

# The proof of the Riemann conjecture

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**Abstract:** In order to strictly prove the hypothesis and conjectures in Riemann's 1859 paper on the Number of Prime Numbers Not greater than  $x$  from a pure mathematical point of view, and in order to strictly prove the Generalized hypothesis and the Generalized conjectures, this paper uses Euler's formula to study the relationship between symmetric and conjugated zeros of Riemann's  $\zeta(s)$  function and Riemann's  $\xi(s)$  function, and proves that Riemann's hypothesis and Riemann's conjecture are completely correct.

**Key words:** Euler's formula, Riemann  $\zeta(s)$  function, Riemann function  $\xi(t)$ , Riemann hypothesis, Riemann conjecture, symmetric zeros, conjugate zeros, uniqueness.

## I. Introduction

Riemann hypothesis and Riemann conjecture are an important and famous mathematical problem left by Riemann in his paper "On the Number of prime Numbers not greater than  $x$ "<sup>[1]</sup>, which is of great significance for the study of prime number distribution and known as the biggest unsolved mystery in mathematics. After years of hard work, I have solved this problem and strictly prove the Generalized hypothesis and the Generalized conjectures, The research shows that the Riemann hypothesis and the Riemann conjecture and the Generalized Riemann hypothesis and the Generalized Riemann conjecture are all completely valid and the Polignac conjecture, twin prime conjecture and Goldbach conjecture are completely true.

## II. Reasoning

### Lemma 1:

$\sum_{n=1}^{\infty} n^{-s} = \prod_p (1 - p^{-s})^{-1}$  ( $n \in \mathbb{Z}_+$ ,  $p \in \mathbb{Z}_+$ ,  $s \in \mathbb{C}$ ,  $n$  goes through all the natural numbers,  $p$  goes through all the prime numbers), this formula was proposed and proved by the Swiss mathematician Leonhard Euler in 1737 in a paper entitled "Some Observations on Infinite Series", Euler's product formula connects a summation expression for natural numbers with a continuative product expression for prime numbers, and contains important information about the distribution of prime numbers. This information was finally deciphered by Riemann after a long gap of 122 years, which led to Riemann's famous paper "On the Number of primes less than a Given Value"<sup>[1]</sup>. In honor of Riemann, the left end of the Euler product formula was named after Riemann, and the notation  $\zeta(s)$  used by Riemann was adopted as the Riemann zeta function .

Because  $e = 2.718281828459045\dots$ ,  $e$  is a natural constant, I use "  $\times$  " for Multiplication, then based on Euler's  $e^{ix} = \cos x + i \sin(x)$  ( $x \in \mathbb{R}$ ),

get  $(e^{3i})^2 = (\cos(3) + i \sin(3))^2 = \cos(2 \times 3) + i \sin(2 \times 3) = \cos(6) + i \sin(6)$ ,

because  $e^{6i} = \cos(6) + i \sin(6)$ ,

so

$$(e^{3i})^2 = e^{6i},$$

In general,

$$(e^{bi})^c = e^{b \times ci} (b \in \mathbb{R}, c \in \mathbb{R}) \text{ is established.}$$

When  $x > 0 (x \in \mathbb{R})$ , suppose  $e^j = x (e = 2.718281828459045\dots, x \in \mathbb{R} \text{ and } x > 0, j \in \mathbb{R})$ , then  $j = \ln(x)$ , based on

euler's  $e^{ix} = \cos(x) + i\sin(x) (x \in \mathbb{R})$ , will get

$$e^{ji} = e^{\ln(x)i} = \cos(\ln x) + i\sin(\ln x) (x \in \mathbb{R} \text{ and } x > 0).$$

suppose  $y \in \mathbb{R}$  and  $y \neq 0$ , now let's figure out expression for  $x^{yi} (x \in \mathbb{R} \text{ and } x > 0, y \in \mathbb{R} \text{ and } y \neq 0)$  is  $x^{yi} = (e^j)^{yi} = (e^{ji})^y = (\cos(\ln x) + i\sin(\ln x))^y$ .

Suppose  $s$  is any complex number, and  $s = \rho + yi (\rho \in \mathbb{R}, y \in \mathbb{R} \text{ and } y \neq 0, s \in \mathbb{C})$ , then let's find the expression of  $x^s (x \in \mathbb{R} \text{ and } x > 0, s \in \mathbb{C})$ ,

You put  $s = \rho + yi (\rho \in \mathbb{R}, y \in \mathbb{R} \text{ and } y \neq 0, s \in \mathbb{C})$  and  $x^{yi} = (e^j)^{yi} = (e^{ji})^y = (\cos(\ln x) + i\sin(\ln x))^y$  into  $x^s$  and you will get

$$x^s = x^{(\rho + yi)} = x^\rho x^{yi} = x^\rho (\cos(\ln x) + i\sin(\ln x))^y = x^\rho (\cos(y \ln x) + i\sin(y \ln x)), \text{ if You put } s = \rho - yi (\rho \in \mathbb{R}, y \in \mathbb{R} \text{ and } y \neq 0, s \in \mathbb{C}) \text{ and } x^{yi} = (e^j)^{yi} = (e^{ji})^y = (\cos(\ln x) + i\sin(\ln x))^y \text{ into } x^s, \text{ you will get } x^{\bar{s}} = x^{(\rho - yi)} = x^\rho (x^{yi})^{-1} = x^\rho (\cos(\ln x) + i\sin(\ln x))^{-y} = x^\rho (\cos(-y \ln x) + i\sin(-y \ln x)) = x^\rho (\cos(y \ln x) - i\sin(y \ln x)),$$

Then

$$\begin{aligned} \zeta(s) &= \sum_{n=1}^{\infty} \frac{1}{x^s} = \sum_{n=1}^{\infty} \frac{1}{n^{\rho + yi}} = \sum_{n=1}^{\infty} \left( \frac{1}{n^\rho} \times \frac{1}{n^{yi}} \right) = \sum_{n=1}^{\infty} (n^{-\rho}) \frac{1}{(\cos(\ln n) + i\sin(\ln n))^y} \\ &= \sum_{n=1}^{\infty} (n^{-\rho} (\cos(\ln n) + i\sin(\ln n))^{-y}) = \sum_{n=1}^{\infty} (n^{-\rho} (\cos(y \ln n) - i\sin(y \ln n))), \end{aligned}$$

$$\zeta(s) = \prod_{p=1}^{\infty} \left( \frac{1}{1 - p^{-s}} \right) = \prod_{p=1}^{\infty} (1 - p^{-s})^{-1} = \prod_{p=1}^{\infty} (1 - p^{-\rho - yi})^{-1} = \prod_{p=1}^{\infty} \left( 1 - \frac{1}{p^{\rho + yi}} \right)^{-1} =$$

$$\prod_{p=1}^{\infty} \left[ 1 - (p^{-\rho}) \frac{1}{(\cos(\ln p) + i\sin(\ln p))^y} \right]^{-1} = \prod_{p=1}^{\infty} \left[ 1 - (p^{-\rho}) (\cos(y \ln p) - i\sin(y \ln p)) \right]^{-1},$$

And

$$\begin{aligned} \zeta(\bar{s}) &= \sum_{n=1}^{\infty} \frac{1}{x^{\bar{s}}} = \sum_{n=1}^{\infty} \frac{1}{n^{\rho - yi}} = \sum_{n=1}^{\infty} \left( \frac{1}{n^\rho} \times \frac{1}{n^{-yi}} \right) = \sum_{n=1}^{\infty} (n^{-\rho}) \frac{1}{(\cos(\ln n) + i\sin(\ln n))^{-y}} \\ &= \sum_{n=1}^{\infty} (n^{-\rho} (\cos(\ln n) + i\sin(\ln n))^y) = \sum_{n=1}^{\infty} (n^{-\rho} (\cos(y \ln n) + i\sin(y \ln n))), \end{aligned}$$

$$\zeta(\bar{s}) = \prod_{p=1}^{\infty} \left( \frac{1}{1 - p^{-\bar{s}}} \right) = \prod_{p=1}^{\infty} (1 - p^{-\bar{s}})^{-1} = \prod_{p=1}^{\infty} (1 - p^{-\rho + yi})^{-1} = \prod_{p=1}^{\infty} \left( 1 - \frac{1}{p^{\rho - yi}} \right)^{-1} =$$

$$\prod_{p=1}^{\infty} \left[ 1 - (x^{-\rho}) \frac{1}{(\cos(\ln p) - i\sin(\ln p))^y} \right]^{-1} = \prod_{p=1}^{\infty} \left[ 1 - (p^{-\rho}) (\cos(y \ln p) + i\sin(y \ln p)) \right]^{-1},$$

and

$$\begin{aligned}\zeta(1-s) &= \sum_{n=1}^{\infty} \frac{1}{n^{1-s}} = \sum_{n=1}^{\infty} \frac{1}{n^{1-\rho-yi}} = \sum_{n=1}^{\infty} (n^{\rho-1}) \frac{1}{(\cos(\ln n) + i \sin(\ln n))^{-y}} \\ &= \sum_{n=1}^{\infty} (n^{\rho-1})(\cos(\ln n) + i \sin(\ln n))^y = \sum_{n=1}^{\infty} (n^{\rho-1})(\cos(y \ln n) + i \sin(y \ln n)),\end{aligned}$$

So

$$X = x^{-\rho}(\cos(y \ln x) - i \sin(y \ln x)) = n^{-\rho}(\cos(y \ln n) - i \sin(y \ln n)),$$

$$Y = x^{-\rho}(\cos(y \ln x) + i \sin(y \ln x)) = n^{-\rho}(\cos(y \ln n) + i \sin(y \ln n)),$$

$$G = [1 - (p^{-\rho})(\cos(y \ln p) - i \sin(y \ln p))]^{-1},$$

$$H = [1 - (p^{-\rho})(\cos(y \ln p) + i \sin(y \ln p))]^{-1},$$

X and Y are complex conjugates of each other, that is

$$X = \overline{Y}, \text{ and}$$

G and H are complex conjugates of each other, that is

$$G = \overline{H}, \text{ so}$$

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{x^s} = \sum_{n=1}^{\infty} X = \prod_{p=1}^{\infty} G, \text{ and } \zeta(\overline{s}) = \sum_{n=1}^{\infty} \frac{1}{x^{\overline{s}}} = \sum_{n=1}^{\infty} Y = \prod_{p=1}^{\infty} H,$$

then

$$\zeta(s) = \overline{\zeta(\overline{s})} \quad (s \in \mathbb{C} \text{ and } s \neq 1).$$

As Riemann said in his paper, x takes all the natural numbers, so  $x=1,2,3\dots n-1,n \dots$ , Let's just plug in all the natural numbers,

Obviously,

$$\zeta(s) = \zeta(\rho + yi) = \sum_{x^s} \frac{1}{x^s} = \sum X = [1^{-\rho} \cos(y \ln 1) + 2^{-\rho} \cos(y \ln 2) + 3^{-\rho} \cos(y \ln 3) + 4^{-\rho} \cos(y \ln 4) + \dots] - i[1^{-\rho} \sin(y \ln 1) + 2^{-\rho} \sin(y \ln 2) + 3^{-\rho} \sin(y \ln 3) + 4^{-\rho} \sin(y \ln 4) + \dots] = U - Vi,$$

$$U = [1^{-\rho} \cos(y \ln 1) + 2^{-\rho} \cos(y \ln 2) + 3^{-\rho} \cos(y \ln 3) + 4^{-\rho} \cos(y \ln 4) + \dots],$$

$$V = [1^{-\rho} \sin(y \ln 1) + 2^{-\rho} \sin(y \ln 2) + 3^{-\rho} \sin(y \ln 3) + 4^{-\rho} \sin(y \ln 4) + \dots],$$

Then

$$\zeta(\overline{s}) = \zeta(\rho - yi) = \sum_{x^{\overline{s}}} \frac{1}{x^{\overline{s}}} = \sum Y = [1^{-\rho} \cos(y \ln 1) + 2^{-\rho} \cos(y \ln 2) + 3^{-\rho} \cos(y \ln 3) + 4^{-\rho} \cos(y \ln 4) + \dots] + i[1^{-\rho} \sin(y \ln 1) + 2^{-\rho} \sin(y \ln 2) + 3^{-\rho} \sin(y \ln 3) + 4^{-\rho} \sin(y \ln 4) + \dots] = U + Vi,$$

$$U = [1^{-\rho} \cos(y \ln 1) + 2^{-\rho} \cos(y \ln 2) + 3^{-\rho} \cos(y \ln 3) + 4^{-\rho} \cos(y \ln 4) + \dots],$$

$$V = [1^{-\rho} \sin(y \ln 1) + 2^{-\rho} \sin(y \ln 2) + 3^{-\rho} \sin(y \ln 3) + 4^{-\rho} \sin(y \ln 4) + \dots],$$

$$\zeta(1-s) = \sum (x^{\rho-1})(\cos(y \ln x) + i \sin(y \ln x)) = [1^{\rho-1} \cos(y \ln 1) + 2^{\rho-1} \cos(y \ln 2) + 3^{\rho-1} \cos(y \ln 3) + 4^{\rho-1} \cos(y \ln 4) + \dots] + i[1^{\rho-1} \sin(y \ln 1) + 2^{\rho-1} \sin(y \ln 2) + 3^{\rho-1} \sin(y \ln 3) + 4^{\rho-1} \sin(y \ln 4) + \dots], \text{ so only when}$$

$$\rho = \frac{1}{2} \text{ and } \zeta(s) = 0, \text{ then It must be true that } \zeta(1-s) = \overline{\zeta(s)} = 0.$$

$$\zeta(s) \text{ and } \zeta(\overline{s}) \text{ are complex conjugates of each other, that is } \zeta(s) = \overline{\zeta(\overline{s})},$$

$$\text{if } \zeta(s) = 0, \text{ then must } \zeta(\overline{s}) = 0, \text{ so if } \zeta(s) = 0, \text{ it must be true that } \zeta(s) = \zeta(\overline{s}) = 0.$$

According to Riemann's paper "On the Number of primes not Greater than x", we can obtain an

expression  $\zeta(1-s) = 2^{1-s} \pi^{-s} \cos\left(\frac{\pi s}{2}\right) \Gamma(s) \zeta(s)$  ( $s \in \mathbb{C}$  and  $s \neq 1$ ) in relation to the Riemann  $\zeta(s)$  function,

which has long been known to modern mathematicians, and which I derive later.

Base on  $\zeta(1-s) = 2^{1-s} \pi^{-s} \cos\left(\frac{\pi s}{2}\right) \Gamma(s) \zeta(s)$  ( $s \in \mathbb{C}$  and  $s \neq 1$ ), then  $\zeta(1-s) = \overline{\zeta(s)} = 0$  ( $s \in \mathbb{C}$  and  $s \neq 1$ ).

Beacause only when  $\rho=\frac{1}{2}$ ,the next three equations,  $\zeta(\rho + yi)=0$ ,  $\zeta(1 - \rho - yi)=0$ , and  $\zeta(\rho-yi)=0$  are all true,so only  $s = \frac{1}{2} + yi$  ( $y \in \mathbb{R}$  and  $y \neq 0, s \in \mathbb{C}$ ) is true, or say  $s = \frac{1}{2} + ti$  ( $t \in \mathbb{R}$  and  $t \neq 0, s \in \mathbb{C}$ ) is true. According the equation  $\zeta(1-s)=2^{1-s}\pi^{-s}\text{Cos}(\frac{\pi s}{2})\Gamma(s)\zeta(s)$  obtained by Riemann, when  $\zeta(s)=0$ , then  $\zeta(1-s)=0$ ,so It must be true that  $\zeta(1-s)=\zeta(s)=0$ , or say It must be true that  $\zeta(1-s)=\zeta(\overline{s})=0$ , so only  $s = \frac{1}{2} + yi$  ( $y \in \mathbb{R}$  and  $y \neq 0, s \in \mathbb{C}$ ) is true, or say only  $s = \frac{1}{2} + ti$  ( $t \in \mathbb{R}$  and  $t \neq 0, s \in \mathbb{C}$ ) is true.

When  $\zeta(1-\overline{s})=\overline{\zeta(1-s)}=0$ , and according  $\zeta(s)=2^s\pi^{s-1}\text{Sin}(\frac{\pi s}{2})\Gamma(1-s)\zeta(1-s)$ , then

Only  $\zeta(s)=\overline{\zeta(\overline{s})}=0$ ,is also say  $\zeta(s)=\zeta(\overline{s})=\zeta(1-\overline{s})=0$ . so only  $\zeta(\rho+yi)=\zeta(\rho-yi)=0$  is true.

According the equation  $\zeta(1-s)=2^{1-s}\pi^{-s}\text{Cos}(\frac{\pi s}{2})\Gamma(s)\zeta(s)$  obtained by Riemann,since Riemann has shown that the Riemann  $\zeta(s)$  function has zero, that is, in  $\zeta(1-s)=2^{1-s}\pi^{-s}\text{Cos}(\frac{\pi s}{2})\Gamma(s)\zeta(s)$ ,  $\zeta(s)=0$  is true, so when  $\zeta(s)=0$ , then only  $\zeta(s)=\zeta(1-s)=0$  is true.

in the process of the Riemann hypothesis proved about  $\zeta(s)=\zeta(1-s)=\zeta(\overline{s})=0$ , is refers to the  $\zeta(s)$  is a functional number? It's not. Does  $\zeta(s)=\zeta(1-s)=\zeta(\overline{s})$  mean the symmetry of the  $\zeta(s)$  function equation?

Does that mean the symmetry of the equation  $s=\overline{s}=1-s$ ? Not really. In my analyst,  $\zeta(s)$ 、 $\zeta(1-s)$  and  $\zeta(\overline{s})$

function expression is the same, are  $\sum_{n=1}^{\infty} n^{-s}$  (n traves all positive integer,  $s \in \mathbb{C}$ ), so according to

$\sum_{n=1}^{\infty} n^{-s}$  (n traves all positive integer,  $s \in \mathbb{C}$ ),  $\zeta(s)$  function of the independent variable  $s$ , the relationship between  $\overline{s}$  and  $1-s$  only  $\mathbb{C}_3^2=3$  kinds, namely  $s=\overline{s}$  or  $s=1-s$  or  $\overline{s}=1-s$ . As follows:

According  $\zeta(s)=\zeta(1-s)=0$  and  $\zeta(s)=\zeta(\overline{s})=\zeta(1-s)=0$ ,then only  $s=\overline{s}$  or  $s=1-s$  or  $\overline{s}=1-s$ ,so  $s \in \mathbb{R}$ , or  $\rho+yi=1-\rho-yi$ , or  $\rho-yi=1-\rho-yi$ , so  $s \in \mathbb{R}$ , or  $\rho=\frac{1}{2}$  and  $y=0$ , or  $\rho = \frac{1}{2}$  and  $y \in \mathbb{R}$  and  $y \neq 0$ ,so  $s \in \mathbb{R}$ , for example  $s=-2n(n \in \mathbb{Z}_+)$ , or  $s=\frac{1}{2}+oi$ , or  $s=\frac{1}{2}+yi(y \in \mathbb{R}$  and  $y \neq 0)$ .  $\zeta(\frac{1}{2}) > \zeta(1) > 0$ ,drop it,  $s=-2n(n \in \mathbb{Z}_+)$ ,It's the trivial zero of the Riemann  $\zeta(s)$  function, drop it.

Beacause only when  $\rho=\frac{1}{2}$ ,the next three equations,  $\zeta(\rho + yi)=0$ ,  $\zeta(1 - \rho - yi)=0$ , and  $\zeta(\rho-yi)=0$  are

all true,  $\zeta(\frac{1}{2}) > \zeta(1) > 0$ ,so only  $s=\frac{1}{2}+yi(y \in \mathbb{R}$  and  $y \neq 0, s \in \mathbb{C})$  is true, or say only  $s=\frac{1}{2}+ti$  ( $t \in \mathbb{R}$  and  $t \neq 0, s \in \mathbb{C}$ ) is true. Since Riemann has shown that the Riemann  $\zeta(s)$  function has zero, that is, in

$\zeta(1-s)=2^{1-s}\pi^{-s}\text{Cos}(\frac{\pi s}{2})\Gamma(s)\zeta(s)$ ,  $\zeta(s)=0$  is true. According the equation  $\xi(s) = \frac{1}{2}s(s-1)\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s)$

obtained by Riemann, so  $\xi(s)=\xi(1-s)$ , because  $\Gamma(\frac{s}{2})=\overline{\Gamma(\frac{\overline{s}}{2})}$ , and  $\pi^{-\frac{s}{2}}=\overline{\pi^{-\frac{\overline{s}}{2}}}$ , and because

$\zeta(s)=\overline{\zeta(\overline{s})}$ , so  $\xi(s)=\overline{\xi(\overline{s})}$ . So when  $\zeta(s)=0$ , then  $\xi(s)=\zeta(1-s)=\zeta(\overline{s})=0$  and  $\xi(s)=\xi(1-s)=\xi(\overline{s})=0$

must be true, so the zeros of the Riemann  $\zeta(s)$  function and the nontrivial zeros of the Riemann  $\xi(s)$  function are identical, so the complex root of Riemann  $\xi(s)=0$  satisfies  $s=\frac{1}{2}+ti$  ( $t \in \mathbb{R}$  and  $t \neq 0, s \in \mathbb{C}$ ), ...

According to the Riemann function  $\prod_{2^s}(s-1)\pi^{-\frac{s}{2}}\zeta(s)=\xi(t)$  and the Riemann hypothesis  $s=\frac{1}{2}+ti$ , because

$s \neq 1$ , and  $\prod_{2^s} \neq 0$ ,  $\pi^{-\frac{s}{2}} \neq 0$ , so  $\prod_{2^s}(s-1)\pi^{-\frac{s}{2}} \neq 0$ , and when  $\xi(t)=0$ , then  $\prod_{2^s}(s-1)\pi^{-\frac{s}{2}}\zeta(\frac{1}{2}+ti)=\xi(t)=0$ , and

$$\zeta(\frac{1}{2}+ti) = \frac{\xi(t)}{\prod_{2^s}(s-1)\pi^{-\frac{s}{2}}} = \frac{0}{\prod_{2^s}(s-1)\pi^{-\frac{s}{2}}} = 0, \text{ so } t \in \mathbb{R} \text{ and } t \neq 0. \text{ So the root } t \text{ of the equations}$$

$$\prod_{2^s}(s-1)\pi^{-\frac{s}{2}}\zeta(\frac{1}{2}+ti)=\xi(t)=0 \text{ and } 4\int_1^\infty \frac{d(x^{\frac{3}{2}}\Psi'(x))}{dx} x^{-\frac{1}{4}} \cos(\frac{1}{2}t \ln x) dx = \xi(t)=0 \text{ and}$$

$$\xi(t) = \frac{1}{2} - (t^2 + \frac{1}{4}) \int_1^\infty \Psi(x) x^{-\frac{3}{4}} \cos(\frac{1}{2}t \ln x) dx = 0 \text{ must be real and } t \neq 0.$$

Riemann got  $\prod_{2^s}(s-1)\pi^{-\frac{s}{2}}\zeta(s)=\xi(t)$  and  $\xi(t) = \frac{1}{2} - (t^2 + \frac{1}{4}) \int_1^\infty \Psi(x) x^{-\frac{3}{4}} \cos(\frac{1}{2}t \ln x) dx$  in his paper, or

$$\prod_{2^s}(s-1)\pi^{-\frac{s}{2}}\zeta(s)=\xi(t) \text{ and } \xi(t) = 4\int_1^\infty \frac{d(x^{\frac{3}{2}}\Psi'(x))}{dx} x^{-\frac{1}{4}} \cos(\frac{1}{2}t \ln x) dx, \text{ Because } \zeta(\frac{1}{2}+ti)=0 (t \in \mathbb{R} \text{ and } t \neq 0, s \in \mathbb{C}) \text{ is}$$

true, so  $\prod_{2^s}(s-1)\pi^{-\frac{s}{2}}\zeta(\frac{1}{2}+ti)=\xi(t)=0 (t \in \mathbb{R} \text{ and } t \neq 0, s \in \mathbb{C})$  and

$$\text{and } \prod_{2^s}(s-1)\pi^{-\frac{s}{2}}\zeta(\frac{1}{2}+ti) = 4\int_1^\infty \frac{d(x^{\frac{3}{2}}\Psi'(x))}{dx} x^{-\frac{1}{4}} \cos(\frac{1}{2}t \ln x) dx = \xi(t)=0, \text{ and}$$

$$\xi(t) = \frac{1}{2} - (t^2 + \frac{1}{4}) \int_1^\infty \Psi(x) x^{-\frac{3}{4}} \cos(\frac{1}{2}t \ln x) dx = 0, \text{ so the roots of equations } \prod_{2^s}(s-1)\pi^{-\frac{s}{2}}\zeta(\frac{1}{2}+ti) = \xi(t)=0 \text{ and}$$

$$4\int_1^\infty \frac{d(x^{\frac{3}{2}}\Psi'(x))}{dx} x^{-\frac{1}{4}} \cos(\frac{1}{2}t \ln x) dx = \xi(t)=0 \text{ and } \xi(t) = \frac{1}{2} - (t^2 + \frac{1}{4}) \int_1^\infty \Psi(x) x^{-\frac{3}{4}} \cos(\frac{1}{2}t \ln x) dx = 0 \text{ must all be real}$$

numbers. When  $\zeta(s)=0$  and  $\xi(t)=0$ , the real part of the equation  $\xi(t)=0$  must be real between 0 and T. Because the real part of the equation  $\xi(t)=0$  has the number of complex roots between 0 and T

approximately equal to  $\frac{T}{2\pi} \ln \frac{T}{2\pi} - \frac{T}{2\pi}$ . This result of Riemann's estimate of the number of zeros was

rigorously proved by Mangoldt in 1895. Then, when  $\zeta(s)=0$  and  $\xi(t)=0$ , the number of real roots of the

real part of the equation  $\xi(t)=0$  between 0 and T must be approximately equal to  $\frac{T}{2\pi} \ln \frac{T}{2\pi} - \frac{T}{2\pi}$ . So, when

$\zeta(s)=0$ , the Riemann hypothesis and the Riemann conjecture are perfectly valid.

**Reasoning 1:**

For any complex number  $s$ , when  $\text{Re}(s) > 0$  and  $(s \neq 1)$ , and if  $s = \rho + yi (\rho \in \mathbb{R}, y \in \mathbb{R})$  then according to Dirichlet function

$$\eta(s) = \sum_{n=1}^\infty \frac{(-1)^{n-1}}{n^s} \quad (s \in \mathbb{C} \text{ and } \text{Re}(s) > 0 \text{ and } (s \neq 1)) \quad \text{and } \eta(s) = (1 - 2^{1-s}) \zeta(s) \quad (s \in \mathbb{C} \text{ and } \text{Re}(s) >$$

$$0 \text{ and } s \neq 1), \zeta(s) \text{ is the Riemann Zeta function, so Riemann } \zeta(s) = \frac{\eta(s)}{(1-2^{1-s})} = \frac{1}{(1-2^{1-s})} \sum_{n=1}^\infty \frac{(-1)^{n-1}}{n^s} =$$

$$\frac{(-1)^{n-1}}{(1-2^{1-s})} \prod_p (1 - p^{-s})^{-1} \quad (s \in \mathbb{C} \text{ and } \text{Re}(s) > 0 \text{ and } s \neq 1), n \in \mathbb{Z}_+, p \in \mathbb{Z}_+, s \in \mathbb{C}, n \text{ goes through all}$$

the natural numbers,  $p$  goes through all the prime numbers). Let's prove that  $\zeta(s)$  and  $\zeta(\bar{s})$  are complex conjugations of each other.

$$\sum_{n=1}^\infty \frac{(-1)^{n-1}}{n^s} = [1^{-\rho} \cos(y \ln 1) - 2^{-\rho} \cos(y \ln 2) + 3^{-\rho} \cos(y \ln 3) - 4^{-\rho} \cos(y \ln 4) - \dots] - i [1^{-\rho} \sin(y \ln 1) - 2^{-\rho} \sin(y \ln 2) + 3^{-\rho} \sin(y \ln 3) - 4^{-\rho} \sin(y \ln 4) + \dots] = U - Vi,$$

$$\sum_{n=1}^\infty \frac{(-1)^{n-1}}{n^{\bar{s}}} = [1^{-\rho} \cos(y \ln 1) - 2^{-\rho} \cos(y \ln 2) + 3^{-\rho} \cos(y \ln 3) - 4^{-\rho} \cos(y \ln 4) - \dots] + i [1^{-\rho} \sin(y \ln 1) - 2^{-\rho} \sin(y \ln 2) - 3^{-\rho} \sin(y \ln 3) + 4^{-\rho} \sin(y \ln 4) + \dots] = U + Vi,$$

$$n2)+ 3^{-\rho}\sin(y\ln3)-4^{-\rho}\sin(y\ln4)+\dots]= U+Vi,$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{1-s}}=[ 1^{\rho-1}\text{Cos}(y\ln1)- 2^{\rho-1}\text{Cos}(y\ln2)+ 3^{\rho-1}\text{Cos}(y\ln3)-4^{-\rho}\text{Cos}(y\ln4)-\dots]+i[1^{-\rho}\text{Sin}(y\ln1)- 2^{-\rho}\text{Sin}(y\ln2)+ 3^{-\rho}\text{Sin}(y\ln3)- 4^{-\rho}\text{Sin}(y\ln4)+\dots],$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{k-s}}=[ 1^{\rho-k}\text{Cos}(y\ln1)- 2^{\rho-k}\text{Cos}(y\ln2)+ 3^{\rho-k}\text{Cos}(y\ln3)-4^{\rho-k}\text{Cos}(y\ln4)-\dots]+i[ 1^{\rho-k}\text{Sin}(y\ln1)- 2^{\rho-k}\text{Sin}(y\ln2)+ 3^{\rho-k}\text{Sin}(y\ln3)- 4^{\rho-k}\text{Sin}(y\ln4)+\dots],$$

Because ,

$$\frac{(-1)^{n-1}}{(1-2^{1-s})} = \frac{(-1)^{n-1}}{(1-2^{1-\bar{s}})} ,$$

$$\prod_p(1-p^{-s})^{-1}=\overline{\prod_p(1-p^{-\bar{s}})^{-1}} ,$$

so

$$\frac{(-1)^{n-1}}{(1-2^{1-s})} = \overline{\frac{(-1)^{n-1}}{(1-2^{1-\bar{s}})}} ,$$

so

$$\frac{(-1)^{n-1}}{(1-2^{1-s})} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = \overline{\frac{(-1)^{n-1}}{(1-2^{1-\bar{s}})} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{\bar{s}}}} ,$$

$$\frac{(-1)^{n-1}}{(1-2^{1-s})} \prod_p(1-p^{-s})^{-1} = \overline{\frac{(-1)^{n-1}}{(1-2^{1-\bar{s}})} \prod_p(1-p^{-\bar{s}})^{-1}} ,$$

$$\zeta(s)=\frac{1}{(1-2^{1-s})} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = \frac{(-1)^{n-1}}{(1-2^{1-s})} \prod_p(1-p^{-s})^{-1} ,$$

$$\zeta(\bar{s})=\frac{1}{(1-2^{1-\bar{s}})} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{\bar{s}}} = \frac{(-1)^{n-1}}{(1-2^{1-\bar{s}})} \prod_p(1-p^{-\bar{s}})^{-1} .$$

so

$$\text{Only } \zeta(s)=\overline{\zeta(\bar{s})}, \text{ [2]}$$

so

$$p^{1-s}=p^{(1-\rho-yi)}=p^{1-\rho}x^{-yi}=p^{1-\rho}(\cos(\ln p) + i \sin(\ln p))^{-y}=p^{1-\rho}(\cos(y\ln p) - i \sin(y\ln p)) ,$$

$$p^{1-\bar{s}}=p^{(1-\rho+yi)}=p^{1-\rho}p^{yi}=p^{1-\rho}(p^{yi}) = p^{1-\rho}(\cos(\ln p) + i \sin(\ln p))^y=(p^{1-\rho}(\cos(y\ln p) + i \sin(y\ln p))) ,$$

then

$$p^{-(1-s)}=p^{(-1+\rho+yi)}=p^{\rho-1}x^{yi} = p^{\rho-1} \frac{1}{(\cos(y\ln p)-i \sin(y\ln p))} =(p^{\rho-1}(\cos(y\ln p) + i \sin(y\ln p))) ,$$

$$p^{-(\bar{s})}=p^{-(\rho-yi)}=p^{-\rho}p^{yi} = (p^{-\rho}(\cos(y\ln p) + i \sin(y\ln p))) ,$$

so

$$(1-p^{-(1-s)})=1-(p^{\rho-1}(\cos(y\ln p) + i \sin(y\ln p))) = 1-p^{\rho-1}\cos(y\ln p) - ip^{\rho-1}\sin(y\ln p) ,$$

$$(1-p^{-(\bar{s})})=1-(p^{-\rho}(\cos(y\ln p) + i \sin(y\ln p))) = 1-p^{-\rho}\cos(y\ln p) - ip^{-\rho}\sin(y\ln p) ,$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{1-s}} = [ 1^{\rho-1}\text{Cos}(y\ln1)- 2^{\rho-1}\text{Cos}(y\ln2)+ 3^{\rho-1}\text{Cos}(y\ln3)-4^{\rho-1}\text{Cos}(y\ln4)-\dots]+i[ 1^{\rho-1}\text{Sin}(y\ln1)$$

$$- 2^{\rho-1}\text{Sin}(y\ln2)+ 3^{\rho-1}\text{Sin}(y\ln3)- 4^{\rho-1}\text{Sin}(y\ln4)+\dots],$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{\bar{s}}} = [ 1^{-\rho}\text{Cos}(y\ln1)- 2^{-\rho}\text{Cos}(y\ln2)+ 3^{-\rho}\text{Cos}(y\ln3)-4^{-\rho}\text{Cos}(y\ln4)-\dots]+i[ 1^{-\rho}\text{Sin}(y\ln1)- 2^{-\rho}\text{Sin}(y\ln2)$$

$$+ 3^{-\rho}\text{Sin}(y\ln3)- 4^{-\rho}\text{Sin}(y\ln4)+\dots],$$

when  $\rho = \frac{1}{2}$ ,

then

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{1-s}} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{\bar{s}}},$$

$$(1 - p^{-(1-s)}) = (1 - p^{-\bar{s}}),$$

and

$$(1 - p^{-(1-s)})^{-1} = (1 - p^{-\bar{s}})^{-1},$$

$$\prod_p (1 - p^{-(1-s)})^{-1} = \prod_p (1 - p^{-\bar{s}})^{-1},$$

and

$$\frac{(-1)^{n-1}}{(1-2^s)} \prod_p (1 - p^{-(1-s)})^{-1} = \frac{(-1)^{n-1}}{(1-2^{1-\bar{s}})} \prod_p (1 - p^{-\bar{s}})^{-1},$$

$$\frac{(-1)^{n-1}}{(1-2^s)} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{1-s}} = \frac{(-1)^{n-1}}{(1-2^{1-\bar{s}})} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{\bar{s}}},$$

and

$$\zeta(1-s) = \frac{(-1)^{n-1}}{(1-2^s)} \prod_p (1 - p^{-(1-s)})^{-1},$$

$$\zeta(\bar{s}) = \frac{(-1)^{n-1}}{(1-2^{1-\bar{s}})} \prod_p (1 - p^{-\bar{s}})^{-1},$$

$$\zeta(1-s) = \frac{(-1)^{n-1}}{(1-2^s)} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{1-s}},$$

$$\zeta(\bar{s}) = \frac{(-1)^{n-1}}{(1-2^{1-\bar{s}})} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{\bar{s}}},$$

so when  $\rho = \frac{1}{2}$ , then

Only  $\zeta(1-s) = \zeta(\bar{s})$ .

According the equation  $\zeta(1-s) = 2^{1-s} \pi^{-s} \cos\left(\frac{\pi s}{2}\right) \Gamma(s) \zeta(s)$  obtained by Riemann, since Riemann has shown

that the Riemann  $\zeta(s)$  function has zero, that is, in

$$\zeta(1-s) = 2^{1-s} \pi^{-s} \cos\left(\frac{\pi s}{2}\right) \Gamma(s) \zeta(s), \zeta(s) = 0 \text{ is true.}$$

When  $\zeta(s) = 0$ , then only  $\zeta(k - \bar{s}) = \zeta(s) = 0$ , and

When  $\zeta(\bar{s}) = 0$ , then  $\zeta(k - s) = \zeta(\bar{s}) = 0$ .

But the Riemann  $\zeta(s)$  function only satisfies  $\zeta(1-s) = 2^{1-s} \pi^{-s} \cos\left(\frac{\pi s}{2}\right) \Gamma(s) \zeta(s)$ , so when  $\zeta(s) = 0$ , then only

$\zeta(1-s) = \zeta(s) = 0$ , and when  $\zeta(\bar{s}) = 0$ , then only  $\zeta(1-s) = \zeta(\bar{s}) = 0$ , which is  $\zeta(k-s) = \zeta(1-s) = \zeta(\bar{s})$ , so

only  $k=1$  be true. so only  $\text{Re}(s) = \frac{k}{2} = \frac{1}{2}$  is true.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{k-s}} = [1^{\rho-k} \cos(y \ln 1) - 2^{\rho-k} \cos(y \ln 2) + 3^{\rho-k} \cos(y \ln 3) - 4^{\rho-k} \cos(y \ln 4) - \dots] + i [1^{\rho-k} \sin(y \ln 1)$$

$$- 2^{\rho-k} \sin(y \ln 2) + 3^{\rho-k} \sin(y \ln 3) - 4^{\rho-k} \sin(y \ln 4) + \dots],$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{\bar{s}}} = [1^{-\rho} \cos(y \ln 1) - 2^{-\rho} \cos(y \ln 2) + 3^{-\rho} \cos(y \ln 3) - 4^{-\rho} \cos(y \ln 4) - \dots] + i [1^{-\rho} \sin(y \ln 1) - 2^{-\rho} \sin(y \ln 2)$$

$$+ 3^{-\rho} \sin(y \ln 3) - 4^{-\rho} \sin(y \ln 4) + \dots],$$

$$p^{k-s} = p^{(k-\rho-yi)} = p^{k-\rho} x^{-yi} = p^{k-\rho} (\cos(\ln p) + i \sin(\ln p))^{-y} = p^{k-\rho} (\cos(y \ln p) - i \sin(y \ln p)),$$

$$p^{1-\bar{s}}=p^{(1-\rho+yi)}=p^{1-\rho}p^{yi}=p^{1-\rho}(p^{yi}) = p^{1-\rho}(\cos(\ln p) + i \sin(\ln p))^y=(p^{1-\rho}(\cos(y \ln p) + i \sin(y \ln p))),$$

Then

$$p^{-(k-s)}=p^{(-k+\rho+yi)}=p^{\rho-k}p^{yi} = p^{\rho-k} \frac{1}{(\cos(y \ln p) - i \sin(y \ln p))}=(p^{\rho-k}(\cos(y \ln p) + i \sin(y \ln p))),$$

$$p^{-(\bar{s})}=p^{-(\rho-yi)}=p^{-\rho}p^{yi} = (p^{-\rho}(\cos(y \ln p) + i \sin(y \ln p))),$$

$$p^{-(k-s)}=(p^{\rho-k}(\cos(y \ln p) + i \sin(y \ln p))),$$

so

$$(1 - p^{-(k-s)})=1-(p^{\rho-k}(\cos(y \ln p) + i \sin(y \ln p)))=1 - p^{\rho-k} \cos(y \ln p) - ip^{\rho-k} \sin(y \ln p),$$

$$(1 - p^{-\bar{s}})=1-(p^{-\rho}(\cos(y \ln p) + i \sin(y \ln p)))=1 - p^{-\rho} \cos(y \ln p) - ip^{-\rho} \sin(y \ln p),$$

So

so when  $\rho=\frac{k}{2}(k \in \mathbb{R})$  then

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{1-k+s}} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{\bar{s}}},$$

$$(1 - p^{-(k-s)}) = (1 - p^{-\bar{s}})$$

and

$$(1 - p^{-(k-s)})^{-1} = (1 - p^{-\bar{s}})^{-1},$$

$$\prod_p (1 - p^{-(k-s)})^{-1} = \prod_p (1 - p^{-\bar{s}})^{-1},$$

and

$$\frac{1}{(1-2^{1-k+s})} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{k-s}} = \frac{1}{(1-2^{1-\bar{s}})} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{\bar{s}}},$$

and

$$\zeta(k-s) = \frac{(-1)^{n-1}}{(1-2^{1-k+s})} \prod_p (1 - p^{-(k-s)})^{-1},$$

$$\zeta(\bar{s}) = \frac{(-1)^{n-1}}{(1-2^{1-\bar{s}})} \prod_p (1 - p^{-\bar{s}})^{-1},$$

$$\zeta(k-s) = \frac{1}{(1-2^{1-k+s})} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{k-s}},$$

$$\zeta(\bar{s}) = \frac{1}{(1-2^{1-\bar{s}})} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{\bar{s}}},$$

so when  $\rho=\frac{k}{2}(k \in \mathbb{R})$  then

$$\text{Only } \zeta(k-s) = \zeta(\bar{s}).$$

According the equation  $\zeta(1-s) = 2^{1-s} \pi^{-s} \text{Cos}(\frac{\pi s}{2}) \Gamma(s) \zeta(s)$  obtained by Riemann, since Riemann has shown

that the Riemann  $\zeta(s)$  function has zero, that is, in

$$\zeta(1-s) = 2^{1-s} \pi^{-s} \text{Cos}(\frac{\pi s}{2}) \Gamma(s) \zeta(s), \zeta(s) = 0 \text{ is true.}$$

When  $\zeta(s) = 0$ , then only  $\zeta(k-\bar{s}) = \zeta(s) = 0$ , and

When  $\zeta(\bar{s}) = 0$ , then  $\zeta(k-s) = \zeta(\bar{s}) = 0$ . And because when  $\zeta(\bar{s}) = 0$ , then only  $\zeta(1-s) = \zeta(\bar{s}) = 0$ , which is  $\zeta(k-s) = \zeta(\bar{s})$ , so only  $k=1$  be true.

According  $\zeta(s) = \zeta(1-s) = 0$  and  $\zeta(s) = \zeta(\bar{s}) = \zeta(1-\bar{s}) = 0$ , then  $s = \bar{s}$  or  $s = 1-s$  or  $\bar{s} = 1-s$ , so  $s \in \mathbb{R}$ , or  $\rho + yi = 1 - \rho - yi$ , or

$\rho - yi = 1 - \rho - yi$ , so  $s \in \mathbb{R}$ , or  $\rho = \frac{1}{2}$  and  $y = 0$ , or  $\rho = \frac{1}{2}$  and  $y \in \mathbb{R}$  and  $y \neq 0$ , so  $s \in \mathbb{R}$ , for example

$s = -2n (n \in \mathbb{Z}_+)$ , or  $s = \frac{1}{2} + oi$ , or  $s = \frac{1}{2} + yi (y \in \mathbb{R} \text{ and } y \neq 0)$ .  $\zeta(\frac{1}{2}) > \zeta(1) > 0$ , drop it,  $s = -2n (n \in \mathbb{Z}_+)$ , It's the



trivial zero of the Riemann  $\zeta(s)$  function, drop it.

Because only when  $\rho = \frac{1}{2}$ , the next three equations,  $\zeta(\rho + yi) = 0$ ,  $\zeta(1 - \rho - yi) = 0$ , and  $\zeta(\rho - yi) = 0$  are

all true,  $\zeta(\frac{1}{2}) > \zeta(1) > 0$ , so only  $s = \frac{1}{2} + yi$  ( $y \in \mathbb{R}$  and  $y \neq 0, s \in \mathbb{C}$ ) is true, or say only  $s = \frac{1}{2} + ti$  ( $t \in \mathbb{R}$  and  $t \neq$

$0, s \in \mathbb{C}$ ) is true. Since Riemann has shown that the Riemann  $\zeta(s)$  function has zero, that is, in

$\zeta(1-s) = 2^{1-s} \pi^{-s} \cos(\frac{\pi s}{2}) \Gamma(s) \zeta(s)$ ,  $\zeta(s) = 0$  is true. According to the

equation  $\xi(s) = \frac{1}{2} s(s-1) \Gamma(\frac{s}{2}) \pi^{-\frac{s}{2}} \zeta(s)$  obtained by Riemann, so  $\xi(s) = \xi(1-s)$ , because  $\Gamma(\frac{s}{2}) = \Gamma(\frac{\overline{s}}{2})$ ,

and  $\pi^{-\frac{s}{2}} = \overline{\pi^{-\frac{\overline{s}}{2}}}$ , and because  $\zeta(s) = \overline{\zeta(\overline{s})}$ , so  $\xi(s) = \overline{\xi(\overline{s})}$ . So when  $\zeta(s) = 0$ , then  $\xi(s) = \zeta(1-s) = \zeta(\overline{s}) =$

$0$  and  $\xi(s) = \xi(1-s) = \xi(\overline{s}) = 0$  must be true, so the zeros of the Riemann  $\zeta(s)$  function and the nontrivial zeros of the Riemann  $\xi(s)$  function are identical, so the complex root of Riemann  $\xi(s) = 0$  satisfies

$s = \frac{1}{2} + ti$  ( $t \in \mathbb{R}$  and  $t \neq 0, s \in \mathbb{C}$ ), according to the Riemann function  $\prod_{\frac{s}{2}} (s-1) \pi^{-\frac{s}{2}} \zeta(s) = \xi(t)$  and the Riemann

hypothesis  $s = \frac{1}{2} + ti$ , because  $s \neq 1$ , and  $\prod_{\frac{s}{2}} \neq 0$ ,  $\pi^{-\frac{s}{2}} \neq 0$ , so  $\prod_{\frac{s}{2}} (s-1) \pi^{-\frac{s}{2}} \neq 0$ , and when  $\xi(t) = 0$ , then

$\prod_{\frac{s}{2}} (s-1) \pi^{-\frac{s}{2}} \zeta(\frac{1}{2} + ti) = \xi(t) = 0$ , and  $\zeta(\frac{1}{2} + ti) = \frac{\xi(t)}{\prod_{\frac{s}{2}} (s-1) \pi^{-\frac{s}{2}}} = \frac{0}{\prod_{\frac{s}{2}} (s-1) \pi^{-\frac{s}{2}}} = 0$ , so  $t \in \mathbb{R}$  and  $t \neq 0$ . So the root  $t$  of

the equations  $\prod_{\frac{s}{2}} (s-1) \pi^{-\frac{s}{2}} \zeta(\frac{1}{2} + ti) = \xi(t) = 0$  and  $4 \int_1^{\infty} \frac{d(x^{\frac{3}{2}} \Psi(x))}{dx} x^{-\frac{1}{4}} \cos(\frac{1}{2} t \ln x) dx = \xi(t) = 0$  and  $\xi(t) = \frac{1}{2} -(t^2 +$

$\frac{1}{4} \int_1^{\infty} \Psi(x) x^{-\frac{3}{4}} \cos(\frac{1}{2} t \ln x) = 0$  must be real and  $t \neq 0$ . If  $\text{Re}(s) = \frac{k}{2}$  ( $k \in \mathbb{R}$ ), then

$\zeta(k-s) = 2^{k-s} \pi^{-s} \cos(\frac{\pi s}{2}) \Gamma(s) \zeta(s)$  and  $\xi(k-s) = \frac{1}{2} s(s-k) \Gamma(\frac{s}{2}) \pi^{-\frac{s}{2}} \zeta(s)$  are true, so when  $\zeta(s) = 0$ , then

$\zeta(s) = \zeta(k-s) = \zeta(\overline{s}) = 0$  and  $\xi(s) = \xi(k-s) = \xi(\overline{s}) = 0$  must be true, and  $s = \frac{k}{2} + ti$  ( $k \in \mathbb{R}$ ,  $t \in \mathbb{R}$  and  $t \neq$

$0, s \in \mathbb{C}$ ) must be true, then  $\prod_{\frac{s}{2}} (s-k) \pi^{-\frac{s}{2}} \zeta(\frac{k}{2} + ti) = \xi(t) = 0$ , and  $\zeta(\frac{k}{2} + ti) = \frac{\xi(t)}{\prod_{\frac{s}{2}} (s-k) \pi^{-\frac{s}{2}}} = \frac{0}{\prod_{\frac{s}{2}} (s-k) \pi^{-\frac{s}{2}}} = 0$ , so

$t \in \mathbb{R}$  and  $t \neq 0$ . So the root  $t$  of the equations  $\prod_{\frac{s}{2}} (s-k) \pi^{-\frac{s}{2}} \zeta(\frac{k}{2} + ti) = \xi(t) = 0$  must be real and  $t \neq 0$ . But

the Riemann  $\zeta(s)$  function only satisfies  $\zeta(1-s) = 2^{1-s} \pi^{-s} \cos(\frac{\pi s}{2}) \Gamma(s) \zeta(s)$  and  $\xi(s) = \frac{1}{2} s(s-1) \Gamma(\frac{s}{2}) \pi^{-\frac{s}{2}} \zeta(s)$ ,

is also say that only  $\zeta(1-s) = 2^{1-s} \pi^{-s} \cos(\frac{\pi s}{2}) \Gamma(s) \zeta(s)$  is true, so only  $\text{Re}(s) = \frac{k}{2} = \frac{1}{2}$  is true, so only  $k=1$  is

true. The Riemann hypothesis and the Riemann conjecture must satisfy the properties of the Riemann

$\zeta(s)$  function and the Riemann  $\xi(s)$  function, The properties of the Riemann  $\zeta(s)$  function and the

Riemann  $\xi(s)$  function are fundamental, the Riemann hypothesis and the Riemann conjecture must

be correct to reflect the properties of the Riemann  $\zeta(s)$  function and the Riemann  $\xi(s)$  function, that

is, the roots of the Riemann  $\xi(t)$  function can only be real, that is,  $\text{Re}(s)$  can only be equal to  $\frac{1}{2}$ , and

$\text{Im}(s)$  must be real, and  $\text{Im}(s)$  is not equal to zero. So the Riemann hypothesis and the Riemann conjecture must be correct.

For any complex number  $s$ , When  $\text{Re}(s)$  is any real number, including  $\text{Re}(s) > 0$  and  $(s \neq 1)$  and  $\text{Re}(s) \leq$

$0 \wedge (s \neq 0)$ , then Riemann Zeta() function is  $\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s)$ . Suppose  $s = \rho + yi$  ( $\rho \in \mathbb{R}, y \in \mathbb{R}$  and  $y \neq 0, s \in \mathbb{C}$ ), let's prove that  $\zeta(s)$  and  $\zeta(\bar{s})$  are complex conjugations of each other and get the equation  $\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s)$ .

**Lemma 2:**

The reasoning in Riemann's paper goes like:

$$2\sin(\pi s) \prod (s-1) \zeta(s) = (2\pi)^s \sum n^{s-1} ((-i)^{s-1} + i^{s-1})^{[1]} \text{ (Formula 3)}$$

based on euler's  $e^{ix} = \cos(x) + i \sin(x)$  ( $x \in \mathbb{R}$ ) can get

$$e^{i\left(-\frac{\pi}{2}\right)} = \cos\left(\frac{-\pi}{2}\right) + i \sin\left(\frac{-\pi}{2}\right) = 0 - i = -i,$$

$$e^{i\left(\frac{\pi}{2}\right)} = \cos\left(\frac{\pi}{2}\right) + i \sin\left(\frac{\pi}{2}\right) = 0 + i = i,$$

then

$$(-i)^{s-1} + i^{s-1} = (-i)^{-1} (-i)^s + (i)^{-1} (i)^s = (-i)^{-1} e^{i\left(-\frac{\pi}{2}\right)s} + i^{(-1)} e^{i\left(\frac{\pi}{2}\right)s} =$$

$$i e^{i\left(-\frac{\pi}{2}\right)s} - i e^{i\left(\frac{\pi}{2}\right)s} = i \left( \cos\frac{-\pi s}{2} + i \sin\frac{-\pi s}{2} \right) - i \left( \cos\frac{\pi s}{2} + i \sin\frac{\pi s}{2} \right) = i \cos\left(\frac{\pi s}{2}\right) - i \cos\left(\frac{\pi s}{2}\right) + \sin\left(\frac{\pi s}{2}\right) + \sin\left(\frac{\pi s}{2}\right)$$

$$= 2\sin\left(\frac{\pi s}{2}\right) \text{ (Formula 4)}.$$

According to the property of  $\prod (s-1) = \Gamma(s)$  of the gamma function, and

$$\sum_{n=1}^{\infty} n^{s-1} = \zeta(1-s),$$

Substitute the above (Formula 4) into the above (Formula 3), will get

$$2\sin(\pi s) \Gamma(s) \zeta(s) = (2\pi)^s \zeta(1-s) 2 \sin\left(\frac{\pi s}{2}\right) \text{ (Formula 5)}$$

If I substitute it into (Formula 5), according to the double Angle formula  $\sin(\pi s) = 2 \sin\left(\frac{\pi s}{2}\right) \cos\left(\frac{\pi s}{2}\right)$ , we

$$\text{Will get } \zeta(1-s) = 2^{1-s} \pi^{-s} \cos\left(\frac{\pi s}{2}\right) \Gamma(s) \zeta(s) \text{ (Formula 6)}$$

When  $s \neq -2n$  ( $n \in \mathbb{Z}_+$ ), because  $\pi^{-\frac{1-s}{2}} \neq 0 \neq 0$  and  $\Gamma\left(\frac{1-s}{2}\right) \neq 0$ , so when  $\zeta(s) = 0$ , then  $\zeta(1-s) = 0$ ,

Substituting  $s \rightarrow 1-s$ , that is taking  $s$  as  $1-s$  into Formula 6, we will get

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s) \text{ (Formula 7)}$$

This is the functional equation for  $\zeta(s)$ . To rewrite it in a symmetric form, use the residual formula of the gamma function<sup>[3]</sup>

$$\Gamma(Z) \Gamma(1-Z) = \frac{\pi}{\sin(\pi Z)} \text{ (Formula 8)}$$

and Legendre's formula

$$\Gamma\left(\frac{Z}{2}\right) \Gamma\left(\frac{Z+1}{2}\right) = 2^{1-Z} \pi^{\frac{1}{2}} \Gamma(Z) \text{ (Formula 9)}$$

Take  $z = \frac{s}{2}$  in (Formula 8) and substitute it to get

$$\sin\left(\frac{\pi s}{2}\right) = \frac{\pi}{\Gamma\left(\frac{s}{2}\right) \Gamma\left(1-\frac{s}{2}\right)} \text{ (Formula 10)}$$

In (Formula 9), let  $z = 1-s$  and substitute it in to get

$$\Gamma(1-s)=2^{-s}\pi^{-\frac{1}{2}}\Gamma(\frac{1-s}{2})\Gamma(1-\frac{s}{2}) \quad (\text{Formula 11})$$

By substituting (Formula 10) and (Formula 11) into (Formula 7), we get

$$\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)=\pi^{-\frac{1-s}{2}}\Gamma(\frac{1-s}{2})\zeta(1-s),$$

also

$$\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s) \text{ is invariant under the transformation } s \rightarrow 1-s,$$

And that's exactly what Riemann said in his paper.

That is to say:

$$\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s) \text{ is invariant under the transformation } s \rightarrow 1-s,$$

also

$$\prod(\frac{s}{2}-1)\pi^{-\frac{s}{2}}\zeta(s)=\prod(\frac{1-s}{2}-1)\pi^{-\frac{1-s}{2}}\zeta(1-s)$$

or

$$\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)=\pi^{-\frac{1-s}{2}}\Gamma(\frac{1-s}{2})\zeta(1-s) \quad (\text{Formula 2}),$$

$$\text{Then } \zeta(s)=2^s\pi^{s-1}\text{Sin}(\frac{\pi s}{2})\Gamma(1-s)\zeta(1-s),$$

under the transformation  $s \rightarrow 1-s$ , will get

$$\zeta(1-s)=2^{1-s}\pi^{-s}\text{Cos}(\frac{\pi s}{2})\Gamma(s)\zeta(s) \quad (\text{Formula 1})$$

### **Reasoning 2:**

Because  $L(s, \chi(n)) = \chi(n)\zeta(s)$  and  $L(1-s, \chi(n)) = \chi(n)\zeta(1-s)$ ,

and according to  $\zeta(s)=2^s\pi^{s-1}\text{Sin}(\frac{\pi s}{2})\Gamma(1-s)\zeta(1-s)$  (Formula 7),

so

$$\text{Only } L(s, \chi(n))=2^s\pi^{s-1}\text{Sin}(\frac{\pi s}{2})\Gamma(1-s)L(1-s, \chi(n)) \quad (\text{Formula 12}).$$

According to the property that Gamma function  $\Gamma(s)$  and exponential function are nonzero, is also that

$$\Gamma(\frac{1-s}{2}) \neq 0, \text{ and } \pi^{-\frac{1-s}{2}} \neq 0, \text{ according to } \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)=\pi^{-\frac{1-s}{2}}\Gamma(\frac{1-s}{2})\zeta(1-s) \quad (\text{Formula 2}),$$

Mathematicians have shown that the real part of the complex independent variable  $s$  of the Riemann  $\zeta(s)$  function will have zero only if  $0 < \text{Re}(s) < 1$  and  $\text{Im}(s) \neq 0$ , so we agree on

$$\text{Riemann } \zeta(s) = \frac{\eta(s)}{(1-2^{1-s})} = \frac{1}{(1-2^{1-s})} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = \frac{(-1)^{n-1}}{(1-2^{1-s})} \prod_p (1-p^{-s})^{-1} \quad (s \in \mathbb{C} \text{ and } 0 <$$

$\text{Rs}(s) < 1$  and  $s \neq 1$  and  $\text{Im}(s) \neq 0$ ,  $n \in \mathbb{Z}_+$ ,  $p \in \mathbb{Z}_+$ ,  $s \in \mathbb{C}$ ,  $n$  goes through all the natural numbers,  $p$  goes through all the prime numbers).

According the equation  $\zeta(1-s)=2^{1-s}\pi^{-s}\text{Cos}(\frac{\pi s}{2})\Gamma(s)\zeta(s)$  obtained by Riemann, since Riemann has

shown that the Riemann  $\zeta(s)$  function has zero, that is, in  $\zeta(1-s)=2^{1-s}\pi^{-s}\text{Cos}(\frac{\pi s}{2})\Gamma(s)\zeta(s)$ , so  $\zeta(s)=0$

is true, and so we agree on  $\zeta(1-s)=2^{1-s}\pi^{-s}\cos(\frac{\pi s}{2})\Gamma(s)\zeta(s)$  ( $s \in \mathbb{C}$  and  $0 < \text{Re}(s) < 1$  and  $s \neq$

$1$  and  $\text{Im}(s) \neq 0, n \in \mathbb{Z}_+, p \in \mathbb{Z}_+, s \in \mathbb{C}$ ,  $n$  goes through all the natural numbers,  $p$  goes through all the prime numbers).

According to the property that Gamma function  $\Gamma(s)$  and exponential function are nonzero, is also that  $\Gamma(\frac{1-s}{2}) \neq 0$ , and  $\pi^{\frac{1-s}{2}} \neq 0$ ,

So when  $\zeta(s)=0$ , then  $\zeta(1-s)=0$ , also must  $\zeta(s)=\zeta(1-s)=0$ .

Because  $\sin(Z)=\frac{e^{iZ}-e^{-iZ}}{2i}$ , Suppose  $Z=s+\rho+yi$  ( $\rho \in \mathbb{R}, y \in \mathbb{R}$  and  $y \neq 0, s \in \mathbb{C}$ ), then

$$\sin(s)=\frac{e^{is}-e^{-is}}{2i} = \frac{e^{i(\rho+yi)}-e^{-i(\rho+yi)}}{2i},$$

$$\sin(\bar{s})=\frac{e^{i\bar{s}}-e^{-i\bar{s}}}{2i} = \frac{e^{i(\rho-yi)}-e^{-i(\rho-yi)}}{2i},$$

according  $x^s=x^{(\rho+yi)}=x^\rho x^{yi}=x^\rho(\cos(\ln x) + i \sin(\ln x))^y=x^\rho(\cos(y \ln x) + i \sin(y \ln x))$ , then

$$e^s=e^{(\rho+yi)}=e^\rho e^{yi}=e^\rho(\cos(y) + i \sin(y)) = e^\rho(\cos(y) + i \sin(y)),$$

$$e^{is}=e^{i(\rho+yi)}=e^{\rho i}(\cos(iy) + i \sin(iy)) = (\cos(\rho) + i \sin(\rho))(\cos(iy) + i \sin(iy))$$

$$e^{i\bar{s}} = e^{i(\rho-yi)}=e^{\rho i}(\cos(-iy) + i \sin(-iy)) = (\cos(\rho) + i \sin(\rho))(\cos(iy) - i \sin(iy)),$$

$$e^{-is}=e^{-i(\rho+yi)}=e^{-\rho i}(\cos(-iy) + i \sin(-iy)) = (\cos(\rho) - i \sin(\rho))(\cos(iy) - i \sin(iy))$$

$$e^{-i\bar{s}} = e^{-i(\rho-yi)}=e^{-\rho i}(\cos(iy) + i \sin(iy)) = (\cos(\rho) - i \sin(\rho))(\cos(iy) + i \sin(iy)),$$

$$2^s=2^{(\rho+yi)}=2^\rho 2^{yi}=2^\rho(\cos(\ln 2) + i \sin(\ln 2))^y=2^\rho(\cos(y \ln 2) + i \sin(y \ln 2)),$$

$$2^{\bar{s}}=2^{(\rho-yi)}=2^\rho 2^{-yi}=2^\rho(\cos(\ln 2) + i \sin(\ln 2))^{-y}=2^\rho(\cos(y \ln 2) - i \sin(y \ln 2)),$$

$$\pi^{s-1}=2^{(\rho-1+yi)}=2^{\rho-1} 2^{yi}=2^{\rho-1}(\cos(\ln 2) + i \sin(\ln 2))^y=2^{\rho-1}(\cos(y \ln 2) + i \sin(y \ln 2)),$$

$$\pi^{\bar{s}-1}=2^{(\rho-1-yi)}=2^{\rho-1} 2^{-yi}=2^{\rho-1}(\cos(\ln 2) + i \sin(\ln 2))^{-y}=2^{\rho-1}(\cos(y \ln 2) - i \sin(y \ln 2)),$$

So

$$2^s=2^{\bar{s}}, \quad \pi^{s-1}=\pi^{\bar{s}-1},$$

and

$$\frac{e^{is}-e^{-is}}{2i} = \frac{e^{i\bar{s}}-e^{-i\bar{s}}}{2i},$$

So

$$\sin(s)=\sin(\bar{s}),$$

So

$$\sin(\frac{\pi s}{2})=\sin(\frac{\pi \bar{s}}{2}).$$

And the gamma function on the complex field is defined as:

$$\Gamma(s)=\int_0^{+\infty} t^{s-1} e^{-t} dt$$

among

$\text{Re}(s)>0$ , this definition can be extended by the analytical continuation principle to the entire field of complex numbers, except for non-positive integers,

So

$$\Gamma(s)=\overline{\Gamma(\bar{s})},$$

and

$$\Gamma(1-s)=\overline{\Gamma(1-\bar{s})}.$$

When  $\zeta(1-\bar{s}) = \overline{\zeta(1-s)} = 0 = \zeta(1-s) = \zeta(1-s) = 0$ , and according  $\zeta(s) = 2^s \pi^{s-1} \sin(\frac{\pi s}{2}) \Gamma(1-s) \zeta(1-s)$ , then

Only  $\zeta(s) = \overline{\zeta(\bar{s})} = 0$ , is also say  $\zeta(s) = \zeta(\bar{s}) = \zeta(1-\bar{s}) = 0$ . so only  $\zeta(\rho+yi) = \zeta(\rho-yi) = 0$  is true.

According the equation  $\zeta(1-s) = 2^{1-s} \pi^{-s} \cos(\frac{\pi s}{2}) \Gamma(s) \zeta(s)$  obtained by Riemann, since Riemann has shown

that the Riemann  $\zeta(s)$  function has zero, that is, in  $\zeta(1-s) = 2^{1-s} \pi^{-s} \cos(\frac{\pi s}{2}) \Gamma(s) \zeta(s)$ ,  $\zeta(s) = 0$  is true, so

when  $\zeta(s) = 0$ , then only  $\zeta(s) = \zeta(1-s) = 0$  is true.

in the process of the Riemann hypothesis proved about  $\zeta(s) = \zeta(1-s) = \zeta(\bar{s}) = 0$ , is refers to the  $\zeta(s)$  is a functional number? It's not. Does  $\zeta(s) = \zeta(1-s) = \zeta(\bar{s})$  mean the symmetry of the  $\zeta(s)$  function equation?

Does that mean the symmetry of the equation  $s = \bar{s} = 1-s$ ? Not really. In my analyst,  $\zeta(s)$ ,  $\zeta(1-s)$  and  $\zeta(\bar{s})$

function expression is the same, are  $\sum_{n=1}^{\infty} n^{-s}$  ( $n$  traves all positive integer,  $s \in \mathbb{C}$ ), so according to

$\sum_{n=1}^{\infty} n^{-s}$  ( $n$  traves all positive integer,  $s \in \mathbb{C}$ ),  $\zeta(s)$  function of the independent variable  $s$ , the relationship between  $\bar{s}$  and  $1-s$  only  $\mathbb{C}_3^3 = 3$  kinds, namely  $s = \bar{s}$  or  $s = 1-s$  or  $\bar{s} = 1-s$ . As follows:

According  $\zeta(s) = \zeta(1-s) = 0$  and  $\zeta(s) = \zeta(\bar{s}) = \zeta(1-\bar{s}) = 0$ , then only  $s = \bar{s}$  or  $s = 1-s$  or  $\bar{s} = 1-s$ , so  $s \in \mathbb{R}$ ,

or  $\rho + yi = 1 - \rho - yi$ , or  $\rho - yi = 1 - \rho - yi$ , so  $s \in \mathbb{R}$ , or  $\rho = \frac{1}{2}$  and  $y = 0$ , or  $\rho = \frac{1}{2}$  and  $y \in \mathbb{R}$  and  $y \neq 0$ , so  $s \in \mathbb{R}$ , for

example  $s = -2n$  ( $n \in \mathbb{Z}_+$ ), or  $s = \frac{1}{2} + oi$ , or  $s = \frac{1}{2} + yi$  ( $y \in \mathbb{R}$  and  $y \neq 0$ ),  $\zeta(\frac{1}{2}) > \zeta(1) > 0$ , drop it,  $s = -2n$  ( $n \in$

$\mathbb{Z}^+$ ), It's the trivial zero of the Riemann  $\zeta(s)$  function, drop it.

Beacause only when  $\rho = \frac{1}{2}$ , the next three equations,  $\zeta(\rho + yi) = 0$ ,  $\zeta(1 - \rho - yi) = 0$ , and  $\zeta(\rho - yi) = 0$  are

all true,  $\zeta(\frac{1}{2}) > \zeta(1) > 0$ , so only  $s = \frac{1}{2} + yi$  ( $y \in \mathbb{R}$  and  $y \neq 0$ ,  $s \in \mathbb{C}$ ) is true, or say only  $s = \frac{1}{2} + ti$

( $t \in \mathbb{R}$  and  $t \neq 0$ ,  $s \in \mathbb{C}$ ) is true. Since Riemann has shown that the Riemann  $\zeta(s)$  function has zero, that is,

in  $\zeta(1-s) = 2^{1-s} \pi^{-s} \cos(\frac{\pi s}{2}) \Gamma(s) \zeta(s)$ ,  $\zeta(s) = 0$  is true. According the equation  $\xi(s) = \frac{1}{2} s(s-1) \Gamma(\frac{s}{2}) \pi^{-\frac{s}{2}} \zeta(s)$

obtained by Riemann, so  $\xi(s) = \xi(1-s)$ , because  $\Gamma(\frac{s}{2}) = \overline{\Gamma(\frac{\bar{s}}{2})}$ , and  $\pi^{-\frac{s}{2}} = \overline{\pi^{-\frac{\bar{s}}{2}}}$ , and because

$\zeta(s) = \overline{\zeta(\bar{s})}$ , so  $\xi(s) = \overline{\xi(\bar{s})}$ . So when  $\zeta(s) = 0$ , then  $\xi(s) = \zeta(1-s) = \zeta(\bar{s}) = 0$  and  $\xi(s) = \xi(1-s) = \xi(\bar{s}) = 0$  must

be true, so the zeros of the Riemann  $\zeta(s)$  function and the nontrivial zeros of the Riemann  $\xi(s)$

function are identical, so the complex root of Riemann  $\xi(s) = 0$  satisfies  $s = \frac{1}{2} + ti$  ( $t \in \mathbb{R}$  and  $t \neq 0$ ,  $s \in \mathbb{C}$ ), ...

According to the Riemann function  $\prod_{2}^s (s-1) \pi^{-\frac{s}{2}} \zeta(s) = \xi(t)$  and he Riemann hypothesis  $s = \frac{1}{2} + ti$ , because

$s \neq 1$ , and  $\prod_{2}^s \neq 0$ ,  $\pi^{-\frac{s}{2}} \neq 0$ , so  $\prod_{2}^s (s-1) \pi^{-\frac{s}{2}} \neq 0$ , and when  $\xi(t) = 0$ , then  $\prod_{2}^s (s-1) \pi^{-\frac{s}{2}} \zeta(\frac{1}{2} + ti) = \xi(t) = 0$ , and

$\zeta(\frac{1}{2} + ti) = \frac{\xi(t)}{\prod_{2}^s (s-1) \pi^{-\frac{s}{2}}} = \frac{0}{\prod_{2}^s (s-1) \pi^{-\frac{s}{2}}} = 0$ , so  $t \in \mathbb{R}$  and  $t \neq 0$ . So the root  $t$  of the equations

$\prod_{2}^s (s-1) \pi^{-\frac{s}{2}} \zeta(\frac{1}{2} + ti) = \xi(t) = 0$  and  $4 \int_1^{\infty} \frac{d(x^2 \Psi'(x))}{dx} x^{-\frac{1}{4}} \cos(\frac{1}{2} t \ln x) dx = \xi(t) = 0$  and

$\xi(t) = \frac{1}{2} -(t^2 + \frac{1}{4}) \int_1^{\infty} \Psi(x) x^{-\frac{3}{4}} \cos(\frac{1}{2} t \ln x) = 0$  must be real and  $t \neq 0$ . If  $\text{Re}(s) = \frac{k}{2}$  ( $k \in \mathbb{R}$ ), then

## The proof of the Riemann conjecture

$\zeta(k-s)=2^{k-s}\pi^{-s}\text{Cos}(\frac{\pi s}{2})\Gamma(s)\zeta(s)$  and  $\xi(k-s)=\frac{1}{2}s(s-k)\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s)$  are true, so when  $\zeta(s)=0$ , then  $\zeta(s)=\zeta(k-s)=\zeta(\bar{s})=0$  and  $\xi(s)=\xi(k-s)=\xi(\bar{s})=0$  must be true, and  $s=\frac{k}{2}+ti$  ( $k \in \mathbb{R}$ ,  $t \in \mathbb{R}$  and  $t \neq 0$ ,  $s \in \mathbb{C}$ ) must be true, then  $\prod_{\frac{s}{2}}(s-k)\pi^{-\frac{s}{2}}\zeta(\frac{k}{2}+ti)=\xi(t)=0$ , and  $\zeta(\frac{k}{2}+ti)=\frac{\xi(t)}{\prod_{\frac{s}{2}}(s-k)\pi^{-\frac{s}{2}}}=\frac{0}{\prod_{\frac{s}{2}}(s-k)\pi^{-\frac{s}{2}}}=0$ , so  $t \in \mathbb{R}$  and  $t \neq 0$ . So the root of the equations  $\prod_{\frac{s}{2}}(s-k)\pi^{-\frac{s}{2}}\zeta(\frac{k}{2}+ti)=\xi(t)=0$  must be real and  $t \neq 0$ . But the Riemann  $\zeta(s)$  function only satisfies  $\zeta(1-s)=2^{1-s}\pi^{-s}\text{Cos}(\frac{\pi s}{2})\Gamma(s)\zeta(s)$  and  $\xi(s)=\frac{1}{2}s(s-1)\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s)$ , is also say that only  $\zeta(1-s)=2^{1-s}\pi^{-s}\text{Cos}(\frac{\pi s}{2})\Gamma(s)\zeta(s)$  is true, so only  $\text{Re}(s)=\frac{k}{2}=\frac{1}{2}$  ( $k \in \mathbb{R}$ ) is true, so only  $k=1$  is true. The Riemann hypothesis and the Riemann conjecture must satisfy the properties of the Riemann  $\zeta(s)$  function and the Riemann  $\xi(s)$  function, The properties of the Riemann  $\zeta(s)$  function and the Riemann  $\xi(s)$  function are fundamental, the Riemann hypothesis and the Riemann conjecture must be correct to reflect the properties of the Riemann  $\zeta(s)$  function and the Riemann  $\xi(s)$  function, that is, the roots of the Riemann  $\xi(t)$  function can only be real, that is,  $\text{Re}(s)$  can only be equal to  $\frac{1}{2}$ , and  $\text{Im}(s)$  must be real, and  $\text{Im}(s)$  is not equal to zero. So the Riemann hypothesis and the Riemann conjecture must be correct.

Riemann found in his paper that

$$\begin{aligned} \prod\left(\frac{s}{2}-1\right)\pi^{-\frac{s}{2}}\zeta(s) &= \int_1^\infty \psi(x) x^{\frac{s}{2}-1} dx + \int_1^\infty \psi\left(\frac{1}{x}\right) x^{\frac{s-3}{2}} dx \\ &\quad + \frac{1}{2} \int_0^1 \left(x^{\frac{s-3}{2}} - x^{\frac{s}{2}-1}\right) dx \\ &= \frac{1}{s(s-1)} + \int_1^\infty \psi(x) \left(x^{\frac{s}{2}-1} + x^{-\frac{1+s}{2}}\right) dx, \end{aligned}$$

Because  $\frac{1}{s(s-1)}$  and  $\int_1^\infty \psi(x) \left(x^{\frac{s}{2}-1} + x^{-\frac{1+s}{2}}\right) dx$  are all invariant under the transformation  $s \rightarrow 1-s$ . If I

introduce the auxiliary function  $\psi(s) = \prod\left(\frac{s}{2}-1\right)\pi^{-\frac{s}{2}}\zeta(s)$ , So I can just write it as  $\psi(s) = \psi(1-s)$ . But it

would be more convenient to add the factor  $s(s-1)$  to  $\psi(s)$  and introduce the coefficient  $\frac{1}{2}$ , which is

exactly what Riemann did, is that to take  $\xi(s) = \frac{1}{2}s(s-1)\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s)$ . Because the factor  $(s-1)$

cancel out the first pole of  $\zeta(s)$  at  $s=1$ , And the factor  $s$  cancel out the pole of  $\Gamma\left(\frac{s}{2}\right)$  at  $s=0$ , and  $s$

is equal to  $-2, -4, -6, \dots$ , the rest of the poles of  $\Gamma\left(\frac{s}{2}\right)$  cancel out. So  $\xi(s)$  is an integral function. And

notice that the factor  $s(s-1)$  obviously doesn't change under the transformation  $s \rightarrow 1-s$ , So we

also have the function  $\xi(s) = \xi(1-s)$ . Base on  $\zeta(1-s) = 2^{1-s}\pi^{-s}\text{Cos}(\frac{\pi s}{2})\Gamma(s)\zeta(s)$  (Formula 1). When

$\sin(\frac{\pi s}{2})=0$ , then if  $s = -2n$  ( $n \in \mathbb{Z}_+$ ),  $\xi(s)$  is going to take the zero. At the same time, according to

$\zeta(1-s) = 2^{1-s}\pi^{-s}\text{Cos}(\frac{\pi s}{2})\Gamma(s)\zeta(s)$ , when  $s \neq 1+2n$  ( $n \in \mathbb{Z}_+$ ), and if  $\zeta(s)=0$ , then must  $\zeta(1-s)=0$ , is that to say

## *The proof of the Riemann conjecture*

$\zeta(s)=\zeta(1-s)=0$ . According to Riemann's hypothesis  $s=\frac{1}{2}+ti(t\in\mathbb{C},s\in\mathbb{C}$  and  $t\neq 0$ ),  $s$  and  $t$  differ by a linear transformation . It's a 90 degree rotation plus a translation of  $\frac{1}{2}$ . So line  $\text{Re}(s)=\frac{1}{2}$  in the  $s$  plane corresponds to the real number line in the  $t$  plane,the zero of Riemann  $\zeta(s)$  on the critical line  $\text{Re}(s)=\frac{1}{2}$  corresponds to the real root of  $\xi(t)(t\in\mathbb{C}$  and  $t\neq 0)$ . In Riemann function  $\xi(t)$ , the function equation  $\xi(s)=\xi(1-s)$  becomes equation  $\xi(t)=\xi(-t)(t\in\mathbb{C}$  and  $t\neq 0)$  is an even function, an even function is a symmetric function, it's zeros are distributed symmetrically with respect to  $t=0$  .The function  $\xi(t)(t\in\mathbb{C}$ , and  $t\neq 0)$  designed by Riemann and Riemann's hypothesis  $s=\frac{1}{2}+ti(t\in\mathbb{C},s\in\mathbb{C}$ , and  $t\neq 0)$  and  $\xi(s)=\xi(1-s)$  are equivalent to  $\xi(t)=\xi(-t)$ .So the function  $\xi(s)$  is also an even function.The zero points on the graph of an even function  $\xi(s)$  with respect to the coordinates of its argument on the real number line equal to some value are symmetrically distributed on the line perpendicular to the real number line of the complex plane. When  $\xi(t)=0$ , is also that  $\xi(t)=\xi(-t)=0$ ,the zeros of  $\xi(t)$ are symmetrically distributed with respect to  $t$  equals 0.When  $\xi(s)=0$ ,is also that  $\xi(s)=\xi(1-s)=0$ ,the zeros of  $\xi(s)$  are symmetrically distributed with respect to point  $(\frac{1}{2},0i)$  on a line perpendicular to the real number line of the complex plane.So when  $\xi(s)=\xi(1-s)=0$ ,  $s$  and  $1-s$  are pair of zeros of the function  $\xi(s)$  symmetrically distributed in the complex plane with respect to point  $(\frac{1}{2},0i)$  on a line perpendicular to the real number line of the complex plane.When  $\zeta(s)=0$ ,  $\zeta(1-s)=0$  is also that  $\zeta(s)=\zeta(1-s)=0$ . We find  $\zeta(s)=\zeta(1-s)=0$  and  $\xi(s)=\xi(1-s)=0$  are just the name of the function is idifferent,the independent variable  $s$  is equal to  $\frac{1}{2}+ti(t\in\mathbb{C},s\in\mathbb{C})$ ,that means that the zero arguments of function  $\zeta(s)$  and function  $\xi(s)$  are exactly the same,so the zeros of the  $\zeta(s)$  function in the complex plane also correspond to the symmetric distribution of point  $(\frac{1}{2},0i)$  on a line perpendicular to the real number line in the complex plane,so When  $\zeta(s) = \zeta(1-s) = 0$ , $s$  and  $1-s$  are pair of zeros of the function  $\zeta(s)$  symmetrically distributed in the complex plane with respect to point  $(\frac{1}{2},0i)$  on a line perpendicular to the real number line of the complex plane.We got  $\overline{\zeta(s)} = \zeta(\bar{s})(s= \rho + yi, \rho \in \mathbb{R}, y \in \mathbb{R}$  and  $y \neq 0)$  before,When  $t$  in Riemann's hypothesis  $s=\frac{1}{2}+ti(t\in\mathbb{C},s\in\mathbb{C}$  and  $t\neq 0)$  is a complex number, and  $s=\frac{1}{2}+ti=\rho+yi$ , then  $s$  in  $\overline{\zeta(s)}=\zeta(\bar{s})(s=\rho+yi,\rho \in \mathbb{R}, y \in \mathbb{R}$  and  $y \neq 0)$  is consistent with  $s$  in Riemann's hypothesis  $s=\frac{1}{2}+ti(t\in\mathbb{C},s\in\mathbb{C}$  and  $t\neq 0)$  .If  $\zeta(s)=\zeta(\bar{s})=0(s=\rho+yi,\rho \in \mathbb{R}, y \in \mathbb{R}$  and  $y \neq 0)$ ,Since  $s$  and  $\bar{s}$  are a pair of conjugate complex numbers,So  $s$  and  $\bar{s}$  must be a pair of zeros of the function  $\zeta(s)$  in the complex plane with respect to point  $(\rho,0i)$  on a line perpendicular to the real number line. $s$  is a symmetric zero of  $1-s$ , and a symmetric zero of  $\bar{s}$ . By the definition of complex numbers, how can a symmetric zero of the same function  $\zeta(s)$  of the same zero independent variable  $s$  on a line perpendicular to the real number axis of the complex plane be both a symmetric zero of  $1-s$  on a line perpendicular to the real number axis of the complex plane with respect to point  $(\frac{1}{2},0i)$  and a symmetric zero of  $\bar{s}$  on a line perpendicular to the real

number axis of the complex plane with respect to point  $(\rho, 0i)$ ? Unless  $\rho$  and  $\frac{1}{2}$  are the same value, is also that  $\rho = \frac{1}{2}$ , and only  $1-s=\bar{s}$  is true, and  $1-s=s$  is wrong. Otherwise it's impossible, this is determined by the uniqueness of the zero of the function  $\zeta(s)$  on the line passing through that point perpendicular to the real number axis of the complex plane with respect to the vertical foot symmetric distribution of the zero of the line and the real number axis of the complex plane. Only one line can be drawn perpendicular from the zero independent variable  $s$  of the function  $\zeta(s)$  to the real number line of the complex plane, the vertical line has only one point of intersection with the real number axis of the complex plane. In the same complex plane, the same zero point of the function  $\zeta(s)$  on the line passing through that point perpendicular to the real number line of the complex plane there will be only one zero point about the vertical foot symmetric distribution of the line and the real number line of the complex plane. Because  $\overline{\zeta(s)} = \zeta(\bar{s})$  ( $s = \rho + yi, \rho \in \mathbb{R}, y \in \mathbb{R}$  and  $y \neq 0$ ), then if  $\zeta(\rho + yi) = 0$ , then  $\zeta(\rho - yi) = 0$ , and because  $\zeta(s) = \zeta(1-s) = 0$ , then  $\zeta(1-\rho-yi) = 0$ , and because  $\zeta(s) = \zeta(1-s) = 0$ , then  $\zeta(1-\rho-yi) = 0$ . The next three equations,  $\zeta(\rho + yi) = 0$ ,  $\zeta(\rho - yi) = 0$ , and  $\zeta(1-\rho-yi) = 0$ , are all true, so only  $1-\rho = \rho$  is true, only  $s = \frac{1}{2} + ti$  ( $t \in \mathbb{R}$  and  $t \neq 0, s \in \mathbb{C}$ ) is true. Since the harmonic series  $\zeta(1)$  diverges, it has been proved by the late medieval French scholar Orem (1323-1382). The Riemann hypothesis and the Riemann conjecture must satisfy the properties of the Riemann  $\zeta(s)$  function and the Riemann  $\xi(s)$  function, The properties of the Riemann  $\zeta(s)$  function and the Riemann  $\xi(s)$  function are fundamental, the Riemann hypothesis and the Riemann conjecture must be correct to reflect the properties of the Riemann  $\zeta(s)$  function and the Riemann  $\xi(s)$  function, that is, the roots of the Riemann  $\xi(t)$  function must only be real, that is,  $\text{Re}(s)$  can only be equal to  $\frac{1}{2}$ , and  $\text{Im}(s)$  must be real, and  $\text{Im}(s)$  is not equal to zero. So the Riemann hypothesis and the Riemann conjecture must be correct. Riemann got  $\prod_2^s (s-1) \pi^{-\frac{s}{2}} \zeta(s) = \xi(t)$  ( $t \in \mathbb{C}, s \in \mathbb{C}$ ), and  $\xi(t) = \frac{1}{2} - (t^2 + \frac{1}{4}) \int_1^\infty \Psi(x) x^{-\frac{3}{4}} \cos(\frac{1}{2} t \ln x) dx$  ( $t \in \mathbb{C}$ ) in his paper, or  $\prod_2^s (s-1) \pi^{-\frac{s}{2}} \zeta(s) = \xi(t)$  and ( $t \in \mathbb{C}, s \in \mathbb{C}$ ),  $\xi(t) = 4 \int_1^\infty \frac{d(x^{\frac{3}{2}} \Psi'(x))}{dx} x^{-\frac{1}{4}} \cos(\frac{1}{2} t \ln x) dx$  ( $t \in \mathbb{C}$ ) [1]. Because  $\zeta(\frac{1}{2} + ti) = 0$  ( $t \in \mathbb{R}$  and  $t \neq 0, s \in \mathbb{C}$ ), so  $\zeta(\frac{1}{2} + ti) = 0$  ( $t \in \mathbb{R}$  and  $t \neq 0, s \in \mathbb{C}$ ) is true, so  $\prod_2^s (s-1) \pi^{-\frac{s}{2}} \zeta(\frac{1}{2} + ti) = \xi(t) = 0$  ( $t \in \mathbb{R}$  and  $t \neq 0, s \in \mathbb{C}$ ) and  $\prod_2^s (s-1) \pi^{-\frac{s}{2}} \zeta(\frac{1}{2} + ti) = 4 \int_1^\infty \frac{d(x^{\frac{3}{2}} \Psi'(x))}{dx} x^{-\frac{1}{4}} \cos(\frac{1}{2} t \ln x) dx = \xi(t) = 0$  ( $t \in \mathbb{R}$  and  $t \neq 0, s \in \mathbb{C}$ ) and  $\xi(t) = \frac{1}{2} - (t^2 + \frac{1}{4}) \int_1^\infty \Psi(x) x^{-\frac{3}{4}} \cos(\frac{1}{2} t \ln x) dx = 0$  ( $t \in \mathbb{C}$ ), so the roots of equations  $\prod_2^s (s-1) \pi^{-\frac{s}{2}} \zeta(\frac{1}{2} + ti) = \xi(t) = 0$  ( $t \in \mathbb{C}$ ) and  $4 \int_1^\infty \frac{d(x^{\frac{3}{2}} \Psi'(x))}{dx} x^{-\frac{1}{4}} \cos(\frac{1}{2} t \ln x) dx = \xi(t) = 0$  ( $t \in \mathbb{C}$ ) and  $\xi(t) = \frac{1}{2} - (t^2 + \frac{1}{4}) \int_1^\infty \Psi(x) x^{-\frac{3}{4}} \cos(\frac{1}{2} t \ln x) dx = 0$  ( $t \in \mathbb{C}$ ) must all be real numbers. When  $\zeta(s) = 0$  and  $\xi(t) = 0$ , the real part of the equation  $\xi(t) = 0$  ( $t \in \mathbb{C}$ ) must be real between 0 and T. Because the real part of the equation  $\xi(t) = 0$  has the number of complex roots between 0 and T approximately equal to  $\frac{T}{2\pi} \ln \frac{T}{2\pi} - \frac{T}{2\pi}$  [1], This result of Riemann's estimate of the number of zeros was rigorously proved by Mangoldt in 1895. Then, when  $\zeta(s) = 0$  ( $s \in \mathbb{C}$ ) and  $\xi(t) = 0$  ( $t \in \mathbb{C}$  and  $t \neq 0$ ), the number of real roots of the real part of the equation  $\xi(t) = 0$  ( $t \in \mathbb{C}$  and  $t \neq 0$ )



between 0 and T must be approximately equal to  $\frac{T}{2\pi} \ln \frac{T}{2\pi} - \frac{T}{2\pi} \{1\}$ , so when the Riemann  $\zeta(s)$  function has nontrivial zeroes, then the Riemann hypothesis and the Riemann conjecture are perfectly valid.

Definition:

Assuming that  $a(n)$  is a uniprimitive function, then the Dirichlet series  $\sum_{n=1}^{\infty} a(n)n^{-s}$  is equal to the Euler product  $\prod_p P(p, s)$ . Where the product is applied to all prime numbers  $p$ , it can be expressed

as:  $1+a(p)p^{-s}+a(p^2)p^{-2s}+\dots$ , this can be seen as a formal generating function, where the existence of

a formal Euler product expansion and  $a(n)$  being a product function are mutually sufficient and necessary conditions. When  $a(n)$  is a completely integrative function, an important special case is

obtained, where  $P(p, s)$  is a geometric series, and  $P(p, s) = \frac{1}{1-a(p)p^{-s}}$ . When  $a(n)=1$ , it is the

Riemann zeta function, and more generally the Dirichlet feature.

Euler's product formula: for any complex number  $s$ ,

$\text{Re}(s) > 1$ , then  $\sum_{n=1}^{\infty} n^{-s} = \prod_p (1 - p^{-s})^{-1}$ , and when  $\text{Re}(s) > 1$  Riemann Zeta function  $\zeta(s) =$

$$\sum_{n=1}^{\infty} n^{-s} = \prod_p (1 - p^{-s})^{-1} \quad (s \in \mathbb{C} \text{ and } \text{Re}(s) > 0 \text{ and } (s \neq 1), n \in \mathbb{Z}_+, p \in \mathbb{Z}_+, s \in \mathbb{C}, n \text{ goes through all}$$

the natural numbers,  $p$  goes through all the prime numbers).

Riemann zeta function expression:

$\zeta(s) = 1/1^s + 1/2^s + 1/3^s + \dots + 1/m^s$  ( $m$  tends to infinity, and  $m$  is always even).

(1) Multiply both sides of the expression by  $(1/2^s)$ , \*for multiplication

$$(1/2^s) * \zeta(s) = 1/1^s * (1/2^s) + 1/2^s * (1/2^s) + 1/3^s * (1/2^s) + \dots + 1/m^s *$$

$$(1/2^s) = 1/2^s + 1/4^s + 1/6^s + \dots + 1/(2 * m)^s$$

This is given by (1) - (2)

$$\zeta(s) - (1/2^s) * \zeta(s) = 1/1^s + 1/2^s + 1/3^s + \dots + 1/m^s - [1/2^s + 1/4^s + 1/6^s + \dots + 1/(2 * m)^s]$$

The derivation of Euler product formula is as follows:

$$\zeta(s) - (1/2^s) * \zeta(s) = 1/1^s + 1/3^s + 1/5^s + \dots + 1/(m-1)^s.$$

Generalized Euler product formula:

Suppose  $f(n)$  is a function that satisfies  $f(n_1)f(n_2) = f(n_1n_2)$  and  $\sum_n |f(n)| < +\infty$  ( $n_1$  and  $n_2$  are both natural numbers), then  $\sum_n f(n) = \prod_p [1 + f(p) + f(p^2) + f(p^3) + \dots]$ .

Proof:

The proof of Euler product formula is very simple, the only caution is to deal with infinite series and infinite products, can not arbitrarily use the properties of finite series and finite products. What I prove below is a more general result, and the Euler product formula will appear as a special case of this result.

Due to  $\sum_{n=1}^{\infty} |f(n)| < \infty$ , so  $1 + f(p) + f(p^2) + f(p^3) + \dots$  absolute convergence. Consider the part of  $p < N$  in the continued product (finite product), since the series is absolutely convergent and the product has only finite terms, the same associative and distributive laws can be used as ordinary finite summations and products.

Using the product property of  $f(n)$ , we can obtain:

$\prod_{p < N} [1 + f(p) + f(p^2) + f(p^3) + \dots] = \sum f(n)$ . The right end of the summation is performed on all natural numbers with only prime factors below  $N$  (each such natural number occurs only once in the

summation, because the prime factorization of the natural numbers is unique). Since all natural

numbers that are themselves below N obviously contain only prime factors below N, So  $\sum_{n < N} f(n) =$

$\sum_{n < N} f(n) + R(N)$ , Where  $R(N)$  is the result of summing all natural numbers that are greater than or equal to N but contain only prime factors below N. From this we get:  $\prod_{p < N} [1 + f(p) + f(p^2) + f(p^3) + \dots] = \sum_{n < N} f(n) + R(N)$ . For the generalized Euler product formula to hold, it is only necessary to prove  $\lim_{n \rightarrow \infty} R(N) = 0$ , and this is obvious, because  $|R(N)| \leq \sum_{n \geq N} |f(n)|$ , and  $\sum_n |f(n)| < +\infty$  sign of

$\lim_{n \rightarrow \infty} \sum_{n \geq N} |f(n)| = 0$ , thus  $\lim_{n \rightarrow \infty} R(N) = 0$ . Because

$1 + f(p) + f(p^2) + f(p^3) + \dots = 1 + f(p) + f(p)^2 + f(p)^3 + \dots = [1 - f(p)]^{-1}$ , so the generalized Euler product formula can also be written as:

$\sum_n f(n) = \prod_p [1 - f(p)]^{-1}$ . In the generalized Euler product formula, take  $f(n) = n^{-s}$ , Then obviously  $\sum_n |f(n)| < \infty$  corresponds to the condition  $\text{Re}(s) > 1$  in the Euler product formula, and the generalized Euler product formula is reduced to the Euler product formula.

From the above proof, we can see that the key to the Euler product formula is the basic property that every natural number has a unique prime factorization, that is, the so-called fundamental theorem of arithmetic.

For any complex number  $s$ ,  $\chi(n)$  is the Dirichlet characteristic and satisfies the following properties:

- 1: There exists a positive integer  $q$  such that  $\chi(n+q) = \chi(n)$ ;
- 2: when  $n$  and  $q$  are not mutual prime,  $\chi(n) = 0$ ;
- 3:  $\chi(a)\chi(b) = \chi(ab)$  for any integer  $a$  and  $b$ ;

**Reasoning 3:**

If  $0 < \text{Re}(s) < 1$ , then

$$L(s, \chi(n)) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \quad (n \in \mathbb{Z}_+, p \in \mathbb{Z}_+, s \in \mathbb{C}, n \text{ goes through all the natural numbers, } p \text{ goes through all the prime numbers, } \chi(n) \in \mathbb{R} \text{ and } (\chi(n) \neq 0), a(n) = a(p) = \chi(n), P(p, s) = \frac{1}{1 - a(p)p^{-s}}).$$

the prime numbers,  $\chi(n) \in \mathbb{R}$  and  $(\chi(n) \neq 0), a(n) = a(p) = \chi(n), P(p, s) = \frac{1}{1 - a(p)p^{-s}}$ .

Next we prove the generalized Riemann conjecture when the Dirichlet eigen function  $\chi(n)$  is any real number that is not equal to zero,

and

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} \quad (s \in \mathbb{C} \text{ and } \text{Re}(s) > 0 \text{ and } (s \neq 1)) \text{ and } \eta(s) = (1 - 2^{1-s}) \zeta(s) \quad (s \in \mathbb{C} \text{ and } \text{Re}(s) >$$

0 and  $s \neq$

$$1, \zeta(s) \text{ is the Riemann Zeta function) ,so Riemann } \zeta(s) = \frac{\eta(s)}{(1 - 2^{1-s})} = \frac{1}{(1 - 2^{1-s})} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} =$$

$$\frac{(-1)^{n-1}}{(1 - 2^{1-s})} \prod_p (1 - p^{-s})^{-1} \quad (s \in \mathbb{C} \text{ and } \text{Re}(s) > 0 \text{ and } (s \neq 1), n \in \mathbb{Z}_+, p \in \mathbb{Z}_+, s \in \mathbb{C}, n \text{ goes through all the natural numbers, } p \text{ goes through all the prime numbers}), \text{so}$$

$$\text{GRH}(s, \chi(n)) = L(s, \chi(n)) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \sum_{n=1}^{\infty} a(n)n^{-s} = \prod_p P(p, s) = \prod_p \left( \frac{1}{1 - a(p)p^{-s}} \right) \quad (n \in$$

$\mathbb{Z}_+, p \in \mathbb{Z}_+, s \in \mathbb{C}, n$  goes through all the natural numbers,  $p$  goes through all the prime numbers,  $\chi$

$$(n) \in \mathbb{R} \text{ and } (\chi(n) \neq 0), a(n) = a(p) = \chi(n), P(p, s) = \frac{1}{1 - a(p)p^{-s}}).$$

$$a(p)p^{-s} = a(p)p^{-\rho} \frac{1}{(\cos(y \ln p) + i \sin(y \ln p))} = a(p)(p^{-\rho} (\cos(y \ln p) - i \sin(y \ln p))),$$

$$(1 - a(p)p^{-s}) = 1 - a(p)(p^{-\rho}(\cos(y \ln p) - i \sin(y \ln p))) = 1 - a(p)p^{-\rho} \cos(y \ln p) + a(p)p^{-\rho} i \sin(y \ln p),$$

$$a(p)p^{-\bar{s}} = a(p)p^{-\rho} \frac{1}{(\cos(y \ln p) - i \sin(y \ln p))} = a(p)(p^{-\rho}(\cos(y \ln p) + i \sin(y \ln p))),$$

$$(1 - a(p)p^{-\bar{s}}) = 1 - a(p)p^{-\rho} \cos(y \ln p) - i a(p)p^{-\rho} \sin(y \ln p),$$

because

$$(1 - a(p)p^{-s}) = \overline{1 - a(p)p^{-\bar{s}}},$$

so

$$(1 - a(p)p^{-s})^{-1} = \overline{(1 - a(p)p^{-\bar{s}})^{-1}},$$

so

$$\prod_p (1 - a(p)p^{-s})^{-1} = \overline{\prod_p (1 - a(p)p^{-\bar{s}})^{-1}},$$

because  $L(s, \chi(n)) = \sum_{n=1}^{\infty} a(n)n^{-s} = \prod_p (1 - a(p)p^{-s})^{-1}$  and  $L(\bar{s}, \chi(n)) = \sum_{n=1}^{\infty} a(n)n^{-\bar{s}} =$

$$\prod_p (1 - a(p)p^{-\bar{s}})^{-1}, \text{ for the Generalized Riemann function } L(s, \chi(n)) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \sum_{n=1}^{\infty} a(n)n^{-s} =$$

$$\prod_p \frac{1}{1 - a(p)p^{-s}} \quad (n \in \mathbb{Z}_+, p \in \mathbb{Z}_+, s \in \mathbb{C}, n \text{ goes through all the natural numbers, } p \text{ goes through all the}$$

prime numbers,  $\chi(n) \in \mathbb{R}$  and  $(\chi(n) \neq 0), a(n) = a(p) = \chi(n), P(p, s) = \frac{1}{1 - a(p)p^{-s}}).$

so

$$L(s, \chi(n)) = \overline{L(\bar{s}, \chi(n))}.$$

$$a(p)p^{1-s} = a(p)p^{(1-\rho-yi)} = a(p)p^{1-\rho} x^{-yi} = a(p)p^{1-\rho} (\cos(\ln p) + i \sin(\ln p))^{-y} = a(p)p^{1-\rho} (\cos(y \ln p) - i \sin(y \ln p)),$$

$$a(p)p^{1-\bar{s}} = a(p)p^{(1-\rho+yi)} = a(p)p^{1-\rho} p^{yi} = a(p)p^{1-\rho} (p^{yi}) = a(p)p^{1-\rho} (\cos(\ln p) + i \sin(\ln p))^y = a(p)p^{1-\rho} (\cos(y \ln p) + i \sin(y \ln p)),$$

then

$$a(p)p^{-(1-s)} = a(p)p^{\rho-1} \frac{1}{(\cos(y \ln p) - i \sin(y \ln p))} = a(p)(p^{\rho-1}(\cos(y \ln p) + i \sin(y \ln p))),$$

$$(1 - a(p)p^{-(1-s)}) = 1 - a(p)p^{\rho-1}(\cos(y \ln p) + i \sin(y \ln p)) =$$

$$1 - a(p)p^{\rho-1} \cos(y \ln p) - a(p)p^{\rho-1} i \sin(y \ln p),$$

$$(1 - a(p)p^{-\bar{s}}) = 1 - a(p)(p^{-\rho}(\cos(y \ln p) + i \sin(y \ln p))) = 1 - a(p)p^{-\rho} \cos(y \ln p) - i a(p)p^{-\rho} \sin(y \ln p),$$

When  $\rho = \frac{1}{2}$ , then

$$(1 - a(p)p^{-(1-s)}) = (1 - a(p)p^{-\bar{s}}),$$

$$(1 - a(p)p^{-(1-s)})^{-1} = (1 - a(p)p^{-\bar{s}})^{-1},$$

so

$$\prod_p (1 - a(p)p^{-(1-s)})^{-1} = \prod_p (1 - a(p)p^{-\bar{s}})^{-1},$$

because  $L(1-s, \chi(n)) = \prod_p (1 - a(p)p^{-(1-s)})^{-1}$  and  $L(\bar{s}, \chi(n)) = \prod_p (1 - a(p)p^{-\bar{s}})^{-1}, n \in \mathbb{Z}_+, p \in$

$\mathbb{Z}_+, s \in \mathbb{C}, n$  goes through all the natural numbers,  $p$  goes through all the prime numbers,  $\chi$

$(n) \in \mathbb{R}$  and  $(\chi(n) \neq 0), a(n) = a(p) = \chi(n), P(p, s) = \frac{1}{1 - a(p)p^{-s}}.$

so

$$\text{Only } L(1-s, \chi(n)) = L(\bar{s}, \chi(n)),$$

and

$$\text{Only } L(1 - \bar{s}, \mathbf{X}(n)) = L(s, \mathbf{X}(n)).$$

Because  $L(s, \mathbf{X}(n)) = \mathbf{X}(n)\zeta(s)$  and  $L(1 - s, \mathbf{X}(n)) = \mathbf{X}(n)\zeta(1-s)$ , so When only  $\rho = \frac{1}{2}$ , it must be true

that  $L(s, \mathbf{X}(n)) = \overline{L(\bar{s}, \mathbf{X}(n))}$ , and it must be true that  $L(1 - s, \mathbf{X}(n)) = L(\bar{s}, \mathbf{X}(n))$ .

Suppose  $k \in \mathbb{R}$ ,

$$a(p)p^{k-s} = a(p)p^{(k-\rho-yi)} = a(p)p^{k-\rho}x^{-yi} = a(p)p^{k-\rho}(\cos(\ln p) + i \sin(\ln p))^{-y} = a(p)p^{k-\rho}(\cos(y \ln p) - i \sin(y \ln p)),$$

$$a(p)p^{k-\bar{s}} = a(p)p^{(k-\rho+yi)} = a(p)p^{k-\rho}p^{yi} = a(p)p^{k-\rho}(p^{yi}) = a(p)p^{k-\rho}(\cos(\ln p) + i \sin(\ln p))^y =$$

$$a(p)(p^{k-\rho}(\cos(y \ln p) + i \sin(y \ln p))),$$

then

$$a(p)p^{-(k-s)} = a(p)p^{\rho-k} \frac{1}{(\cos(y \ln p) - i \sin(y \ln p))} = a(p)(p^{\rho-k}(\cos(y \ln p) + i \sin(y \ln p))),$$

$$(1 - a(p)p^{-(k-s)}) = 1 - (a(p)p^{\rho-k}(\cos(y \ln p) + i \sin(y \ln p))) = 1 - a(p)p^{\rho-k} \cos(y \ln p) - ip^{\rho-k} \sin(y \ln p),$$

$$(1 - a(p)p^{-\bar{s}}) = 1 - (a(p)p^{-\rho}(\cos(y \ln p) + i \sin(y \ln p))) = 1 - a(p)p^{-\rho} \cos(y \ln p) - ia(p)p^{-\rho} \sin(y \ln p),$$

When  $\rho = \frac{k}{2}$  ( $k \in \mathbb{R}$ ), then

$$(1 - a(p)p^{-(k-s)}) = (1 - a(p)p^{-\bar{s}}),$$

$$(1 - a(p)p^{-(k-s)})^{-1} = (1 - a(p)p^{-\bar{s}})^{-1},$$

so

$$\prod_p (1 - a(p)p^{-(k-s)})^{-1} = \prod_p (1 - a(p)p^{-\bar{s}})^{-1},$$

because  $L(k - s, \mathbf{X}(n)) = \prod_p (1 - a(p)p^{-(k-s)})^{-1}$  and  $L(\bar{s}, \mathbf{X}(n)) = \prod_p (1 - a(p)p^{-\bar{s}})$ , for the

generalized Riemann function  $L(s, \mathbf{X}(n))$  ( $n \in \mathbb{Z}_+$ ,  $p \in \mathbb{Z}_+$ ,  $s \in \mathbb{C}$ ,  $n$  goes through all the natural

numbers,  $p$  goes through all the prime numbers,

$$\mathbf{X}(n) \in \mathbb{R} \text{ and } (\mathbf{X}(n) \neq 0), a(n) = a(p) = \mathbf{X}(n), P(p, s) = \frac{1}{1 - a(p)p^{-s}}.$$

so

$$\text{Only } L(k - s, \mathbf{X}(n)) = L(\bar{s}, \mathbf{X}(n)),$$

and

$$\text{Only } L(k - \bar{s}, \mathbf{X}(n)) = L(s, \mathbf{X}(n)).$$

And because Only  $L(1 - s, \mathbf{X}(n)) = L(\bar{s}, \mathbf{X}(n))$ , so only  $k=1$  be true.

so

$$\text{GRH}(s, \mathbf{X}(n)) = L(s, \mathbf{X}(n)) = \sum_{n=1}^{\infty} \frac{\mathbf{X}(n)}{x^s} = \frac{\mathbf{X}(n)\eta(s)}{(1-2^{1-s})} = \frac{\mathbf{X}(n)}{(1-2^{1-s})} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{x^s} =$$

$$\frac{\mathbf{X}(n)}{(1-2^{1-s})} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{x^{\rho+yi}} = \frac{(-1)^{n-1}}{(1-2^{1-s})} \sum_{n=1}^{\infty} \mathbf{X}(n) \left( \frac{1}{x^{\rho}} \frac{1}{x^{yi}} \right) =$$

$$\frac{(-1)^{n-1}}{(1-2^{1-s})} \sum_{n=1}^{\infty} \mathbf{X}(n) (x^{-\rho}) \frac{1}{(\cos(\ln x) + i \sin(\ln x))^y} = \frac{(-1)^{n-1}}{(1-2^{1-s})} \sum_{n=1}^{\infty} \mathbf{X}(n) (x^{-\rho} (\cos(\ln x) +$$

$$i \sin(\ln x))^{-y}) =$$

$$\frac{(-1)^{n-1}}{(1-2^{1-s})} \sum_{n=1}^{\infty} \mathbf{X}(n) (n^{-\rho} (\cos(y \ln(n)) - i \sin(y \ln(n))) (x =$$

$n, x$  goes through all positive integers,  $n$  goes through all positive integers) ,

$$\begin{aligned} \text{GRH}(\bar{s}, \chi(n)) &= L(\bar{s}, \chi(n)) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{\bar{s}}} = \frac{\chi(n)\eta(\bar{s})}{(1-2^{1-\bar{s}})} = \frac{\chi(n)}{(1-2^{1-\bar{s}})} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{x^{\bar{s}}} \\ &= \frac{\chi(n)}{(1-2^{1-\bar{s}})} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{x^{\rho-yi}} = \frac{(-1)^{n-1}}{(1-2^{1-\bar{s}})} \sum_{n=1}^{\infty} \chi(n) \left(\frac{1}{x^{\rho}} \frac{1}{x^{-yi}}\right) \\ &= \frac{1}{(1-2^{1-\bar{s}})} \sum_{n=1}^{\infty} \left(\chi(n) \frac{1}{x^{\rho}} \frac{1}{(\cos(\ln x) + i \sin(\ln x))^{-y}}\right) \\ &= \frac{1}{(1-2^{1-\bar{s}})} \sum_{n=1}^{\infty} (\chi(n)x^{-\rho}(\cos(\ln x) + i \sin(\ln x))^y) = \end{aligned}$$

$$\frac{1}{(1-2^{1-\bar{s}})} \sum_{n=1}^{\infty} (\chi(n)n^{-\rho}(\cos(y \ln(n)) + i \sin(y \ln(n)))) \quad (x =$$

$n, x$  goes through all positive integers,  $n$  goes through all positive integers) ,

$$\begin{aligned} \text{GRH}(1-s, \chi(n)) &= L(1-s, \chi(n)) = \sum_{n=1}^{\infty} \frac{\chi(n)}{x^s} = \frac{\chi(n)\eta(1-s)}{(1-2^s)} = \frac{\chi(n)}{(1-2^s)} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{x^{1-\rho-yi}} = \\ \frac{(-1)^{n-1}}{(1-2^s)} \sum_{n=1}^{\infty} \chi(n) \left(\frac{1}{x^{1-\rho}} \frac{1}{x^{-yi}}\right) &= \frac{(-1)^{n-1}}{(1-2^s)} \sum_{n=1}^{\infty} (\chi(n)x^{\rho-1}(\cos(y \ln(n)) + i \sin(y \ln(n)))) \quad (x = \end{aligned}$$

$n, x$  goes through all positive integers,  $n$  goes through all positive integers) ,

Suppose

$$U = [\chi(n)1^{-\rho} \cos(y \ln 1) - \chi(n)2^{-\rho} \cos(y \ln 2) + \chi(n)3^{-\rho} \cos(y \ln 3) - \chi(n)4^{-\rho} \cos(y \ln 4) + \dots],$$

$$V = [\chi(n)1^{-\rho} \sin(y \ln 1) - \chi(n)2^{-\rho} \sin(y \ln 2) + \chi(n)3^{-\rho} \sin(y \ln 3) - \chi(n)4^{-\rho} \sin(y \ln 4) + \dots],$$

Then

$$L(s, \chi(n)) = \overline{L(\bar{s}, \chi(n))} .$$

And  $x$  goes through all the natural numbers, so  $x=1,2,3,\dots, n-1, n, \dots$ , let's just plug in, so

$$L(s, \chi(n)) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = [\chi(n)1^{-\rho} \cos(y \ln 1) - \chi(n)2^{-\rho} \cos(y \ln 2) + \chi(n)3^{-\rho} \cos(y \ln 3) - \chi(n)4^{-\rho} \cos$$

$$(y \ln 4) + \dots] - i[\chi(n)1^{-\rho} \sin(y \ln 1) - \chi(n)2^{-\rho} \sin(y \ln 2) + \chi(n)3^{-\rho} \sin(y \ln 3) - \chi(n)4^{-\rho} \sin(y \ln 4) + \dots] = U - Vi,$$

$$U = [\chi(n)1^{-\rho} \cos(y \ln 1) - \chi(n)2^{-\rho} \cos(y \ln 2) + \chi(n)3^{-\rho} \cos(y \ln 3) - \chi(n)4^{-\rho} \cos(y \ln 4) + \dots],$$

$$V = [\chi(n)1^{-\rho} \sin(y \ln 1) - \chi(n)2^{-\rho} \sin(y \ln 2) + \chi(n)3^{-\rho} \sin(y \ln 3) - \chi(n)4^{-\rho} \sin(y \ln 4) + \dots],$$

Then

$$L(\bar{s}, \chi(n)) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{\bar{s}}} = [\chi(n)1^{-\rho} \cos(y \ln 1) - \chi(n)2^{-\rho} \cos(y \ln 2) + \chi(n)3^{-\rho} \cos(y \ln 3) - \chi(n)4^{-\rho} \cos(y \ln 4) +$$

$$\dots] + i[\chi(n)1^{-\rho} \sin(y \ln 1) - \chi(n)2^{-\rho} \sin(y \ln 2) + \chi(n)3^{-\rho} \sin(y \ln 3) - \chi(n)4^{-\rho} \sin(y \ln 4) + \dots] = U + Vi,$$

$$U = [\chi(n)1^{-\rho} \cos(y \ln 1) - \chi(n)2^{-\rho} \cos(y \ln 2) + \chi(n)3^{-\rho} \cos(y \ln 3) - \chi(n)4^{-\rho} \cos(y \ln 4) + \dots],$$

$$V = [\chi(n)1^{-\rho} \sin(y \ln 1) - \chi(n)2^{-\rho} \sin(y \ln 2) + \chi(n)3^{-\rho} \sin(y \ln 3) - \chi(n)4^{-\rho} \sin(y \ln 4) + \dots],$$

$L(s, \chi(n))$  and  $\overline{L(\bar{s}, \chi(n))}$  are complex conjugates of each other, that is  $L(s, \chi(n)) = \overline{L(\bar{s}, \chi(n))}$ .

When  $\chi(n)\rho = \frac{1}{2}$ ,

then

$$L(s, \chi(n)) = L(1-s, \chi(n)) = U - Vi,$$

$$U = [X(n)1^{-\rho} \cos(y \ln n) - X(n)2^{-\rho} \cos(y \ln 2) + X(n)3^{-\rho} \cos(y \ln 3) - X(n)4^{-\rho} \cos(y \ln 4) + \dots],$$

$$V = [X(n)1^{-\rho} \sin(y \ln n) - X(n)2^{-\rho} \sin(y \ln 2) + X(n)3^{-\rho} \sin(y \ln 3) - X(n)4^{-\rho} \sin(y \ln 4) + \dots].$$

and When  $\rho = \frac{1}{2}$ ,

then

$$\text{Only } L(1-s, X(n)) = L(\bar{s}, X(n)).$$

$$\text{GRH}(k-s, X(n)) = L(k-s, X(n)) = \frac{X(n)\eta(k-s)}{(1-2^{1-k+s})} = \frac{X(n)}{(1-2^{1-k+s})} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{x^{k-\rho-yi}} =$$

$$\frac{(-1)^{n-1}}{(1-2^{1-k+s})} \sum_{n=1}^{\infty} X(n) \left( \frac{1}{x^{k-\rho}} \frac{1}{x^{-yi}} \right) = \frac{(-1)^{n-1}}{(1-2^{1-k+s})} \sum_{n=1}^{\infty} (X(n)n^{\rho-k} (\cos(y \ln(n)) + i \sin(y \ln(n)))) (x =$$

$n, x$  goes through all positive integers,  $n$  goes through all positive integers),

$$W = [X(n)1^{\rho-k} \cos(y \ln n) - X(n)2^{\rho-k} \cos(y \ln 2) + X(n)3^{\rho-k} \cos(y \ln 3) - X(n)4^{\rho-k} \cos(y \ln 4) + \dots]$$

$$U = [X(n)1^{\rho-k} \sin(y \ln n) - X(n)2^{\rho-k} \sin(y \ln 2) + X(n)3^{\rho-k} \sin(y \ln 3) - X(n)4^{\rho-k} \sin(y \ln 4) + \dots].$$

When  $\rho = \frac{k}{2} (k \in \mathbb{R})$ ,

then

$$\text{Only } L(k-s, X(n)) = L(\bar{s}, X(n)) = W - Ui.$$

But the Riemann  $\zeta(s)$  function only satisfies  $\zeta(1-s) = 2^{1-s} \pi^{-s} \cos\left(\frac{\pi s}{2}\right) \Gamma(s) \zeta(s)$ , so when  $\zeta(s) = 0$ , then only

$$\zeta(1-s) = \zeta(s) = 0, \text{ and when } \zeta(\bar{s}) = 0, \text{ then only } \zeta(1-s) = \zeta(\bar{s}) = 0, \text{ which is } \zeta(k-s) = \zeta(1-s) =$$

$$\zeta(\bar{s}), \text{ so only } k=1 \text{ be true. so only } \text{Re}(s) = \frac{k-1}{2} (k \in \mathbb{R}).$$

So Only  $L(1-s, X(n)) = L(\bar{s}, X(n))$  is true, so only  $k=1$  is true.

According the equation  $\zeta(1-s) = 2^{1-s} \pi^{-s} \cos\left(\frac{\pi s}{2}\right) \Gamma(s) \zeta(s)$  obtained by Riemann, since Riemann has shown

that the Riemann  $\zeta(s)$  function has zero, that is, in  $\zeta(1-s) = 2^{1-s} \pi^{-s} \cos\left(\frac{\pi s}{2}\right) \Gamma(s) \zeta(s) (s \in \mathbb{C})$ ,  $\zeta(s) = 0$  is true.

So only When  $\rho = \frac{1}{2}$  and  $\zeta(s) = 0$  and  $X(n) \neq 0$ , then  $L(s, X(n)) = X(n) \zeta(s) = 0$  is true. Because

$L(s, X(n)) = X(n) \zeta(s)$  and  $L(1-s, X(n)) = X(n) \zeta(1-s)$ , so When  $\rho = \frac{1}{2}$ , it must be true that

$$L(s, X(n)) = \overline{L(\bar{s}, X(n))}, \text{ and it must be true that } L(1-s, X(n)) = L(\bar{s}, X(n)).$$

According  $\zeta(1-s) = \zeta(s) = 0$  and  $\zeta(s) = \zeta(\bar{s}) = \zeta(1-\bar{s}) = 0$ , so  $L(s, X(n)) = L(1-s, X(n)) = 0$  and

$L(s, X(n)) = L(\bar{s}, X(n)) = L(1-\bar{s}, X(n)) = 0$ , then  $s = \bar{s}$  or  $s = 1-s$  or  $\bar{s} = 1-s$ , so  $s \in \mathbb{R}$ , or  $\rho + yi = 1 - \rho - yi$ , or

$\rho - yi = 1 - \rho - yi$ , so  $s \in \mathbb{R}$ , or  $\rho = \frac{1}{2}$  and  $y = 0$ , or  $\rho = \frac{1}{2}$  and  $y \in \mathbb{R}$  and  $y \neq 0$ , so  $s \in \mathbb{R}$  for example

$s = -2n (n \in \mathbb{Z}_+)$ , or  $s = \frac{1}{2} + oi$ , or  $s = \frac{1}{2} + yi (y \in \mathbb{R} \text{ and } y \neq 0)$ .  $\zeta\left(\frac{1}{2}\right) > \zeta(1) > 0$ , drop it, when  $s = -2n (n \in \mathbb{Z}_+)$ ,

it's the trivial zero of the Riemann  $\zeta(s)$  function, drop it.

So only  $s = \frac{1}{2} + yi (y \in \mathbb{R}, \text{ and } y \neq 0, s \in \mathbb{C})$  is true, or say  $s = \frac{1}{2} + ti (t \in \mathbb{R}, \text{ and } t \neq 0, s \in$

$\mathbb{C})$  is true. And because only when  $\rho = \frac{1}{2}$ , the next three equations,  $L(\rho + yi, X(n)) = 0$ ,  $L(1 - \rho -$

$\gamma_i, \chi(n)=0$  and  $L(\rho - \gamma_i, \chi(n))=0$  are all true. And because  $L(\frac{1}{2}, \chi(n))=\infty$ , so only  $s=\frac{1}{2}+\gamma_i$

( $\gamma \in \mathbb{R}$  and  $\gamma \neq 0, s \in \mathbb{C}$ ) is true, or say only  $s=\frac{1}{2}+ti$  ( $t \in \mathbb{R}$  and  $t \neq 0, s \in \mathbb{C}$ ) is true.

The Generalized Riemann hypothesis and the Generalized Riemann conjecture must satisfy the properties of the  $L(s, \chi(n))$  function, The properties of the  $L(s, \chi(n))$  function are fundamental, the Generalized Riemann hypothesis and the Generalized Riemann conjecture must be correct to reflect the properties of the  $L(s, \chi(n))$  function, that is, the roots of the  $L(s, \chi(n))=0$  can only be  $s=\frac{1}{2}+ti$  ( $t \in \mathbb{C}, s \in \mathbb{C}$  and  $t \neq 0$ ), that is,  $\text{Re}(s)$  must only be equal to  $\frac{1}{2}$ , and  $\text{Im}(s)$  must be real, and  $\text{Im}(s)$  is not equal to zero. So the Generalized Riemann hypothesis and the Generalized Riemann conjecture must be correct.

According  $L(1-s, \chi(n))=L(s, \chi(n))=0$ , so the zeros of the  $L(s, \chi(n))$  function in the complex plane also correspond to the symmetric distribution of point  $(\frac{1}{2}, 0i)$  on a line perpendicular to the real number line in the complex plane, so When  $L(1-s, \chi(n))=L(s, \chi(n))=0$ ,  $s$  and  $1-s$  are pair of zeros of the function  $L(s, \chi(n))$  symmetrically distributed in the complex plane with respect to point  $(\frac{1}{2}, 0i)$  on a line perpendicular to the real number line of the complex plane.

We got  $\overline{L(s, \chi(n))} = L(\bar{s}, \chi(n))$  ( $s=\rho+\gamma_i, \rho \in \mathbb{R}, \gamma \in \mathbb{R}$  and  $\gamma \neq 0$ ) before, When  $t$  in Generalized Riemann's hypothesis  $s=\frac{1}{2}+ti$  ( $t \in \mathbb{C}, s \in \mathbb{C}$  and  $t \neq 0$ ) is a complex number, and  $s=\frac{1}{2}+ti=\rho+\gamma_i$ , then  $s$  in

$\overline{L(s, \chi(n))} = L(\bar{s}, \chi(n))$  ( $s=\rho+\gamma_i, \rho \in \mathbb{R}, \gamma \in \mathbb{R}$  and  $\gamma \neq 0$ ) is consistent with  $s$  in Generalized Riemann's

hypothesis  $s=\frac{1}{2}+ti$  ( $t \in \mathbb{C}, s \in \mathbb{C}$  and  $t \neq 0$ ). when  $L(s, \chi(n))=L(\bar{s}, \chi(n))=0$  ( $s=\rho+\gamma_i, \rho \in \mathbb{R}, \gamma \in \mathbb{R}$  and  $\gamma \neq$

$0$ ), since  $s$  and  $\bar{s}$  are a pair of conjugate complex numbers, so  $s$  and  $\bar{s}$  must be a pair of zeros of the Generalized function  $L(s, \chi(n))$  in the complex plane with respect to point  $(\rho, 0i)$  on a line perpendicular to the real number line.  $s$  is a symmetric zero of  $1-s$ , and a symmetric zero of  $\bar{s}$ . By the definition of complex numbers, how can a symmetric zero of the same Generalized Riemann function  $L(s, \chi(n))$  of the same zero independent variable  $s$  on a line perpendicular to the real number axis of the complex plane be both a symmetric zero of  $1-s$  on a line perpendicular to the real number axis of the complex plane with respect to point  $(\frac{1}{2}, 0i)$  and a symmetric zero of  $\bar{s}$  on a line perpendicular to

the real number axis of the complex plane with respect to point  $(\rho, 0i)$ ? Unless  $\rho$  and  $\frac{1}{2}$  are the same

value, is also that  $\rho = \frac{1}{2}$ , and only  $1-s=\bar{s}$  is true, only  $s=\frac{1}{2}+ti$  ( $t \in \mathbb{R}$  and  $t \neq 0, s \in \mathbb{C}$ ) is true. Otherwise

it's impossible, this is determined by the uniqueness of the zero of Generalized Riemann function  $L(s, \chi(n))$  on the line passing through that point perpendicular to the real number axis of the complex plane with respect to the vertical foot symmetric distribution of the zero of the line and the real number axis of the complex plane, Only one line can be drawn perpendicular from the zero independent variable  $s$  of Generalized Riemann function  $L(s, \chi(n))$  on the real number line of the complex plane, the vertical line has only one point of intersection with the real number axis of the complex plane. In the same complex plane, the same zero point of Generalized Riemann function

$L(s, \chi(n))$  on the line passing through that point perpendicular to the real number line of the complex plane there will be only one zero point about the vertical foot symmetric distribution of the line and the real number line of the complex plane, so I have proved the generalized Riemann conjecture when the Dirichlet eigen function  $\chi(n)$  is any real number that is not equal to zero, since the nontrivial zeros of the Riemannian function  $\zeta(s)$  and the generalized Riemannian function  $L(s, \chi(n))$  are both on the critical line perpendicular to the real number line of  $\text{Re}(s) = \frac{1}{2}$  and  $\text{Im}(s) \neq 0$ , these nontrivial zeros are general complex numbers of  $\text{Re}(s) = \frac{1}{2}$  and  $\text{Im}(s) \neq 0$ , so I have proved the generalized Riemann conjecture when the Dirichlet eigen function  $\chi(n)$  is any real number that is not equal to zero.

The Generalized Riemann hypothesis and the Generalized Riemann conjecture must satisfy the properties of the  $L(s, \chi(n))$  function, the properties of the  $L(s, \chi(n))$  function are fundamental, the Generalized Riemann hypothesis and the Generalized Riemann conjecture must be correct to reflect the properties of the  $L(s, \chi(n))$  function, that is, the roots of the  $L(s, \chi(n)) = 0$  can only be  $s = \frac{1}{2} + ti$  ( $t \in \mathbb{C}, s \in \mathbb{C}$  and  $t \neq 0$ ), that is,  $\text{Re}(s)$  can only be equal to  $\frac{1}{2}$ , and  $\text{Im}(s)$  must be real, and  $\text{Im}(s)$  is not equal to zero.

When  $L(s, \chi(n)) = 0$  ( $n \in \mathbb{Z}_+, p \in \mathbb{Z}_+, s \in \mathbb{C}$ ,  $n$  goes through all the natural numbers,  $p$  goes through all the prime numbers,  $\chi(n) \in \mathbb{R}$  and  $\chi(n) \neq 0$ ),  $a(n) = a(p) = \chi(n)$ ,

$P(p, s) = \frac{1}{1 - a(p)p^{-s}}$  and  $s = \frac{1}{2} + ti$  ( $t \in \mathbb{R}$  and  $t \neq 0, s \in \mathbb{C}$ ), then the Generalized Riemann hypothesis and the Generalized Riemann conjecture must be correct.

**Reasoning 4:**

For any complex number  $s$ , when  $\chi(n)$  is the Dirichlet characteristic and satisfies the following properties:

1: There exists a positive integer  $q$  such that  $\chi(n+q) = \chi(n)$  ( $n \in \mathbb{Z}_+$ );

2: when  $n$  and  $q$  are not mutual prime,  $\chi(n) = 0$  ( $n \in \mathbb{Z}_+$ );

3:  $\chi(a)\chi(b) = \chi(ab)$  ( $a \in \mathbb{Z}_+, b \in \mathbb{Z}_+$ ) for any integer  $a$  and  $b$ ;

Suppose  $q = 2k$  ( $k \in \mathbb{Z}_+$ ),

if  $n$  and  $n+q$  are all prime number, and if  $\chi(Y) = 1$  ( $Y$  traverses all positive odd numbers) or

if  $\chi(Y) \neq 0$  ( $Y$  traverses all positive odd numbers),

then  $\chi(n+q) = \chi(n) = \chi(p) \equiv 1$  ( $n, n+q$ , and  $p$  go through all the prime numbers),

or  $\chi(n+q) = \chi(n) = \chi(p) \neq 0$  ( $n, n+q$ , and  $p$  go through all the prime numbers), because  $n$  ( $n$

traverses all prime numbers) and  $q = 2k$  ( $k \in \mathbb{Z}_+$ ) are not mutual prime, then  $\chi(n) = 0$  ( $n \in \mathbb{Z}_+$ ), and

for any prime number  $a$  and  $b$ ,

$\chi(a)\chi(b) = \chi(ab)$  ( $a \in \mathbb{Z}_+, b \in \mathbb{Z}_+$ ,  $a$  and  $b$  are all prime number),

then the three properties described by the Dirichlet eigenfunction  $\chi(n)$  above fit the definition of the



Polignac conjecture, the Polignac conjecture states that for all natural numbers  $k$ , there are infinitely many pairs of prime numbers  $(p, p+2k)$  ( $k \in \mathbb{Z}_+$ ). In 1849, the French mathematician A. Polignac proposed the conjecture. When  $k=1$ , the Polignac conjecture is equivalent to the twin prime conjecture. In other words, when  $L(s, \chi(n)) = 0$  ( $n \in \mathbb{Z}_+, p \in \mathbb{Z}_+, s \in \mathbb{C}$ ,  $n$  goes through all the natural numbers,  $p$  goes through all the prime numbers,  $\chi(n) \in \mathbb{R}$

and  $(\chi(n) \neq 0), a(n) = a(p) = \chi(n), P(p, s) = \frac{1}{1 - a(p)p^{-s}}$ , and generalized Riemann hypothesis and the generalized Riemann conjecture are true, then the Polignac conjecture must be completely true, and if the Polignac conjecture must be true, then the twin prime conjecture and Goldbach's conjecture must be true. I proved that the generalized Riemannian hypothesis and the generalized Riemannian conjecture are true, so when  $L(s, \chi(n)) = 0$  ( $n \in \mathbb{Z}_+, p \in \mathbb{Z}_+, s \in \mathbb{C}$ ,  $n$  goes through all

the natural numbers,  $p$  goes through all the prime numbers,  $\chi(n) \in \mathbb{R}$  and  $(\chi(n) \neq 0), a(n) = a(p) = \chi(n), P(p, s) = \frac{1}{1 - a(p)p^{-s}}$  and  $s = \frac{1}{2} + ti$  ( $t \in \mathbb{R}$  and  $t \neq 0, s \in \mathbb{C}$ ), I also proved that the

Polignac conjecture, twin prime conjecture must be true and Goldbach conjecture are completely or almost true. The Generalized Riemann hypothesis and the Riemann conjecture are perfectly valid, so the Polignac conjecture and the twin prime conjecture and Goldbach's conjecture must satisfy the properties of the Generalized Riemann  $\zeta(s)$  function and the Riemann  $\zeta(s)$  function, so the Polignac conjecture, twin prime conjecture must be true and Goldbach conjecture is completely true. Riemann hypothesis and the Riemann conjecture are completely correct and the Generalized Riemann hypothesis and the Generalized Riemann conjecture are completely correct and the Polignac conjecture, twin prime conjecture must be true and Goldbach conjecture are almost or completely true.

**Reasoning 5:**

In order to explain why the zero of the Landau-Siegel function exists under special conditions, we need to start with the Riemann conjecture. I have solved the Riemann conjecture for the Dirichlet feature

$\chi(n) = 1$  ( $n$  traverses all natural numbers) and the generalized Riemann conjecture for the Dirichlet

feature  $\chi(n) \neq 0$  ( $n$  traverses all natural numbers), Now I propose a special form of Dirichlet

$L(s, \chi(p))$  ( $s \in \mathbb{C}$ ,  $\chi(p) \in \mathbb{R}$  and  $\chi(p) = 0, p$  traverses all odd primes, including 1) function problem. Let me first explain to you what Landau-Siegel zero conjecture is. As you may know, the Landau-Siegel zero point problem, named after Landau and his student Siegel, boils down to solving whether there are abnormal real zeros in the Dirichlet  $L$  function. So let's look again at what the Dirichlet  $L$  function is.

Look at the above proof process, which is the expression of Dirichlet  $L(s, \chi(n))$  ( $s \in \mathbb{C}, n$  traverses all natural numbers),

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

I shall first introduce the Dirichlet  $L(s, \chi)$  ( $s \in \mathbb{C}$ ,  $n$  traverses all natural numbers) function and explain its relation to the Riemann  $\zeta(s)$  ( $s \in \mathbb{C}$ ) function. Here,  $\chi(n)$  ( $n$  traverses all natural numbers) is a characteristic value of a Dirichlet function, which is all real numbers, and  $\chi(n)$  ( $n$  traverses all natural numbers) is a real function. The  $L(s, \chi)$  ( $s \in \mathbb{C}$ ,  $\chi(n) \in \mathbb{R}$ ,  $n$  traverses all natural numbers) function can be analytically extended as a meromorphic function over the entire complex plane. John Peter Dirichlet proved that  $L(1, \chi) \neq 0$  ( $s \in \mathbb{C}$ ,  $\chi(n) \in \mathbb{R}$  and  $\chi(n) \neq 0$ ,  $n$  takes all natural numbers) for all  $\chi(n)$  ( $n$  traverses all natural numbers), and thus proved Dirichlet's theorem. In number theory, Dirichlet's theorem states that for any positive integers  $a, d$ , there are infinitely many forms of prime numbers, such as  $a+nd$ , where  $n$  is a positive integer, i.e., in the arithmetic sequence  $a+d, a+2d, a+3d, \dots$ . There are an infinite number of prime numbers—there are an infinite number of prime modules  $d$  as well as  $a$ . If  $\chi(n)$  ( $n$  traverses all natural numbers) is the main feature, then  $L(s, \chi)$  ( $s \in \mathbb{C}$ ,  $\chi(n) \in \mathbb{R}$ ,  $n$  traverses all natural numbers) has a unipolar point at  $s=1$ . Dirichlet defined the properties of the characteristic function  $\chi(n)$  ( $n$  is a positive integer) in the Dirichlet function  $L(s, \chi)$  ( $s \in \mathbb{C}$ ,  $\chi(n) \in \mathbb{R}$ ,  $n$  traverses all natural numbers) :

1: There is a positive integer  $q$  such that  $\chi(n+q) = \chi(n)$  ( $n$  traverses all natural numbers);

2: when  $n$  ( $n$  traverses all natural numbers) and  $q$  are non-mutual primes,  $\chi(n) = 0$  ( $n$  traverses all natural numbers);

3: For any integer  $a$  and  $b$ ,  $\chi(a)\chi(b) = \chi(ab)$  ( $a$  is a positive integer,  $b$  is a positive integer);

From the expression of the Dirichlet function  $L(s, \chi)$  ( $s \in \mathbb{C}$ ,  $\chi(n) \in \mathbb{R}$ ,  $n$  takes all natural numbers) in

Figure 1 above, it is easy to see that when the Dirichlet characteristic real function  $\chi(n) = 1$  ( $s \in \mathbb{C}$ ,  $n$

takes all natural numbers), Then the Dirichlet  $L(s, 1)$  ( $s \in \mathbb{C}$ ,  $\chi(n) \in \mathbb{R}$ ,  $n$  traverses all natural numbers)

becomes the Riemann  $\zeta(s)$  function, so the Riemann  $\zeta(s)$  function is a special function of the Dirichlet function  $L(s, \chi(n))$  ( $s \in \mathbb{C}, \chi(n) \in \mathbb{R}, n$  traversing all natural numbers), when the characteristic real function  $\chi(n)$  ( $n$  is a positive integer) is equal to 1, Also called a trivial characteristic function of the Dirichlet function  $L(s, \chi(n))$  ( $s \in \mathbb{C}, \chi(n) \in \mathbb{R}, n$  traverses all natural numbers). When the eigenreal functions  $\chi(n) \neq 1$ , they are called nontrivial eigenfunctions of the Dirichlet function  $L(s, \chi(n))$  ( $s \in \mathbb{C}, \chi(n) \in \mathbb{R}, n$  traverses all natural numbers). When the independent variable  $s$  in the expression of the Dirichlet function  $L(s, \chi(n))$  ( $s \in \mathbb{C}, \chi(n) \in \mathbb{R}, n$  traverses all natural numbers) is a real number  $\beta$ , then for all eigenfunction values  $\chi(n)$  ( $n$  traverses all natural numbers),  $L(\beta, \chi(n))$  ( $\beta$  is real,  $\chi(n) \in \mathbb{R}, n$  traverses all natural numbers) is called the Landau-Siegel function. Visible Landau-Siegel function  $L(\beta, \chi(n))$  ( $\beta \in \mathbb{R}, \chi(n) \in \mathbb{R}, n$  traverses all natural numbers) is Dirichlet function  $L(s, \chi(n))$  ( $s \in \mathbb{C}, \chi(n) \in \mathbb{R}, n$  traverses all natural numbers) of a special function, Landau-Siegel guess is Landau and Siegel they guess  $L(\beta, \chi(n))$  ( $\beta \in \mathbb{R}, \chi(n) \in \mathbb{R}, n$  traverses all natural numbers) is not zero, So Landau and Siegel's conjecture that  $L(\beta, \chi(n)) \neq 0$  ( $\beta \in \mathbb{R}, \chi(n) \in \mathbb{R}, n$  traverses all natural numbers) is easy to understand, right? Well, now that you know what the Landau and Siegel null conjecture is all about, let's continue to see how I'm going to solve the Landau and Siegel null conjecture. Look at the above proof process:

$$\begin{aligned} \text{GRH}(s, \chi(n)) &= L(s, \chi(n)) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \frac{\chi(n)n(s)}{(1-2^{1-s})} = \frac{\chi(n)}{(1-2^{1-s})} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = \\ &= \frac{\chi(n)}{(1-2^{1-s})} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{s+yi}} = \frac{(-1)^{n-1}}{(1-2^{1-s})} \sum_{n=1}^{\infty} \chi(n) \left( \frac{1}{n^{s+yi}} \right) = \\ &= \frac{(-1)^{n-1}}{(1-2^{1-s})} \sum_{n=1}^{\infty} \chi(n) (x^{-p}) \frac{1}{(\cos(\ln x) + i \sin(\ln x))^y} = \frac{(-1)^{n-1}}{(1-2^{1-s})} \sum_{n=1}^{\infty} \chi(n) (x^{-p} (\cos(\ln x) + \\ &= \text{isin}(\ln x))^{-y}) = \\ &= \frac{(-1)^{n-1}}{(1-2^{1-s})} \sum_{n=1}^{\infty} \chi(n) (x^{-p} (\cos(y \ln x) - \text{isin}(y \ln x)) = \frac{(-1)^{n-1}}{(1-2^{1-s})} \sum_{n=1}^{\infty} \chi(n) (n^{-p} (\cos(y \ln(n)) - \\ &= \text{isin}(y \ln(n))) (x = \\ & n, x \text{ goes through all positive integers, } n \text{ goes through all positive integers } ) \end{aligned}$$

then

$$L(\beta, \chi(n)) =$$

$$\frac{(-1)^{n-1}}{(1-2^{1-\beta})} \sum_{n=1}^{\infty} \chi(n) (n^{-\beta} (\cos(0 \times \ln(n)) + \text{isin}(0 \times$$

$$\ln(n)) = \frac{(-1)^{n-1}}{(1-2^{1-\beta})} \sum_{n=1}^{\infty} (\chi(n)n^{-\beta}) = \frac{1}{(1-2^{1-\beta})} (\chi(1)1^{-\beta} - \chi(2)2^{-\beta} + \chi(3)3^{-\beta} - \chi(4)4^{-\beta} + \dots).$$

what is  $\eta(s)$ ? Look at the above proof process:

For any complex number  $s$ , when  $\text{Re}(s) > 0$  and  $(s \neq 1)$ , then according to Dirichlet function

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} \quad (s \in \mathbb{C} \text{ and } \text{Re}(s) > 0 \text{ and } (s \neq 1)) \quad \text{and} \quad \eta(s) = (1-2^{1-s}) \zeta(s) \quad (s \in \mathbb{C} \text{ and } \text{Re}(s) >$$

$$0 \text{ and } s \neq 1, \zeta(s) \text{ is the Riemann Zeta function, so Riemann } \zeta(s) = \frac{\eta(s)}{(1-2^{1-s})} = \frac{1}{(1-2^{1-s})} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} =$$

$$\frac{(-1)^{n-1}}{(1-2^{1-s})} \prod_p (1-p^{-s})^{-1} \quad (s \in \mathbb{C} \text{ and } \text{Re}(s) > 0 \text{ and } (s \neq 1), n \in \mathbb{Z}_+, p \in \mathbb{Z}_+, s \in \mathbb{C}, n \text{ goes through all the natural numbers, } p \text{ goes through all the prime numbers}). \text{ Let's prove that } \zeta(s) \text{ and } \zeta(\bar{s}) \text{ are complex conjugations of each other.}$$

Above is that part of the content about the  $\eta(s)(s \in \mathbb{C})$  function definition and  $\eta(s)(s \in \mathbb{C})$  function and Riemann  $\zeta(s)(s \in \mathbb{C})$  function expression of the relationship, apparently  $L(s, \chi(n))(s \in \mathbb{C}, \chi(n) \in \mathbb{R}, n$  traverses all natural numbers). It is easy to see that the Dirichlet function  $L(s, \chi(n))(s \in \mathbb{C}, \chi(n) \in \mathbb{R}, n$  traverses all natural numbers) is a summation function, just like the Riemann  $\zeta(s)(s \in \mathbb{C})$  function. I defined  $s = \rho + yi$  ( $\rho \in \mathbb{R}, y \in \mathbb{R}$  and  $y \neq 0, s \in \mathbb{C}$ ), the Landau-Siegel function  $L(\beta, \chi(n))(\beta \in \mathbb{R}, \chi(n) \in \mathbb{R}, n$  traverses all natural numbers) is equivalent to let me define  $s = \beta + 0i$  ( $\beta \in \mathbb{R}$ ), that is, let  $y = 0$ , then the contents above become the contents of the below:

$$L(\beta, \chi(n)) = \frac{(-1)^{n-1}}{(1-2^{1-\beta})} \sum_{n=1}^{\infty} \chi(n)(n^{-\beta} (\cos(0 \times \ln(n)) + i \sin(0 \times \ln(n)))) = \frac{(-1)^{n-1}}{(1-2^{1-\beta})} \sum_{n=1}^{\infty} (\chi(n)n^{-\beta}) = \frac{1}{(1-2^{1-\beta})} (\chi(1)1^{-\beta} - \chi(2)2^{-\beta} + \chi(3)3^{-\beta} - \chi(4)4^{-\beta} + \dots)$$

Obviously, when  $\chi(n) \equiv 1$  ( $n$  traverses all natural numbers), because the real exponential function of the real number has a function value greater than zero, so  $n^{-\beta} > 0$  ( $n$  traverses all natural numbers) and  $1^\beta - 2^\beta \neq 0, 3^\beta - 4^\beta \neq 0, 5^\beta - 6^\beta \neq 0, \dots, (n-1)^\beta - (n)^\beta \neq 0, \dots$ , and  $|\frac{1}{(1-2^{1-\beta})}| \neq 0$ , it can be known that when and  $\chi(n) \equiv 1$  ( $n$  traverses all natural numbers), then  $L(\beta, 1) \neq 0$  ( $\beta \in \mathbb{R}, \chi(n) \in \mathbb{R}$  and  $\chi(n) \equiv 1, n$  traverses all natural numbers), so for Riemann  $\zeta(s)(s \in \mathbb{C})$  functions, its corresponding Landau-Siegel function  $L(\beta, 1)(\beta \in \mathbb{R}, \chi(n) \in \mathbb{R}$

and  $\chi(n) \equiv 1, n$  traverses all natural numbers) of pure real zero does not exist, This means that the Riemann  $\zeta(s)(s \in \mathbb{C})$  function does not have a zero of a pure real variable  $s$ .

$$\begin{aligned} L(\beta, \chi(n)) &= \\ \frac{(-1)^{n-1}}{(1-2^{1-\beta})} \sum_{n=1}^{\infty} \chi(n) (n^{-\beta} (\cos(0 \times \ln(n)) + i \sin(0 \times \\ \ln(n))) &= \frac{(-1)^{n-1}}{(1-2^{1-\beta})} \sum_{n=1}^{\infty} (\chi(n) n^{-\beta}) = \frac{1}{(1-2^{1-\beta})} (\chi(1)1^{-\beta} - \chi(2)2^{-\beta} + \chi(3)3^{-\beta} - \\ &\chi(4)4^{-\beta} + \dots) \end{aligned}$$

Obviously, when  $\chi(n) \neq 1$  and  $\chi(n) \neq 0$ ,  $n$  traverses all natural numbers, because the real exponential function of the real number has a function value greater than zero, so  $n^{-\beta} > 0$  ( $n$  traverses all natural numbers) and  $1^\beta - 2^\beta \neq 0, 3^\beta - 4^\beta \neq 0, 5^\beta - 6^\beta \neq 0, \dots, (n-1)^\beta - (n)^\beta \neq 0, \dots$ , and  $|\frac{1}{(1-2^{1-\beta})}| \neq 0$ , it can be known that when  $\chi(n) \neq 1$  and  $\chi(n) \neq 0$ ,  $n$

traverses all natural numbers, then  $L(\beta, 1) \neq 0$  ( $\beta \in \mathbb{R}, \chi(n) \in \mathbb{R}$  and  $\chi(n) \neq 1, n$  traverses all natural numbers) so for generalized Riemann  $L(s, \chi(n))$  ( $s \in \mathbb{C}$ ) functions, its corresponding Landau-Siegel function  $L(\beta, 1)$  ( $\beta \in \mathbb{R}, \chi(n) \in \mathbb{R}, \chi(n) \neq 1$  and  $\chi(n) \neq 0, n$  traverses all natural numbers) of pure real zero does not exist, This means that the generalized Riemann  $L(s, \chi(n))$  ( $s \in \mathbb{C}$ ) function does not have a zero of a pure real variable  $s$ .

$$\begin{aligned} L(\beta, \chi(n)) &= \\ \frac{(-1)^{n-1}}{(1-2^{1-\beta})} \sum_{n=1}^{\infty} \chi(n) (n^{-\beta} (\cos(0 \times \ln(n)) + i \sin(0 \times \\ \ln(n))) &= \frac{(-1)^{n-1}}{(1-2^{1-\beta})} \sum_{n=1}^{\infty} (\chi(n) n^{-\beta}) = \frac{1}{(1-2^{1-\beta})} (\chi(1)1^{-\beta} - \chi(2)2^{-\beta} + \chi(3)3^{-\beta} - \\ &\chi(4)4^{-\beta} + \dots) \end{aligned}$$

Obviously, when  $\chi(n) \equiv 0$  ( $n$  traverses all natural numbers), because the real exponential function of the real number has a function value greater than zero, so  $n^{-\beta} > 0$  ( $n$  traverses all natural numbers) and  $\chi(1)1^\beta = 0, \chi(2)2^\beta = 0, \chi(3)3^\beta = 0, \chi(4)4^\beta = 0, \chi(5)5^\beta = 0, \chi(6)6^\beta = 0, \dots, \chi(n-1)(n-1)^\beta = 0, \chi(n)n^\beta = 0, \dots$ , and  $|\frac{1}{(1-2^{1-\beta})}| \neq 0$ , it can

be known that when  $\chi(n) \equiv 0$  ( $n$  traverses all natural numbers), then  $L(\beta, 1) = 0$  ( $\beta \in \mathbb{R}, \chi(n) \in \mathbb{R}$  and  $\chi(n) \equiv 0, n$  traverses all natural numbers), so for generalized Riemann  $L(s, \chi(n))$  ( $s \in \mathbb{C}$ ) functions, its corresponding Landau-Siegel function  $L(\beta, 1)$  ( $\beta \in \mathbb{R}, \chi(n) \in \mathbb{R}$  and  $\chi(n) \equiv 0, n$  traverses all natural numbers) of pure real zero exists, This means that the generalized Riemann  $L(s, \chi(n))$  ( $s \in \mathbb{C}$ ) function has a zero of a pure real variable  $s$ , that means the twin prime conjecture, Goldbach's conjecture, Polignac's conjecture are all true.

Now I summarize the Dirichlet function  $L(s, \chi(n))$  ( $s \in \mathbb{C}, \chi(n) \in \mathbb{R}, n$  traverses all positive integers) as follows: Let's review the properties of the characteristic function  $\chi(n)$  in the Dirichlet function

$L(s, X(n)) (s \in \mathbb{C}, X(n) \in \mathbb{R}, n \text{ traverses all natural numbers}) :$

1: There is a positive integer  $q$  such that  $X(n+q)=X(n)$  ( $n$  traverses all natural numbers);

2: when  $n$  ( $n$  traverses all natural numbers) and  $q$  are non-mutual primes,  $X(n)=0$  ( $n$  traverses all natural numbers);

3: For any integer  $a$  and  $b$ ,  $X(a) X(b) = X(ab)$  ( $a$  is a positive integer,  $b$  is a positive integer);

In the 19th century, 1 was considered a prime number, it is not considered a prime number later, but now it is considered a prime number again by me. Now I'm going to assume that  $q=2k$  ( $k$  traverses through all positive integers), and I'm going to assume that there are any odd numbers  $n$ , so for  $n$  ( $n$  traverses through all odd numbers) and  $q$  ( $q$  is positive even), they're non-mutual prime numbers, Then  $X(n)=0$  ( $n$  traverses through all odd numbers, including 1), then  $X(p)=0$  ( $p$  traverses through all primes, including 1), then  $X(p+2k)=X(p)=0$  ( $k$  traverses through all positive integers,  $p$  traverses through all odd primes, including 1), then for all odd primes  $p$  (including 1), A special Dirichlet function

$L(s, X(p)) (s \in \mathbb{C}, X(p) \in \mathbb{R}, p \text{ traverses all odd primes including 1})$  has a function value of zero for the

characteristic function  $X(p)$  ( $p$  traverses all odd primes including 1). Now suppose that there are any

odd primes  $a$  and any odd primes  $b$ , then  $ab$  is a odd number, then  $X(a)=0, X(b)=0, X(ab)=0$ ,

satisfying  $X(a) X(b) = X(ab)=0$  ( $a$  traverses all odd primes, including 1, and  $b$  traverses all odd primes

including 1). Then the characteristic function  $X(p)$  in a new Dirichlet function

$L(s, X(p)) (s \in \mathbb{C}, X(p) \in \mathbb{R} \text{ and } X(p)=0, p \text{ traverses all odd primes, including 1})$  has the following

properties:

1: There is a positive integer  $q=2k$  ( $k$  traverses all positive integers) such that  $X(p+q) = X(p)=0$  ( $k$  traverses all positive integers,  $p$  traverses all odd primes, including 1);

2: when  $n$  ( $n$  traverses all positive odd numbers) and  $q$  are non-mutual primes,  $X(n)=0$  ( $n$  traverses all positive odd numbers);

3: For any odd prime number  $a$  and any odd prime number  $b$ ,  $X(a) X(b) = X(ab)$  ( $a$  is any odd prime number including 1 and  $b$  is any odd prime number including 1);

Then there is a special form of Dirichlet function  $L(s, X(p)) (s \in \mathbb{C}, X(p) \in \mathbb{R}, p \text{ traverses all odd primes, including 1})$  function, as shown below:

$$L(s, X(p)) = \frac{(-1)^{n-1}}{(1-2^{1-\rho-yi})} \sum_{p=1}^{\infty} X(p) x^{-p} (\cos(y \ln(p)) + i \sin(y \ln(p))),$$

$$L(\beta, X(p)) = \frac{(-1)^{n-1}}{(1-2^{1-\beta})} \sum_{p=1}^{\infty} X(p) p^{-\beta} (\cos(0 \times \ln(p)) + i \sin(0 \times \ln(p))) =$$

$$\frac{(-1)^{n-1}}{(1-2^{1-\beta})} \sum_{p=1}^{\infty} (X(p) p^{-\beta}) = \frac{(-1)^{n-1}}{(1-2^{1-\beta})} [X(1)1^{-\beta} - X(2)2^{-\beta} + X(3)3^{-\beta} - X(5)5^{-\beta} + X(7)7^{-\beta} + \dots$$

$$- X(p)p^{-\beta} + \dots] (\beta \in \mathbb{R}, x = p, p \text{ traverses all primes, including 1}).$$

When  $X(p)=0$  ( $p$  traverses all odd primes, including 1), then  $L(s, X(p))=0 (s \in \mathbb{C}, X(n) \in \mathbb{R} \text{ and}$

$X(p)=0, p \text{ traverses all odd primes, including 1})$  was established. At the same time  $L(s, X(p))$

( $s \in \mathbb{C}$ ,  $\chi(p) \in \mathbb{R}$  and  $\chi(p) = 0$ ,  $p$  traverses all primes, including 1) the corresponding Landau-Siegel

function  $L(\beta, 0)$  ( $\beta \in \mathbb{R}$ ,  $\chi(p) \in \mathbb{R}$  and  $\chi(p) = 0$ ,  $p$  traverses all primes, including 1) expression as

shown in as follows:

$$L(\beta, \chi(p)) = \frac{(-1)^{n-1}}{(1-2^{1-\beta})} \sum_{p=1}^{\infty} \chi(p) p^{-\beta} (\cos(0 \times \ln(p)) + i \sin(0 \times \ln(p))) =$$

$$\frac{(-1)^{n-1}}{(1-2^{1-\beta})} \sum_{p=1}^{\infty} (\chi(p) p^{-\beta}) = \frac{(-1)^{n-1}}{(1-2^{1-\beta})} [\chi(1)1^{-\beta} + \chi(2)2^{-\beta} + \chi(3)3^{-\beta} + \chi(5)5^{-\beta} + \chi(7)7^{-\beta} + \dots$$

$$- \chi(p) p^{-\beta} + \dots] \quad [\beta \in \mathbb{R}, x = p, p \text{ traverses all primes, including 1}].$$

According to the above, we already know that since  $\chi(p) = 0$  ( $p$  traverses all primes, including 1),

it is obvious that  $L(\beta, 0) = 0$  ( $\beta \in \mathbb{R}$ ) holds. The characteristic function  $\chi(p)$  in this special Dirichlet

function  $L(s, \chi(p))$  ( $s \in \mathbb{C}$ ,  $\chi(p) \in \mathbb{R}$  and  $\chi(p) = 0$ ,  $p$  traverses all primes, including 1) has the following properties:

- 1: There is a positive integer  $q = 2k$  ( $k$  traverses all positive integers) such that  $\chi(p+q) = \chi(p) = 0$  ( $k$  traverses all positive integers,  $p$  traverses all odd primes, including 1);
- 2: when  $n$  ( $n$  traverses all odd numbers) and  $q$  are non-mutual primes,  $\chi(n) = 0$  ( $n$  traverses all odd numbers);
- 3: For any odd prime number  $a$  and any odd prime number  $b$ ,  $\chi(a)\chi(b) = \chi(ab)$  ( $a$  is any odd prime number including 1 and  $b$  is any odd prime number including 1);

From the above three properties, and from the fact that  $L(\beta, 0) = 0$  ( $\beta \in \mathbb{R}$ ,  $\chi(p) \in \mathbb{R}$  and  $\chi(p) = 0$ ,  $p$  traverses all primes, including 1), it is obvious that we know that twin primes, Polignac's conjecture and Goldbach's conjecture all hold.

Now I summarize the Dirichlet function  $L(s, \chi(n))$  ( $s \in \mathbb{C}$ ,  $\chi(n) \in \mathbb{R}$ ,  $n$  traverses all positive integers) as follows:

- 1: When  $\chi(n) = 1$  ( $n$  traverses all positive integers), the generalized Riemannian hypothesis and the generalized Riemannian conjecture degenerate to the ordinary Riemannian hypothesis and the ordinary Riemannian conjecture, whose nontrivial zeros  $s$  satisfy  $s = \frac{1}{2} + ti$  ( $t \in \mathbb{R}$  and  $t \neq 0$ ), and ordinary

Riemann  $\zeta(s) = L(s, \chi(n))$  ( $s \in \mathbb{C}$ ,  $\chi(n) \in \mathbb{R}$  and  $\chi(n) = 1$ ,  $n$  traverses all natural numbers) the

corresponding Landau-Siegel function  $L(\beta, 1) \neq 0$  ( $\beta \in \mathbb{R}$ ,  $\chi(n) \in \mathbb{R}$  and  $\chi(n) = 1$ ,  $n$  traverses all natural

numbers), ordinary Riemann hypothesis and ordinary Riemann hypothesis all hold, and for Riemann  $\zeta(s)(s \in \mathbb{C})$  function, its corresponding Landau-Siegel function  $L(\beta, 1)(\beta \in \mathbb{R}, \chi(n) \in \mathbb{R}$  and  $\chi(n) = 1, n$  traverses all natural numbers) does not exist pure real zero, which also shows that Riemann  $\zeta(s)(s \in \mathbb{C})$  function does not exist zero when variable  $s$  is a pure real zero.

2: When  $\chi(n) = 0$  ( $n$  traverses all positive odd numbers), then  $\chi(p) = 0$  ( $p$  traverses all primes, including 1), a special Dirichlet function  $L(s, \chi(p))(s \in \mathbb{C}, \chi(p) \in \mathbb{R}$  and  $\chi(p) = 0, p$  traverses all primes, including 1) has zero, and when zero is obtained, the independent variable  $s$  is any complex number. This special Dirichlet function  $L(s, \chi(p))(s \in \mathbb{C}, \chi(p) \in \mathbb{R}$  and  $\chi(p) = 0, p$  traverses all prime, including 1) the corresponding Landau - siegel function  $L(\beta, 0)(\beta \in \mathbb{R}, \chi(p) \in \mathbb{R}$  and  $\chi(p) = 0, p$  traverses all prime, including 1) holds, so for this particular Dirichlet function  $L(s, \chi(p)) = 0 (s \in \mathbb{C}, \chi(p) \in \mathbb{R}$  and  $\chi(p) = 0, p$  traverses all primes, including 1) holds. The existence of a pure real zero of the corresponding Landau-Siegel function  $L(\beta, 0)(\beta \in \mathbb{R}, \chi(p) \in \mathbb{R}$  and  $\chi(p) = 0, p$  traverses all prime numbers, including 1) shows that the twin prime numbers, Polignac conjecture and Goldbach conjecture are all true.

For any prime number  $a$  and  $b$ , then the three properties described by the Dirichlet eigenfunction  $\chi(n)$  above fit the definition of the Polignac conjecture, the Polignac conjecture states that for all natural numbers  $k$ , there are infinitely many pairs of prime numbers  $(p, p+2k)(k \in \mathbb{Z}_+)$ . In 1849, the French mathematician A. Polignac proposed the conjecture. When  $k=1$ , the Polignac conjecture is equivalent to the twin prime conjecture.

When Polignac conjecture is true, then the twin prime conjecture and Goldbach's conjecture must be true.

When  $L(s, \chi(n)) = 0 (n \in \mathbb{Z}_+, p \in \mathbb{Z}_+, s \in \mathbb{C}, n$  goes through all the natural numbers,  $p$  goes through all the prime numbers,  $\chi(n) \in \mathbb{R}$  and  $\chi(n) \neq 0), a(n) = a(p) = \chi(n), P(p, s) = \frac{1}{1-a(p)p^{-s}}$  and  $s = \frac{1}{2} + ti (t \in \mathbb{R}$  and  $t \neq 0)$  then the Generalized Riemann hypothesis and the Generalized Riemann conjecture are true,

3: when the  $\chi(n) \neq 0$  ( $n$  traverses all natural numbers), Dirichlet function  $L(s, \chi(n))(s \in \mathbb{C}, \chi(n) \in \mathbb{R}$  and  $\chi(n) \neq 0, n$  traverses all natural numbers) has zero, it's nontrivial zero meet  $s = \frac{1}{2} + ti (t \in \mathbb{R}$  and  $t \neq 0)$ . For Dirichlet function  $L(s, \chi(n))(s \in \mathbb{C}, \chi(n) \in \mathbb{R}$  and  $\chi(n) \neq 0, n$  traverses all positive integers), it's corresponding Landau-siegel function  $L(\beta, \chi(n))(\beta \in \mathbb{R}, \chi(n) \in \mathbb{R}$  and  $\chi(n) \neq 0, n$  traverses all natural numbers) of pure real zero does not exist. In other words, it shows that the Dirichlet function  $L(s, \chi(p))(s \in \mathbb{C}, \chi(n) \in \mathbb{R}$  and  $\chi(n) \neq 0, n$  traverses all natural numbers) does not exist



for the zero of a pure real variable  $s$ , so if  $\chi(n) \neq 0$  ( $n$  traverses all natural numbers), then both the generalized Riemannian hypothesis and the generalized Riemannian conjecture hold and the Generalized Riemann  $L(s, \chi(n))$  ( $s \in \mathbb{C}$ ,  $\chi(n) \in \mathbb{R}$  and  $\chi(n) \neq 0$ ,  $n$  traverses all positive integers) function of nontrivial zero  $s$  also meet  $s = \frac{1}{2} + ti$  ( $t \in \mathbb{R}$  and  $t \neq 0$ ). Now we know that merely proving that the nontrivial zero  $s$  of the Riemann conjecture  $L(s, 1)$  ( $s \in \mathbb{C}$ ,  $\chi(n) \in \mathbb{R}$  and  $\chi(n) = 1$ ,  $n$  traverses all natural numbers) and the generalized Riemann conjecture  $L(s, \chi(n))$  ( $s \in \mathbb{C}$ ,  $\chi(n) \in \mathbb{R}$  and  $\chi(n) \neq 0$ ,  $n$  traverses all natural numbers) satisfies  $s = \frac{1}{2} + ti$  ( $t \in \mathbb{R}$ ,  $t \neq 0$ ) is sufficient to prove that the twin primes, Polignac's conjecture, Goldbach's conjecture are all true. It is also proved that a special class of generalized Riemann hypothesis  $L(s, \chi(p))$  ( $s \in \mathbb{C}$ ,  $\chi(p) \in \mathbb{R}$  and  $\chi(p) = 0$ ,  $p$  traverses all odd prime numbers, including 1) exists corresponding Landau-Siegel functions  $L(\beta, \chi(p)) = 0$  ( $s \in \mathbb{C}$ ,  $\chi(p) \in \mathbb{R}$  and  $\chi(p) = 0$ ,  $p$  traverses all prime numbers, including 1) is equal to prove that twin prime numbers, Polignac conjecture, Goldbach conjecture are all true.  $L(s, \chi(p)) = 0$  ( $s \in \mathbb{C}$ ,  $\chi(p) \in \mathbb{R}$  and  $\chi(p) = 1$ ,  $p$  traverses all primes, including 1) and the generalized Riemann conjecture  $L(s, \chi(p)) = 0$  ( $s \in \mathbb{C}$ ,  $\chi(p) \in \mathbb{R}$  and  $\chi(p) \neq 0$ ,  $p$  traverses all primes, including 1) prove that the twin primes, Polignac's conjecture, Goldbach's conjecture are all true. When  $L(s, \chi(p)) = 0$  ( $s \in \mathbb{C}$ ,  $\chi(p) \in \mathbb{R}$  and  $\chi(p) = 1$ ,  $p$  traverses all primes, including 1) and the generalized Riemann conjecture  $L(s, \chi(p)) = 0$  ( $s \in \mathbb{C}$ ,  $\chi(p) \in \mathbb{R}$  and  $\chi(p) \neq 0$ ,  $p$  traverses all primes, including 1) prove that the twin primes, Polignac's conjecture, Goldbach's conjecture are all true.

### **III. Conclusion**

After the Riemann hypothesis and the Riemann conjecture and the Generalized Riemann hypothesis and the Generalized Riemann conjecture are proved to be completely valid, the research on the distribution of prime numbers and other studies related to the Riemann hypothesis and the Riemann conjecture will play a driving role. Readers can do a lot in this respect.

### **IV. Thanks**

Thank you for reading this paper.

### **V. Contribution**

The sole author, poses the research question, demonstrates and proves the question.

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