ON THE TIME COMPLEXITY OF PROBLEM AND SOLUTION SPACES

THEOPHILUS AGAMA

ABSTRACT. We continue with the development of the theory of problems and their solutions spaces [1] and [2]. We introduce and study the notion of verification and resolution time complexity of solutions and problem spaces.

1. **Background**

In [2] and [1] the theory of problems and their solution spaces was studied. Let X denotes a solution (resp. answer) to problem Y (resp. question). Then we call the collection of all problems to be solved to provide solution X to problem Y the problem space induced by providing solution X to problem Y. We denote this space with $\mathcal{P}_Y(X)$. If K is any subspace of the space $\mathcal{P}_Y(X)$, then we denote this relation with $K \subseteq \mathcal{P}_Y(X)$. If the space K is a subspace of the space $\mathcal{P}_Y(X)$ with $K \neq \mathcal{P}_Y(X)$, then we write $K \subset \mathcal{P}_Y(X)$. We say problem V is a sub-problem of problem Y if providing a solution to problem Y furnishes a solution to problem V. If V is a sub-problem of the problem Y, then we write $V \leq Y$. If V is a sub-problem of the problem Y and $V \neq Y$, then we write $V < Y$ and we call V a proper sub-problem of Y.

Similarly, Let X denotes a solution (resp. answer) to problem Y (resp. question). Then we call the collection of all solutions to problems obtained as a result of providing the solution X to problem Y the solution space induced by providing solution X to problem Y. We denote this space with $\mathcal{S}_Y(X)$. If K is any subspace of the space $\mathcal{S}_Y(X)$, then we denote this relation with $K \subset \mathcal{S}_Y(X)$. We make the assignment $T \in \mathcal{S}_Y(X)$ if solution T is also a solution in this space.

Let V be a problem. Then we say V is reducible if there exists a proper sub-problem of V with no proper sub-problem. On the other hand, we say problem V is irreducible if every proper sub-problem of V has a proper sub-problem. Let $\{Y_i\}_{i\geq 1}$ be the sequence of all the sub-problems of V. Then we say V is regular if

$$\cdots \leq Y_3 \leq Y_2 \leq Y_1 \leq V.$$

We say it is irregular if there exists sub-problems Y_j and Y_k of V such that $Y_j \not\leq Y_k$ and $Y_k \not\leq Y_j$. We say problem Y is equivalent to problem V if providing solution to problem Y also provides a solution to problem V and conversely providing a solution to problem V also provides a solution to problem Y. We denote the equivalence with $V \equiv Y$. We say X and U are alternative solutions to Y if and only if U and X both solves Y. We denote this relation with $X \perp U$ or $U \perp X$.

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2. The time complexity

In this section we study the notion of time complexity of problem and solution spaces.

Definition 2.1. The **resolution** complexity of problem T by providing solution U that solves T is the **algorithmic** time required to generate solution U for problem T. We denote this complexity with $\mathcal{C}_r(T,U)$.

Definition 2.2. The **verification** complexity of a solution U to problem T is the **algorithmic** time required to check solution U for correctness. We denote this complexity with $\mathcal{C}_v(T,U)$.

Definition 2.3. Let T be a problem with solution U. We say the time complexity with respect to problem T with solution U is in **equilibrium** if $\mathcal{C}_r(T,U)=\mathcal{C}_v(T,U)$.

It is important to declare that the time complexity is not unique to problems and solutions. More precisely, it is indeed possible that the resolution time complexity and the verification time complexity may differ quite significantly among equivalent problems and alternative solutions. Consequently, it may not be possible to extend an equilibrium to equivalent problems and alternative solutions. Let us suppose that $\mathcal{C}_r(T_1,U_1)<\infty$ and $\mathcal{C}_v(T_1,U_1)<\infty$ with $T_1\equiv T_2$ (equivalent problems) then $U_1\perp U_2$ (alternative solution). It is possible that

$$\mathcal{C}_r(T_1,U_1)\neq \mathcal{C}_r(T_2,U_1)$$

and

$$\mathcal{C}_v(T_1,U_1)\neq \mathcal{C}_v(T_2,U_1)$$

and similarly

$$\mathcal{C}_r(T_2,U_2)\neq \mathcal{C}_r(T_2,U_1)$$

and

$$\mathcal{C}_v(T_2,U_2)\neq \mathcal{C}_v(T_2,U_1).$$

Hence if $\mathcal{C}_r(T_1,U_1)=\mathcal{C}_v(T_1,U_1)$ and $T_1\equiv T_2$ then the equilibrium

$$\mathcal{C}_r(T_2,U_2)=\mathcal{C}_v(T_2,U_2)$$

may only hold under certain condition. We begin by verifying that time complexity can be ordered up to sub-problems and sub-solutions of a given problem.

Proposition 2.1. *Let T be a problem with solution U. Let $\{T_i\}_{i\geq 1}$ and $\{U_i\}_{i\geq 1}$ denotes the sequence of all sub-problems and sub-solutions of T and U, respectively. If $\mathcal{C}_r(T,U)<\infty$ and $\mathcal{C}_v(T,U)<\infty$, then we have*

$$\mathcal{C}_r(T_i,U_i)<\mathcal{C}_r(T,U)$$

and

$$\mathcal{C}_v(T_i,U_i)<\mathcal{C}_v(T,U)$$

for each $i\geq 1$.

Proof. Since $\mathcal{C}_r(T,U)<\infty$ and $\mathcal{C}_v(T,U)<\infty$ and

$$\mathcal{C}_r(T,U):=\sum_{i\geq 1}\mathcal{C}_r(T_i,U_i)$$

and

$$\mathcal{C}_v(T,U) := \sum_{i\geq 1} \mathcal{C}_v(T_i,U_i)$$

the inequality follows easily. □

Remark 2.4. In cases where we do not want to make a reference to the solution and a problem in the notation of the resolution and the verification time complexity, we will write for simplicity $\mathcal{C}_r(T)$ and $C_v(U)$. We will adopt this notation in situations where a reference to a problem or a solution turns out to be irrelevant.

Proving the existence of equilibrium of time complexity of problems is by no means an easy endeavour. In the sequel we prove that assuming equilibrium in the time complexity can be passed down to sub-problems and sub-solutions. We make these ideas formal in the proposition below.

Proposition 2.2. *Let T be a regular problem with solution U such that for any sub-problems T_i, T_j with $i \neq j$, then $\mathcal{C}_r(T_i,U_i) \neq \mathcal{C}_v(T_j,U_j)$. If $\mathcal{C}_r(T,U) = \mathcal{C}_v(T,U)$, then there exists $Q \leq T$ (Q a sub-problem of T) and $L \leq U$ (L a sub-solution of U) that solves Q such that $\mathcal{C}_r(Q,L) = \mathcal{C}_v(Q,L)$.*

Proof. Suppose T is a regular problem with solution U. Let $\{T_i\}_{i\geq 1}$ be the sequence of all sub-problems of T with corresponding sequence of solutions $\{U_i\}_{i\geq 1}$. Suppose on the contrary that $\mathcal{C}_r(T_i,U_i) = \mathcal{C}_v(T_i,U_i)$ for each $i \geq 1$. By virtue of the regularity of T, we can arrange the sequence of sub-problems and sub-solutions in the following way $T_1 \geq T_2 \geq \cdots$ and the corresponding sequence of sub-solutions $U_1 \geq U_2 \geq \cdots$, where each preceding T_i is a sub-problem of T_{i-1} and similarly each U_i is a sub-solution for U_{i-1}. Since problem T is said to be solved by providing a solution to each of the sub-problems, we find under the assumption $\mathcal{C}_r(T,U) = \mathcal{C}_v(T,U)$, that

$$\mathcal{C}_r(T,U) = \sum_{i\geq 1} \mathcal{C}_r(T_i,U_i) = \sum_{i\geq 1} \mathcal{C}_v(T_i,U_i) = \mathcal{C}_v(T,U).$$

Now suppose on the contrary that $\mathcal{C}_r(T_1,U_1) \neq \mathcal{C}_v(T_1,U_1)$, then under the regularity condition, it follows that

$$\sum_{i\geq 2} \mathcal{C}_r(T_i,U_i) \neq \sum_{i\geq 2} \mathcal{C}_v(T_i,U_i)$$

since providing a solution to all sub-problems of T_2 solves problem T_2. Under the requirement that $\mathcal{C}_r(T_i,U_i) \neq \mathcal{C}_v(T_j,U_j)$ for all $i \neq j$, it follows that

$$\mathcal{C}_r(T,U) = \sum_{i\geq 1} \mathcal{C}_r(T_i,U_i) \neq \sum_{i\geq 1} \mathcal{C}_v(T_i,U_i) = \mathcal{C}_v(T,U)$$

violating the assumption that $\mathcal{C}_r(T,U) = \mathcal{C}_v(T,U)$. □

Theorem 2.5. *Let T be a regular problem with a solution K. If M is the maximal sub-problem of T with a solution L and $\mathcal{C}_r(M,L) \ll$ polynomial time and $\mathcal{C}_r(T,K) = \mathcal{C}_v(T,K)$, then $\mathcal{C}_v(T,K) \ll$ polynomial time.*

Proof. Suppose T is a regular problem and let $\{T_i\}_{i\geq 1}$ denotes the sequence of all sub-problems of T with corresponding sequence of sub-solutions $\{K_i\}_{i\geq 1}$ where each K_i solves T_i. We can arrange the sequence of sub-problems in the following way: $T_1 \geq T_2 \geq \cdots$ where $T_1 := M$ is the maximal sub-problem of T and where each sub-problem T_i is a sub-problem of T_{i-1} for $i \geq 2$. Since problem T is solved by solving each of the sub-problems in the sequence, we can write

$$\begin{aligned} C_r(T,K) &= \sum_{i\geq 1} C_r(T_i, K_i) \\ &= C_r(T_1, K_1) + \sum_{i\geq 2} C_r(T_i, K_i). \end{aligned}$$

By the regularity of problem T, we see that

$$\sum_{i\geq 2} C_r(T_i, K_i) = C_r(T_1, K_1) \ll \textit{polynomial time}.$$

Thus $C_r(T,K) \ll$ *polynomial time*. Under the equality $C_r(T,K) = C_v(T,K)$, we deduce that $C_v(T,K) \ll$ *polynomial time*, which completes the proof of the theorem. □

Remark 2.6. Theorem 2.5 is an important ingredient for exploring a deep understanding of the P=NP problem. It purports that once there exist an equilibrium of time complexity of a given problem, it suffices to only investigate the resolution complexity of the maximal sub-problem for a class of well-behaved problems which we refer to as regular problems, introduced and studied in [2].

Although the task of proving equilibrium of resolution and verification time complexity can be very hard, we can often carry out this process from bottom-up. That is to say, proving equilibrium of time complexity for sub-problems can be extended to time complexity equilibrium of the actual problem. The following proposition exemplifies that principle.

Proposition 2.3. *Let Y be a problem with solution X and let $\{Y_i\}_{i\geq 1}$ and $\{X_i\}_{i\geq 1}$ denotes the sequence of all proper sub-problems and a solutions to sub-problems of Y. If $\mathcal{C}_r(Y_i, X_i) = \mathcal{C}_v(Y_i, X_i)$ for each $i \geq 1$, then $\mathcal{C}_r(Y, X) = \mathcal{C}_v(Y, X)$.*

Proof. The sequences $\{Y_i\}_{i\geq 1}$ and $\{X_i\}_{i\geq 1}$ denotes the sequence of all proper sub-problems and a solutions to sub-problems of Y, respectively. Since the solution to problem Y is furnished solving each of the sub-problems in $\{Y_i\}_{1\geq 1}$, it follows under the assumption $\mathcal{C}_r(Y_i, X_i) = \mathcal{C}_v(Y_i, X_i)$ for each $i \geq 1$ that

$$\mathcal{C}_r(Y, X) = \sum_{i\geq 1} \mathcal{C}_r(Y_i, X_i) = \sum_{i\geq 1} \mathcal{C}_v(Y_i, X_i) = \mathcal{C}_v(Y, X).$$

□

We now obtain an important characterization of irreducible problems, a class of problems introduced and studied in [2].

Theorem 2.7. *If X is an irreducible problem, then $\mathcal{C}_r(X) = \infty$ or X is not solvable.*

Proof. Suppose X is an irreducible problem and assume the contrary that $\mathcal{C}_r(X) < \infty$ and that X is solvable. Since X is irreducible, each sub-problem $X_j \leq X$ has a proper sub-problem, and problem X has infinitely many proper sub-problems $X_i < X$. Thus

$$\mathcal{C}_r(X) := \sum_{i=1}^{\infty} \mathcal{C}_r(X_i) < \infty$$

since problem X is solved by providing a solution to each of the sub-problems. This implies that for any $\epsilon > 0$, there exists some $N := N(\epsilon)$ such that for all $i \geq N$ we have

$$\sum_{i=N}^{\infty} \mathcal{C}_r(X_i) < \epsilon.$$

That is, $\mathcal{C}_r(X_i) \longrightarrow 0$ as $i \longrightarrow \infty$. This means the algorithmic time required to solve infinitely many proper sub-problems of problem X converges to zero, which violates the assumption that X is solvable. □

The difficulty of proving equilibrium of time complexity of a given problem may be made easier depending on its structure. Irregular problems seem to be very difficult to understand and unfortunately most problems fall into this category. It is however much easier to establish an equilibrium for a class of well behaved problems that fall into the category of reducible and regular problems. It turns out that once equilibrium is reached for the finest form of this problem, then equilibrium will certainly be attained for the actual problem. We make this discussion formal in the following results.

Theorem 2.8 (extension principle)**.** *Let T be a regular and a reducible problem with solution U. If T_k is a sub-problem of T with solution U_k such that there exist no $T_j \in \{T_i\}_{i\geq 1}$ with $T_j \not< T_k$ and that $\mathcal{C}_r(T_k, U_k) = \mathcal{C}_v(T_k, U_k)$, then $\mathcal{C}_r(T, U) = \mathcal{C}_v(T, U)$.*

Proof. Suppose T is a regular problem with solution U and let $\{T_i\}_{i\geq 1}$ be the sequence of all sub-problems of T with the corresponding sequence of solutions $\{U_i\}_{i\geq 1}$, where each U_i solves T_i for each $i \geq 1$. Since T is reducible, it has a sub-problem with no proper sub-problem. Let T_k be this sub-problem of T, then by the regularity of T, we can arrange the sequence of all sub-problems of T in the following way:

$$T_k \leq T_{k-1} \leq T_{k-2} \leq \cdots \leq T_1$$

with

$$U_k \leq U_{k-1} \leq U_{k-2} \leq \cdots U_1$$

where each T_i is a sub-problem of T_{i-1} and U_i is a sub-solution of U_{i-1}. Under the equilibrium $\mathcal{C}_r(T_k, U_k) = \mathcal{C}_v(T_k, U_k)$ and since problem T_{k-1} is solved by providing a solution to all of its proper sub-problems, it follows that $\mathcal{C}_r(T_{k-1}, U_{k-1}) = \mathcal{C}_v(T_{k-1}, U_{k-1})$. Similarly, problem T_{k-2} is solved by providing a solution to all of its sub-problems and it follows that

$$\begin{aligned}\mathcal{C}_r(T_{k-2}, U_{k-2}) &= \mathcal{C}_r(T_k, U_k) + \mathcal{C}_r(T_{k-1}, U_{k-1}) \\ &= \mathcal{C}_v(T_k, U_k) + \mathcal{C}_v(T_{k-1}, U_{k-1}) \\ &= \mathcal{C}_v(T_{k-2}, U_{k-2}).\end{aligned}$$

We can iterate this process to reach the equilibrium $\mathcal{C}_r(T, U) = \mathcal{C}_v(T, U)$. □

Corollary 2.1. Let T be a regular and a reducible problem with solution U. Let T_k is a sub-problem of T with solution U_k such that there exist no $T_j \in \{T_i\}_{i\geq 1}$ with $T_j \not< T_k$ and that $\mathcal{C}_r(T_k, U_k) = \mathcal{C}_v(T_k, U_k)$. If $\mathcal{C}_v(T, U) \ll$ *polynomial time* then $\mathcal{C}_r(T, U) \ll$ *polynomial time*.

Proof. It follows from Theorem 2.8 that $\mathcal{C}_r(T, U) = C_v(T, U)$ so that under the hypothesis $\mathcal{C}_v(T, U) \ll$ *polynomial time* then $\mathcal{C}_r(T, U) \ll$ *polynomial time*. ☐

Remark 2.9. Corollary 2.1 suggests that under a certain mild condition, if a certain class of well-behaved problems have a solution that are easy to verify for correctness then they must also be easy to solve at the same level.

3. The time complexity of problem and solution spaces

In this section, we study the notion of time complexity on problem and solutions spaces, as opposed to a specific problem and its solution.

Definition 3.1. Let $\mathcal{P}_Y(X)$ and $\mathcal{S}_Y(X)$ be the problem and solution spaces induced by providing solution X to problem Y. Then by the resolution complexity of the problem space $\mathcal{P}_Y(X)$, we mean the sum of each resolution complexity of each problem in the space. For each problem $T \in \mathcal{P}_Y(X)$ there exists a solution $L \in \mathcal{S}_Y(X)$ that solves T. We denote the resolution complexity of the space with

$$\mathcal{P}_Y^r(X) := \sum_{\substack{T \in \mathcal{P}_Y(X) \\ L \in \mathcal{S}_Y(X)}} \mathcal{C}_r(T, L)$$

and the verification complexity with

$$\mathcal{S}_Y^v(X) := \sum_{\substack{L \in \mathcal{S}_Y(X) \\ T \in \mathcal{P}_Y(X)}} \mathcal{C}_v(T, L).$$

Proposition 3.1. *Let $\mathcal{P}_Y(X)$ and $\mathcal{S}_Y(X)$ be the problem and solution spaces induced by providing solution X to problem Y. If for each $T \in \mathcal{P}_Y(X)$ and each $L \in \mathcal{S}_Y(X)$ that solves T, $\mathcal{C}_r(T, L) = \mathcal{C}_v(T, L)$ then $\mathcal{P}_Y^r(X) = \mathcal{S}_Y^v(X)$.*

Proof. This follows trivially from the proof of Proposition 2.3. ☐

This phase of the project is much focused on the time complexity of problems and their solutions, given its profound relation with the well studied P vs NP problem in computer science. The theory as developed in [2] and [1] bears some connection to classical abstract algebra, module theory and functional analysis. As such certain concepts that appears in those areas could have their corresponding analogue in this theory and could offer a better understanding of the problem. The notion of **compactness**, **density**, **convexity** and **boundedness** could have their appropriate analogue in this theory, all of which are relegated to subsequent studies.

References

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Department of Mathematics, African institute for mathematical sciences, Ghana, Cape-coast