Newton's law of gravitation, *GR*, *SR* and the correspondence principle

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Abstract

This paper contains reflections on how Newton's classical inverse-square law of gravitation corresponds to both Einstein's theory of special relativity adapted for an accelerated object, as well as Einstein's theory of general relativity for the free-fall of an object in a curved spacetime. Using a model described in detail in a related paper, *viXra:2409.0004*, it is shown that space and time are regular in the neighbourhood of a static point mass, and that a black hole and event horizon are mathematical artefacts. In addition, since gravity does not diverge to infinity as masses approach each other, there is no singularity at the coordinate origin.

1 Introduction

Newton's law of gravitation, Einstein's theory of special relativity (SR) and Einstein's theory of general relativity (GR) are three different theories, but - if they are going to be used to provide a good physical description of gravitational motion - they must correspond with each other in situations where they overlap in their range of applicability. The three theories are certainly not the same or easily reconciled with each other. They are conceptually different. Even fundamental quantities such as time are not necessarily the same in the three theories. This paper is intended to clarify some of the differences and establish where they are equivalent.

In Newtonian or classical physics, denoting time as τ, this is the time we normally use. It is called absolute time, because it doesn't depend on anything else, such as motion or gravity, and it is considered to be the same everywhere. For example, the time on the Sun at this instant (now) is the same as it is here on Earth, except that we see the Sun as it was 8 minutes ago, due to the time it takes for information in the form of light to reach the Earth. The velocity of a body is given by $dr/d\tau$ and its acceleration $d^2r/d\tau^2$, where dr is an increment of distance and $d\tau$ an increment of time.

Special relativity (SR) considers the relative motion of two inertial or force-free frames of reference. A coordinate frame is imagined to have clocks positioned everywhere that are synchronised with each other to read the same time, which is called the coordinate time t. We can imagine ourselves as an observer in this frame equipped with one of these clocks, so the coordinate time in SR corresponds to time as we measure it where we are. In this sense, we are stationary and the other inertial frame is moving relative to us, but due to Einstein's postulate of the invariance of the speed of light, an identical clock in the moving frame will tick at a different rate from our coordinate frame clock. This different time rate - which is slower than the coordinate time rate - is called the proper time, here t'. The name - proper time - is perhaps a bad translation into English from the German word *Eigenzeit*, which means *own time* or *self time*. There is nothing proper about it in the sense of being correct or right. Minkowski showed that due to Einstein's postulate, the three spatial coordinates and time could be considered mathematically as a flat four-dimensional spacetime manifold with a symmetry referred to as Lorentzian covariance. SR was not initially conceived to deal with accelerations but, as I shall show later, it can indeed be adapted to describe situations where accelerated motion in a gravitational field occurs.

General relativity (GR), where gravity is involved, is different again. An observer can now be imagined to be situated in a coordinate frame of reference which would have been present before any object causing gravity via spacetime curvature is inserted into the space. Alternatively, you could imagine the coordinate frame as an uncurved frame very distant from the mass causing gravity. Now imagine two identical clocks that are not moving relative to each other, one on Earth and the other on a satellite above the Earth. These two clocks will tick at different rates, not like in SR due to relative motion, but due to the curvature of time, an effect called gravitational time dilation. The time on the satellite clock may also be called the proper time t'; it runs at a faster rate, because it is at a higher gravitational potential. As a principle, clocks always show the proper time. In GR, a four-dimensional spacetime manifold is also adopted as in SR, but in GR it is curved or distorted, and the path of a particle is calculated using Hamilton's principle of least action, which involves extremising the path between two points in spacetime, which also extremises the proper time. Thus, in GR, the physical quantities involved are `proper` quantities (not co-ordinate quantities), such as the proper velocity $\dot{r} = dr/dt'$, and the proper acceleration $\ddot{r} = d^2r/dt'^2$.

2 Predictions

The question to be answered is: how do these various quantities relate to each other when we attempt to use the correspondence principle to compare them in a situation where a test body is moving in a gravitational field? We shall always be considering here a test particle with a small mass m falling radially towards a much greater point mass M.

2.1 Classically

In Newtonian mechanics, the radial acceleration a is given by $a = d^2r/d\tau^2 = -GM/r^2$. Referred to zero an infinite distance from M, the potential energy of the test mass is then $-GmM/r$, and its kinetic energy, $\frac{1}{2}mv^2 = GmM/r$, or $v^2 = 2GM/r$, where $v = dr/d\tau$. We see that both v and $a \to \infty$ for $r \to 0$.

2.2 Using *SR*

Next, SR was not conceived for accelerated motion, but nevertheless we can make some deductions about free-fall. A free-falling object in a gravitational field is force-free, and so it is in an inertial frame, except that the velocity keeps increasing according to the principle of conservation of energy. Firstly, write the four-force on a co-moving particle as the rate of change of momentum in the proper frame:

$$\tilde{F} = \frac{d\tilde{p}}{dt'} = \left(\frac{iP}{c}, \vec{F}\right) \tag{1}$$

where P is power, given by $P = dE/dt'$, and E is the energy. We then write

$$\tilde{F}.\tilde{s} = \left(\frac{iP}{c}, F\right).(ic\,dt, dr) = -P\,dt + F\,dr = 0 \tag{2}$$

for conservation of energy. Now rearrange this as $P\,dt = F\,dr$ and we have

$$\frac{dE}{dt'}\,dt = -\frac{GmM}{r^2}\,dr \tag{3}$$

where the term on the right is Newton's law for the gravitational force, and m is the rest mass of the free-falling particle. This expression represents the differential gain in kinetic energy balanced against the differential loss in potential energy. In SR we have the following expression relating proper time increments dt', coordinate time increments dt and the relative speed v:

$$dt'^2 = dt^2 - dr^2/c^2 \tag{4}$$

or

$$\frac{dt'}{dt} = \sqrt{1 - v^2/c^2} = \frac{1}{\gamma} \tag{5}$$

where $v = dr/dt$ is the coordinate velocity. This gives

$$\gamma\, dE = -\frac{GmM}{r^2}\, dr \tag{6}$$

We then have

$$\gamma\, d\gamma = -\frac{GM}{c^2}\frac{dr}{r^2} \tag{7}$$

which on integration from $\infty \rightarrow r$ gives

$$v^2 = \frac{2GM}{r}\left(1 + \frac{2GM}{c^2 r}\right)^{-1} \tag{8}$$

This expression describes how the coordinate velocity v of a free-falling object changes with distance r as it approaches the gravitational mass M. When r is large the Newtonian expression, $v^2 = 2GM/r$, is recovered, but as r decreases, the velocity starts to lag behind the classical result, reaching a limiting value of c. Writing the constant quantity $2GM/c^2 = \alpha$, this may be rewritten as

$$\frac{v^2}{c^2} = \frac{\alpha}{r}\left(1 + \frac{\alpha}{r}\right)^{-1} = \frac{\alpha}{r + \alpha} \tag{9}$$

and by differentiating this expression, the `coordinate` acceleration of free-fall is

$$\frac{d^2 r}{dt^2} = v\,\frac{dv}{dr} = -\frac{1}{2}\,c^2\frac{\alpha}{(r + \alpha)^2} \tag{10}$$

This does not diverge to infinity but reaches a constant value of $-c^4/4GM$ for $r \rightarrow 0$.

On the other hand, it can be shown by rearranging the metric that the proper velocity is given by

$$\frac{\dot{r}^2}{c^2} = \frac{v^2}{c^2}\left(1 - \frac{v^2}{c^2}\right)^{-1} \tag{11}$$

which, using Equation 9, gives

$$\frac{\dot{r}^2}{c^2} = \frac{\alpha}{r} \tag{12}$$

Thus, the proper velocity dr/dt' does diverge to infinity as $r \to 0$. This is obvious really, because, when the coordinate speed approaches c as $r \to 0$, the proper clock stops ticking.

2.3 Using *GR*

The first person to obtain a solution for the gravitational field outside a point mass using general relativity (*GR*) was Karl Schwarzschild in 1916 [1], less than a year after Albert Einstein published his *GR* theory [2]. Due to subsequent work by Droste [3], Weyl [4], and in particular Hilbert [5], a spacetime increment $d\tilde{s}$ is usually written in spherical polar coordinates (t, r, θ, ϕ) in the form:

$$d\tilde{s}^2 = c^2 dt'^2 = A\, c^2 dt^2 - B\, dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{13}$$

where A and B are radially dependent functions describing the curvature of the time and radial metric coefficients, respectively, and r is a radial coordinate, traditionally called the Schwarzschild radial coordinate in honour of Schwarzschild's name. However, as I shall explain later, r in the metric is not identical to the true radial coordinate distance from the point mass M.

Next, the calculus of variations is used to obtain geodesic equations for the four coordinate variables (t, r, θ, ϕ) with the proper time t' as the Lagrangian parameter that extremises the path of a particle in the curved spcetime. This procedure is explained in detail in another paper [6] that could be read in conjunction with this paper, as well as in many standard textbooks.

Knowing the geodesic equations immediately enables the equation of free-fall motion to be determined, viz.

$$\ddot{r} + \frac{A'}{2B}\, c^2 \dot{t}^2 + \frac{B'}{2B}\, \dot{r}^2 = 0 \tag{14}$$

where $\ddot{r}\,(= d^2r/dt'^2)$ is the proper acceleration, $\dot{r}\,(= dr/dt')$ is the proper velocity, $A' = dA/dr$ and $B' = dB/dr$. Using the radial part of the metric in Equation 13 to eliminate $\dot{t}$ (with $d\theta = d\phi = 0$), we can then reformulate Equation 14 to read:

$$\ddot{r} + \frac{A'}{2AB}\, c^2 + \left(\frac{A'}{2A} + \frac{B'}{2B}\right) \dot{r}^2 = 0 \tag{15}$$

The functions A and B are found using Einstein's field equations of GR for the vacuum outside the point mass. This involves obtaining the Christoffel curvature components from the geodesic equations to obtain the Ricci tensor components, which are then all set to zero (see any textbook on gravitation), with the result that

$$A = \frac{1}{B} = 1 - \frac{\alpha}{r} \tag{16}$$

where α is a constant of integration.

To find α in terms of physical quantities, such as Newton's gravitational constant G and the mass of the object causing gravity M, it is customary to make use of Newton's law of gravitation, in what is called a weak-field approximation. Substituting $B = 1/A$, from Equation 16 into the free-fall equation of motion (Equation 15), we obtain:

$$\ddot{r} + \frac{A'c^2}{2} = 0 \tag{17}$$

Then, using $A' = \alpha/r^2$ by differentiating Equation 16, we obtain

$$\ddot{r} = -\frac{\frac{1}{2}\alpha c^2}{r^2} \tag{18}$$

The result appears to show correspondence with the inverse-square law behaviour of Newton's law of gravitation, in which free-fall acceleration is $-GM/r^2$, giving $\alpha = 2GM/c^2$, which is a positive quantity known variously as the gravitational radius or Schwarzschild radius; α turns out to represent only a small distance, about $2.9\,km$ for a star the mass of the Sun and $8.7\,mm$ for Earth. However, for a supermassive object, such as the centre of the Milky Way galaxy it could be approximately 12 million kilometres.

Since α is positive (as opposed to being negative), the solution in Equation 16 predicts that A and B change sign at this radial coordinate $r = \alpha$. For a point (or highly compacted) mass with physical radius less than α, it therefore seems that a discontinuity occurs in spacetime. The distance α defines what is now called the event horizon, inside which the point mass is obscured as a black hole.

Next, by integrating Equation 17 one obtains the proper velocity of free-fall $\dot{r}$, which may be written as

$$\frac{\dot{r}^2}{c^2} = 1 - A \tag{19}$$

and inserting the solution $A = 1 - \alpha/r$ we obtain

$$\frac{\dot{r}^2}{c^2} = \frac{\alpha}{r} \tag{20}$$

This predicts $\dot{r} \to c$ for $r \to \alpha$ and $\dot{r} \to \infty$ for $r \to 0$.

The same GR solution also makes a prediction for the coordinate velocity. This can also be derived from the radial part of the metric:

$$dt'^2 = A\,dt^2 \; - B\,dr^2/c^2 \tag{21}$$

Rearranging this in two ways (writing $\gamma = dt/dt'$) gives

$$\frac{1}{\gamma^2} = A - B\frac{u^2}{c^2} \tag{22}$$

where $u = dr/dt$ is the GR coordinate velocity, and

$$1 = A\,\gamma^2 - B\frac{\dot{r}}{c^2} \tag{23}$$

We thus obtain

$$\frac{u^2}{c^2} = \frac{A\dot{r}^2/c^2}{(1 + B\dot{r}^2/c^2)} \tag{24}$$

Substituting $\dot{r}^2/c^2 = 1 - A$ and $B = 1/A$ we finally obtain

$$\frac{u^2}{c^2} = (1 - A)A^2 = \frac{\alpha}{r}\left(1 - \frac{\alpha}{r}\right)^2 \tag{25}$$

This is a very strange result, because it suggests that the coordinate velocity goes to zero as $r \to \alpha$. As a distant observer, we would then presumably see falling objects stop for ever at the event horizon. This surely cannot be correct?

3 Discussion and further theory

Historically, Schwarzschild [1] recognised there was a mathematical discontinuity in his GR solution for a point mass, but by defining a suitable auxiliary radial coordinate he forced the discontinuity to be at the origin, since he believed it to be non-physical. Shortly afterwards Droste [3] and Weyl [4] provided a solution, but restricted the range of r to $r > \alpha$. Subsequently, Hilbert [5] extended Droste and Weyl's solution to the region $r < \alpha$ on the grounds that a coordinate transformation does not alter the physics of the situation, and GR is supposed to be a generally covariant theory. It is essentially Hilbert's solution allowing for a change in sign of A and B, that is accepted today

However, in a previous related paper [6] I showed there is a simple explanation that falsifies the irregular behaviour of spacetime. The geometry of Newton's inverse-square law of gravitation is undeniably strictly Euclidean (or flat), i.e. spatial curvature plays no part in Newtonian gravity. On the other hand, GR explains gravity through the curvature of both space and time. Logically, then, to relate Newton's law with GR to obtain correspondence, we must recognise that Newtonian gravity is that contribution to gravity resulting exclusively from the curvature of time. In comparing GR with Newton's law it is therefore incorrect to use the reciprocity of space and time curvature dictated by GR. We have to write $B = 1$, as for a Euclidean space, and then the equation of free-fall from Equation 15 becomes modified to read

$$\ddot{r} + \frac{A'}{2A}\left(c^2 + \dot{r}^2\right) = 0 \qquad [B = 1] \tag{26}$$

Solving this differential equation and inserting Newtonian expressions for free-fall acceleration and velocity as before then gives the following expression for A:

$$A = \left(1 + \frac{\alpha}{r}\right)^{-1} \qquad ; \qquad \alpha = 2GM/c^2 \tag{27}$$

where the constant of integration α is again equal to $2GM/c^2$. Thus, we have quantified the insight that the gravitational force or acceleration in Newton's law relates strictly to the curvature of the time coordinate in GR.

Now writing the Schwarzschild radial coordinate in the GR solution of Equation 16 as r^*, so as not to confuse the two coordinates, we may write

$$A = 1 - \frac{\alpha}{r^*} = \left(1 + \frac{\alpha}{r}\right)^{-1} \tag{28}$$

from which it follows that

$$r^* = r + \alpha \tag{29}$$

The difference between r^* and r is extremely small when they are very much greater than α, and distinguishing between the two then becomes irrelevant. But when r is of the order of α the situation is crucially different. While the range of r goes from zero to ∞, the range of r^* is from α to ∞. The spacetime manifold does not exist for $r^* < \alpha$; Hilbert's extension to $r^* < \alpha$ is therefore meaningless, and there is no event horizon.

The solution $A = (1 + \alpha/r)^{-1}; B = 1$ does not accurately satisfy Einstein's vacuum field equations of GR, since Newton's inverse-square law of gravity only describes that aspect of gravity caused exclusively by the curvature of the time coordinate (and not space), and this is manifestly dominant for most cases we consider, such as planetary motion.

However, space curvature becomes significant when speeds approach the speed of light, and distances to the central mass become small, and this will modify gravity from being purely Newtonian. GR then describes correctly phenomena that Newton's law does not describe, such as the perihelion rotation of the planet Mercury, and the bending of starlight passing near the Sun. To satisfy GR, we thus require $B = 1/A$, as before, but the Schwarzschild radial coordinate must be replaced by $(r + \alpha)$.

We then obtain for the proper velocity:

$$\frac{\dot{r}^2}{c^2} = 1 - A = \frac{\alpha}{r + \alpha} \tag{30}$$

This means we have $\dot{r} \to c$ for $r \to 0$, which contrasts fundamentally with the black hole solution, where

$$\frac{\dot{r}^2}{c^2} = 1 - A = \frac{\alpha}{r}$$

which gives $\dot{r} \to c$ for $r \to \alpha$, and $\dot{r} \to \infty$ for $r \to 0$. The solution presented here thus predicts a limiting free-fall `proper` velocity of c, in contrast to the other theories that predict infinite velocity as $r \to 0$. Furthermore, in my model the radial free-fall acceleration is given by

$$\ddot{r} = -\frac{1}{2}\,c^2 \frac{\alpha}{(r + \alpha)^2} = -\frac{GM}{(r + 2GM/c^2)^2} \tag{31}$$

which shows classical Newtonian behaviour $-GM/r^2$ for $r \gg \alpha$ but deviates (decreases) from inverse-square law behaviour for r of the order of α.

4 Further discussion

Looking back to Subsection 2.2, we see that the `coordinate` velocity prediction using SR (Equation 9) is identical to the `proper` velocity prediction in GR using my model (Equation 30). How can this be true? To show it makes sense, I shall use what may be called a `heuristic` argument, as follows (i.e. proceeding by rules that are loosely defined).

Write the radial part of the metric in GR (i.e. with $d\theta, d\phi = 0$), bracketed as

$$c^2 dt'^2 = c^2(\sqrt{A}\, dt)^2 - (\sqrt{B}\, dr)^2 \tag{32}$$

and make the following substitutions:

$$dT = \sqrt{A}\, dt \quad ; \quad d\varrho = \sqrt{B} dr \tag{33}$$

We then have

$$c^2 dt'^2 = c^2 dT^2 - d\varrho^2 \tag{34}$$

which now resembles a spacetime element in SR in terms of coordinates T and ϱ. Writing the proper velocity in SR as $\dot{\varrho} = d\varrho/dt'$ and the coordinate velocity as $v = d\varrho/dT$, we then have the following relationships:

$$\dot{\varrho} = \frac{d\varrho}{dt'} = \frac{\sqrt{B}dr}{dt'} = \sqrt{B}\,\dot{r} \tag{35}$$

and

$$v = \frac{d\varrho}{dT} = \frac{\sqrt{B}\,dr}{\sqrt{A}\,dt} = \frac{\sqrt{B}\,u}{\sqrt{A}} \tag{36}$$

where $u = dr/dt$ is the coordinate velocity in GR.

I have already shown in GR in Equation 19 that

$$\frac{\dot{r}^2}{c^2} = 1 - A \tag{37}$$

which gives

$$\frac{\dot{\varrho}^2}{c^2} = B(1 - A) \tag{38}$$

I have also shown in GR that

$$\frac{u^2}{c^2} = A^2(1 - A) \tag{39}$$

Therefore we have

$$\frac{v^2}{c^2} = AB(1 - A) \tag{40}$$

Now use the point mass solution with $B = 1/A$ and we then have the following relationships between the proper and coordinate velocities in GR compared with their proper and coordinate equivalents in SR for this free-fall thought experiment:

$$\frac{\dot{r}^2}{c^2} = 1 - A \quad ; \quad \frac{u^2}{c^2} = A^2(1 - A) \tag{41}$$

$$\frac{\dot{\varrho}^2}{c^2} = \frac{1 - A}{A} \quad ; \quad \frac{v^2}{c^2} = 1 - A \tag{42}$$

Comparing the last two sets of equations, the point I have wanted to prove is that the proper velocity $\dot{r}$ in GR is equivalent to the coordinate velocity v in SR, which is why Equations 9 and 30 agree with each other.

Finally, the coordinate velocity in GR from my model is given by

$$\frac{u^2}{c^2} = A^2(1 - A) = \frac{\alpha r^2}{(r + \alpha)^3} \tag{43}$$

This quantity falls to zero as $r \rightarrow 0$, which is interesting, because it supports the prediction in my model that gravity disappears as masses approach each other intimately, and that the traditionally expected singularity at the origin is in fact non-existent.

5 Conclusion

It has been shown here how SR can be interpreted for the case of free-fall, and how the various physical quantities in Newton's law, SR and GR correspond to each other for the case of a test object falling towards a point mass. The deductions outlined here deviate markedly from the current paradigm. The idea of a black hole and event horizon, although a mathematical possibility, has been shown to be non-physical by correct correspondence of Newton's law in conjunction with Einstein's GR theory. The present model shows that gravity does not diverge to infinity as masses approach each other, which removes the physically inexplicable issue of a singularity in spacetime.

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