme sparsity of matter across subatomic to cosmological scales. Assumptions (i)–(iv) naturally become more rigorous in the asymptotic limit.

Nevertheless, while Assumptions (i)–(iv) provide a tractable analytical pathway, they represent an idealized regime. In complex or highly dynamical field configurations, these simplifying conditions alone may prove insufficient to uniquely yield this exact relation without further non-linear refinements. In particular, Assumptions (ii) and (iv) appear overly restrictive; nonetheless, they provide a valuable first approximation for the functional form of G .

The complete mathematical derivation is detailed in Appendix O. Based on Assumptions (i)–(iv), Eqs. (10) and (11), and the geodesic equation corresponding to Newtonian gravity, we obtain the following relation for G :

$$G \simeq \frac{c^4}{2 \left(\sum_{n=0}^{\infty} \sqrt{\frac{E(y_n)}{\|x - y_n\|}} \right)^2}. \quad (\text{N.1})$$

In Eq. (N.1), $E(y_n) = C^0(y_n)c$ denotes the total energy of the particle located at y_n . This expression applies specifically to vacuum gravitational waves (VGWs). As shown in Appendix O, Eq. (N.1) serves as a leading-order approximation under Assumptions (i)–(iv).

The convergence behavior of Eq. (N.1) might initially appear more restrictive than that of Eq. (10), as the $1/\sqrt{r}$ dependence of individual terms suggests potential divergence. However, the resulting gravitational force retains its standard $1/r^2$ spatial dependence by construction, reducing any potential divergence to the classical Olbers-type cosmological boundary problem. A cutoff function Φ_{cut} can be introduced to regularize the summation:

$$\Phi_{\text{cut}}(a, b) = \begin{cases} \sqrt{\frac{a}{b}}, & b \leq R_{\text{cut}}, \\ 0, & b > R_{\text{cut}}, \end{cases} \quad (\text{N.2})$$

yielding the regularized expression:

$$G \simeq \frac{c^4}{2 \left(\sum_{n=0}^{\infty} \Phi_{\text{cut}}(E(y_n), \|x - y_n\|) \right)^2}. \quad (\text{N.3})$$

Here, R_{cut} represents the maximum effective propagation distance of the VGWs, with $R_{\text{cut}} = 15$ kpc suggested by surrounding matter fits [1]. This cutoff does not undermine the asymptotic nature of the derivation; at galactic scales, a gradual damping or smooth attenuation profile can replace the sharp threshold in Eq. (N.2).

Appendix O. An equation for G : calculations

In this appendix, the detailed claculation of an equation for G is derived. Assumptions (i), (ii), (iii), and (iiii) are made. The same context and notations as in paragraph 3.12.6 are used. Let's remind the properties of the Schwarzschild metric. The first equation is as follows.

$$g_{00}(x') = 1 - \frac{R}{r} \quad (\text{O.1})$$

Of course, $g_{00}(x')$ is the time-time component of the metric at the x' event, R is the Schwarzschild radius of P , and r is the distance between P and the x' event. The following is another important equation valid in this metric, which relates the metric to the free-fall speed.

$$g_{00}(x') = 1 - \frac{v^2}{c^2} \quad (\text{O.2})$$

Here, v is the speed of the usual P_1 free-falling test particle, located at x' , which was initially located infinitely far from P . The validity of equation (O.2) comes from assumption (i). Indeed, equations (O.1) and (O.2) are valid together because Newton's law is assumed to be valid. Equation (O.4), which follows, can be used to confirm this statement. From equations (11) and (O.2), using $e = D^1(x)/(D^0(x) - D^1(x))$, the following equation is obtained.

$$g_{00}(x') = \frac{1 + 2e}{(1 + e)^2} \quad (\text{O.3})$$

Furthermore, the geodesic equation of P_1 is as follows [10].

$$\frac{\partial^2 r}{\partial \tau^2} = -\frac{c^2}{2} \frac{\partial g_{00}}{\partial r} \quad (\text{O.4})$$

Here, τ is the proper time of P_1 . Substituting g_{00} with its value given by equation (O.3), equation (O.4) becomes the following.

$$\frac{\partial^2 r}{\partial \tau^2} = c^2 \frac{e}{(1+e)^3} \frac{\partial e}{\partial r} \quad (\text{O.5})$$

Then, once again, assumption (i) is used, and Newton's law is assumed valid. The asymptotic formulation of equation (O.5) is derived, relying on Newton's law with the asymptotic value of the right-hand side of equation (O.5).

$$-\frac{M_0 G}{r^2} \simeq c^2 e \frac{\partial e}{\partial r} \quad (\text{O.6})$$

Here, M_0 is the mass of P . Assumption (iii) has been used: the contribution of P in equation (10) is far weaker than the sum of the other contributions. Therefore, asymptotically $D^1(x) \ll D^0(x)$, and asymptotically e is equal to 0 compared to 1. The solution of this differential equation (O.6) gives the following asymptotic value for e .

$$e \simeq \sqrt{\frac{R}{r}} \quad (\text{O.7})$$

From this, it follows that $D^1(x)/D^0(x) = e/(1+e) \simeq \sqrt{R/r}$. This results in the following equation.

$$\frac{1_w(x, y_0) f(x, y_0) C^0(y_0)}{\sum_{n=0}^{\infty} 1_w(x, y_n) f(x, y_n) C^0(y_n)} \simeq \sqrt{\frac{R}{r}} \quad (\text{O.8})$$

It is assumed that the y_0 virtual particle is the only one associated with P . It was used $C^0(y_0) = C^1(y_0)$, which reflects the fact that $C^\mu(y_0)$ is a null four-momentum.

Under assumption (iii), the denominator of equation (O.8) is constant, that is, independent of the x location. Therefore, this equation shows that the $1_w(x, y_0) f(x, y_0) C^0(y_0)$ contribution is proportional to $1/\sqrt{r}$. Under assumption (ii), it can be deduced that the $1_w(x, y_n) f(x, y_n) C^0(y_n)$ contributions are proportional to $1/\sqrt{\|x - y_n\|_3}$. $\|x - y_n\|_3$ is the spatial length calculated covariantly along the geodesic from y_n to x and observed in R_0 .

Also, assuming r constant, and varying $C^0(y_0)$, from equation (O.8), the $1_w(x, y_0) f(x, y_0) C^0(y_0)$ contribution is not proportional to $C^0(y_0)$ but proportional to $\sqrt{R}$, and therefore to $\sqrt{M_0}$ and $\sqrt{C^0(y_0)}$. This contradicts the hypothesis made during the construction of equation (10). Here, a proportionality to the square root of an energy is observed. The answer to this apparent contradiction is that the final picture is coherent. Therefore, one can replace each $1_w(x, y_n) f(x, y_n) C^0(y_n)$ contribution in equation (O.8) by $1_w(x, y_n) \sqrt{C^0(y_n)}/\|x - y_n\|_3$. Now, from $f(x, y_0) C^0(y_0)$ being proportional to $\sqrt{C^0(y_0)}/r$ and assumption (ii), the result is the following.

$$\frac{1_w(x, y_n) \sqrt{\frac{C^0(y_0)}{r}}}{\sum_{n=0}^{\infty} 1_w(x, y_n) \sqrt{\frac{C^0(y_n)}{\|x - y_n\|_3}}} \simeq \sqrt{\frac{R}{r}} \quad (\text{O.9})$$

We can use $R = 2M_0G/c^2$ and $C^0(y_0) = M_0c$, in order to transform equation (O.9) into the following one.

$$G \simeq \frac{c^4}{2 \left(\sum_{n=0}^{\infty} 1_w(x, y_n) \sqrt{\frac{E(y_n)}{\|x - y_n\|_3}} \right)^2} \quad (\text{O.10})$$

Here, $E(y_n) = C^0(y_n)c$ is the total energy of the particle located at y_n . Now, it is possible to use this result with VGWs. This means that the energy distributions of the P_i are replaced by the limits of the S_n^i distributions, as shown by equation (6). The result is equation (N.1), which is equation (O.10), without the $1_w(x, y_n)$ terms. Indeed, in any given x and y_n spacetime events, it is possible to find a m and a S_m^n distribution centered on y_n , such that the gravitational wave (GW) generated by the virtual particle of S_m^n propagates in x . Moreover, this S_m^n distribution can be chosen with a circle's radius as small as desired (remember that those S_m^n distributions are circular-like trajectories of virtual particles moving at the speed of light). In other words, after the limits of equation (6) are taken, each x spacetime event is reached by the VGW of each P_n particle.

Appendix P. Refinement of the equation for G

We now modify Eq. (N.1).

First, replacing the infinite summation with a finite sum over N constituent particles yields:

$$G \simeq \frac{c^4}{2 \left(\sum_{n=1}^N \sqrt{\frac{E(y_n)}{\|x - y_n\|}} \right)^2}. \quad (\text{P.1})$$

The refinement consists of substituting the term $t_1 = \sqrt{\frac{E(y_n)}{\|x - y_n\|}}$ in the denominator of Eq. (P.1) with:

$$t_2 = \frac{k}{N} \sqrt{\frac{\sqrt{E(y_n)}}{\|x - y_n\|}}, \quad (\text{P.2})$$

where k is a scaling constant defined by $k = \sqrt{\frac{c^3}{2\sqrt{a_0G}}}$, and a_0 denotes the characteristic acceleration constant of MOND phenomenology. This substitution yields:

$$G \simeq \frac{c^4}{2 \left(\sum_{n=1}^N \frac{k}{N} \sqrt{\frac{\sqrt{E(y_n)}}{\|x - y_n\|}} \right)^2}. \quad (\text{P.3})$$

Now we evaluate the gravitational acceleration within a galaxy at a radial distance $r = \|x - y_n\|$. The sum is dominated by the galactic core, as distant contributions become negligible. Assuming that the core particles are localized near the center, that each particle

carries an identical energy e , and that the total core mass is $M = Ne/c^2$, the denominator factor evaluates to:

$$2 \left(\sum_{n=1}^N t_2 \right)^2 = 2N^2 t_2^2 = 2k^2 \frac{\sqrt{Mc^2}}{r}. \quad (\text{P.4})$$

Substituting Eq. (P.4) into the expression for Newtonian acceleration $a_N = \frac{G_0 M}{r^2}$ (where G_0 denotes the bare Newtonian constant) yields the effective acceleration:

$$a = \frac{MG}{r^2} = \frac{M}{r^2} \frac{c^4}{2k^2 \frac{\sqrt{Mc^2}}{r}} = \sqrt{\frac{MG_0}{r^2}} a_0 = \sqrt{a_N a_0} = a_{\text{MOND}}, \quad (\text{P.5})$$

where a_{MOND} corresponds precisely to the characteristic MOND acceleration in the deep-MOND regime.

The substitution in Eq. (P.2) preserves the fundamental $1/\sqrt{r}$ spatial dependence while modifying how the energies of individual VGWs combine. Given the inherently non-linear nature of the Lagrangian formulation determining the attenuation function f in Eq. (10), such a non-linear energy combination rule might naturally arise within this framework.

Appendix Q. Revisiting Newton's law

A revisiting of Einstein's field equations is naturally prompted by the preceding results. Einstein's formulation represents the most direct translation of Newton's law from non-relativistic physics into relativity. Consequently, the foundational step consists of re-examining Newtonian gravity. Poisson's formulation of Newton's law serves as the starting point for constructing Einstein's field equations, not merely owing to its historical precedence, but due to its intrinsic physical conservation properties.

First, in vacuum, Poisson's formulation expresses the following fundamental non-relativistic principle:

(i) *The flux of the acceleration vector field is conserved in vacuum.*

This leads to $\nabla \cdot \vec{a} = 0$ in vacuum, where $\vec{a}$ denotes the acceleration field. Within the framework established in this work—where vacuum spacetime structure is calculated exclusively from VGWs generated by theoretical particles propagating at speed c —this condition acquires a physical interpretation: it corresponds directly to the conservation of VGW energy flux in empty space.

Second, matter acts as the source for this interaction field. The attraction exerted on a particle A by a particle B acts on A along the direction of B , tangent to the spatial geodesic connecting them. Conversely, empty space generates no net force. Therefore, the divergence of the force vector field $\vec{F}$ must be a function of the local matter density ρ , denoted here by $\Phi(\rho)$:

$$\nabla \cdot \vec{F} = \Phi(\rho). \quad (\text{Q.1})$$

Applying the fundamental principle of dynamics ($\vec{F} = m\vec{a}$), where m is the inertial mass of particle A located at spatial location M and $\vec{a}$ is the acceleration generated at M by source density ρ , yields:

$$m(\nabla \cdot \vec{a}) = \Phi(\rho). \quad (\text{Q.2})$$

The Weak Equivalence Principle (WEP) dictates that $\vec{a}$ is independent of test mass m for a fixed source configuration ρ . This requires $\Phi(\rho)$ to be strictly proportional to m , leading to:

$$\nabla \cdot \vec{a} = f(\rho), \quad (\text{Q.3})$$

where $f(\rho)$ is derived directly from $\Phi(\rho)$. Poisson's equation is thus recovered: the divergence of the acceleration field is governed by local energy density, vanishing in empty space:

$$f(0) = 0. \quad (\text{Q.4})$$

Equations (Q.3) and (Q.4) do not uniquely fix the functional form or sign of $f(\rho)$. The question of whether $f(\rho)$ is linear in ρ is tied to energy conservation and the principle of action and reaction. For an isolated particle P in an otherwise empty universe, Eq. (Q.3) integrates over a spherical boundary to:

$$a = \frac{f(\rho)V}{4\pi x^2}, \quad (\text{Q.5})$$

where a is the magnitude of acceleration at position M located at distance $x = \|M - O\|$ from the origin O of particle P , with uniform density ρ and volume V . Applying $\vec{F} = m'\vec{a}$ to a test particle P' of mass m' at M gives:

$$F = \frac{m'f(\rho)V}{4\pi x^2}. \quad (\text{Q.6})$$

Newton's third law requires invariance under the permutation of particles P and P' ($m \leftrightarrow m', \rho \leftrightarrow \rho', V \leftrightarrow V'$), yielding:

$$m'f(\rho)V = mf(\rho')V', \quad (\text{Q.7})$$

where m is the mass of particle P . Since this symmetry must hold for arbitrary particle parameters, $f(\rho)V$ must be proportional to m . (Alternatively, this proportionality follows from demanding invariance of the total interaction energy E under particle interchange.) This linearity is expressed as:

$$f(\rho) = K\rho, \quad (\text{Q.8})$$

where K is a coupling constant. Substituting Eq. (Q.8) into Eq. (Q.6) yields the classical inverse-square relation:

$$F = \frac{Kmm'}{4\pi x^2}. \quad (\text{Q.9})$$

Crucially, no pure theoretical argument within Newtonian mechanics fixes the numerical value of $K/(4\pi) \equiv G$; historically, G was determined empirically. Furthermore, this

derivation relies on the assumption of an absolute vacuum surrounding P and P' . If the ambient environment is non-empty, this idealized model breaks down and surrounding energy distributions must be accounted for:

- (1) Background matter and energy density throughout the universe generate additional divergence via Eq. (Q.3), modifying the resulting field strength in Eq. (Q.6).
- (2) Simple two-body action-reaction symmetry is incomplete; rigorous energy conservation requires evaluating the total energy of the universe (or the particles plus their local surrounding environment).

These foundational considerations demonstrate that Newtonian gravity naturally accommodates a variable effective coupling G modulated by the surrounding energy density.

Appendix R. Revisiting Einstein equation

In the general relativistic regime, an analogous argument applies by replacing the acceleration divergence $\nabla \cdot \vec{a}$ with the Einstein tensor $G_{\mu\nu}$ and matter density ρ with the stress-energy tensor $T_{\mu\nu}$. Because General Relativity represents the direct relativistic extension of Poisson's equation, the non-relativistic argument for an environmentally dependent coupling constant carries over to the relativistic field equations.

Treating $K/(4\pi)$ in Eq. (Q.9) as an immutable universal constant lacks first-principles justification. Instead, the total background energy distribution of the universe plays a direct physical role in determining effective local gravitational coupling. In vacuum ($\rho = 0$), Eqs. (Q.3) and (Q.4) reduce to $\nabla \cdot \vec{a} = 0$, whose exact relativistic counterpart is:

$$R_{\mu\nu} = 0. \quad (\text{R.1})$$

Equation (R.1) remains exact and requires no modification within this framework.

When coupling curvature to matter, introducing a multiplicative, energy-dependent tensor X_ν^μ between $T_{\mu\nu}$ and the Einstein tensor represents a natural generalization of the field equations. Evaluating such a tensor structure against physical observables indicates that X_ν^μ must depend on the surrounding energy density at the location where the gravitational interaction is evaluated, recovering the surrounding matter framework.

Furthermore, in standard General Relativity, setting G to a universal solar-system constant within the matter Lagrangian is an empirical prescription rather than a theoretical necessity. In contrast, the vacuum Einstein-Hilbert Lagrangian—based purely on the scalar curvature R —is theoretical sound, requires no predefined coupling constant, and naturally yields Eq. (R.1) via the variational principle.

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