Proof of the Collatz conjecture based on a directed graph

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September 9, 2024

Abstract

The Collatz conjecture considers recursively sequences of positive integers where n is succeeded by $\frac{n}{2}$, if n is even or $\frac{3n+1}{2}$, if n is odd. The conjecture states that for all starting values n the Collatz sequence eventually reaches a trivial cycle 1, 2, 1, 2……. If the Collatz conjecture is false, then either there is a nontrivial cycle, or one sequence goes to infinity. In this paper, we construct a directed graph based on the union of infinite number of basic Collatz directed graphs. Each basic Collatz directed graph relates to each positive integer. We show that the directed graph is connected and covers all positive integers. There is only a trivial cycle and no sequence goes to infinity.

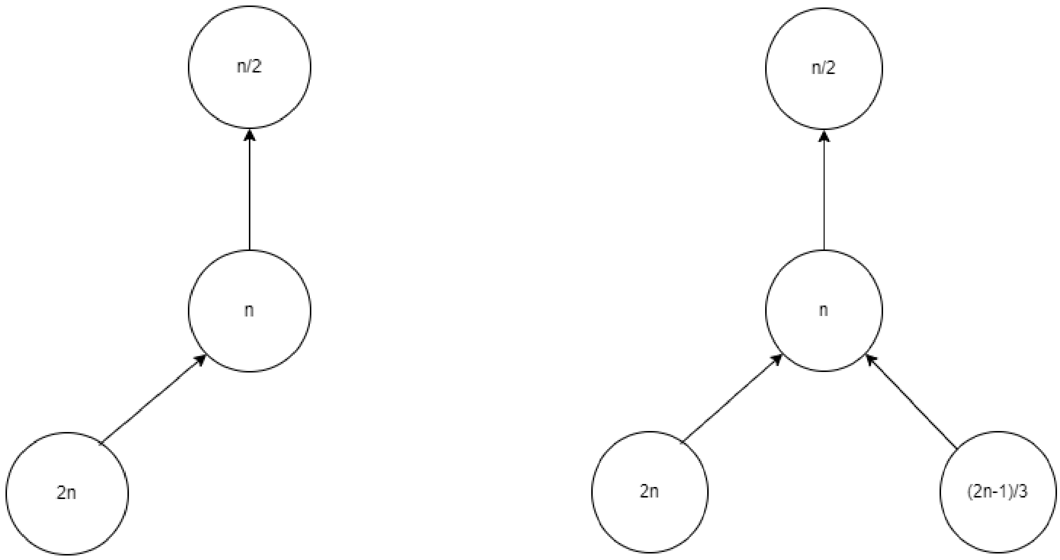
1. Introduction

The Collatz conjecture considers recursively sequences of positive integers where n is succeeded by $\frac{n}{2}$, if n is even, or $\frac{3n+1}{2}$, if n is odd. The conjecture states that for all starting values n the sequence eventually reaches the trivial cycle 1, 2, 1, 2……. If the Collatz conjecture is false, then either there is a nontrivial cycle, or one sequence goes to infinity [1-2].

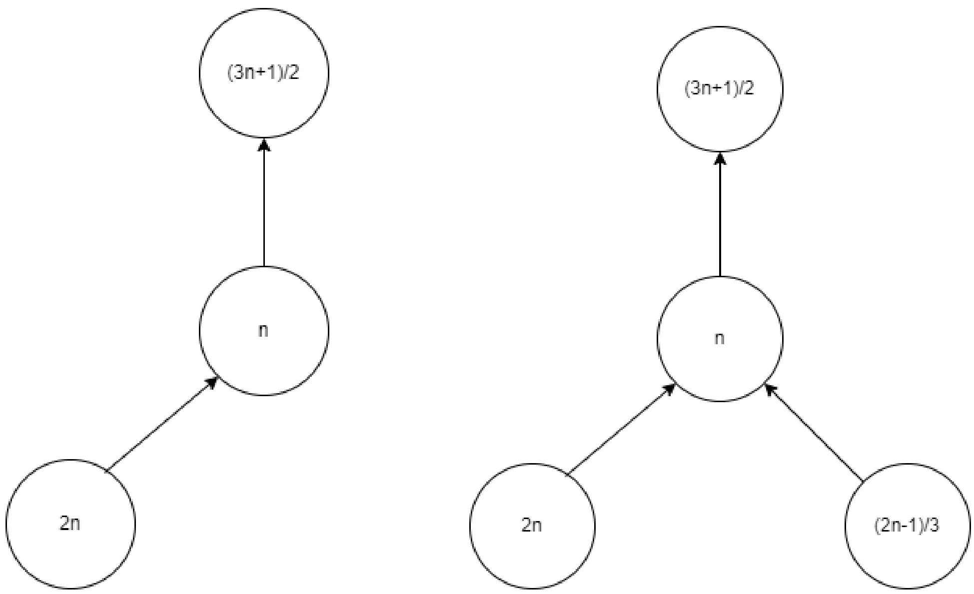
2. A basic Collatz directed graph

The basic directed graph is constructed for each natural number as follows:

Let n be a positive integer node. Its parent node is $\frac{n}{2}$, if n is even or $\frac{3n+1}{2}$, if n is odd. Its left child is 2n. Its right child is $\frac{2n-1}{3}$, if n $\equiv$ 2 mod 3, or no right child , if n $\neq$ 2 mod 3. Thus there are four types of basis Collatz directed graph as shown in Figure 1.



(a) n is even and not equal to 2 mod 3 (b) n is even and equals to 2 mod 3



(c) n is odd and not equal to 2 mod 3 (d) n is odd and equals to 2 mod 3

Figure 1, Four types of basic Collatz directed graphs

Examples of basic Collatz directed graphs shown in Figure 2.

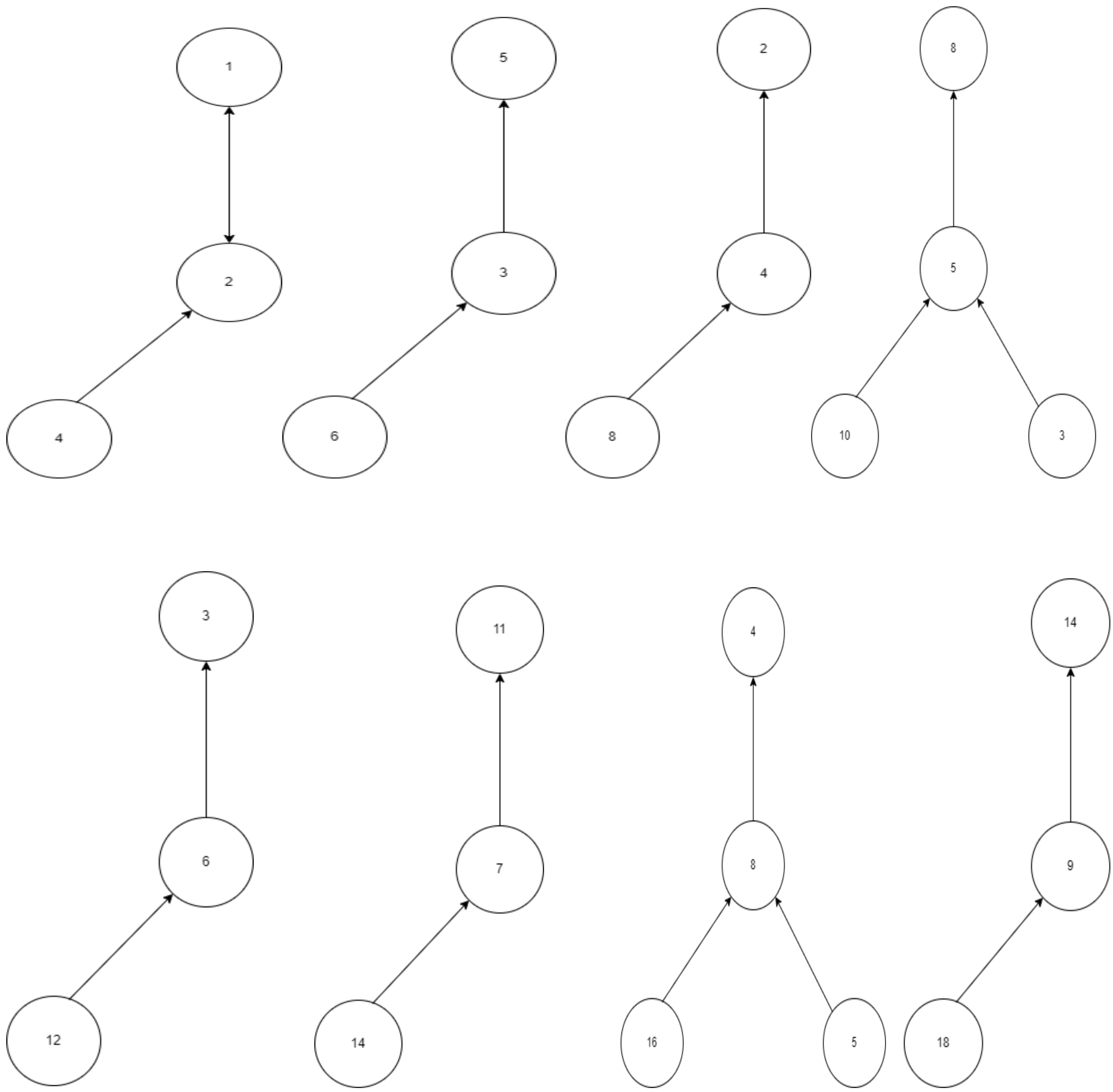


Figure 2. Basic Collatz directed graphs of 2, 3, 4, 5, 6, 7, 8, and 9

2. **The union of two basic Collatz directed graphs**

The union of two basic Collatz directed graphs for different cases shown in Figure 3.

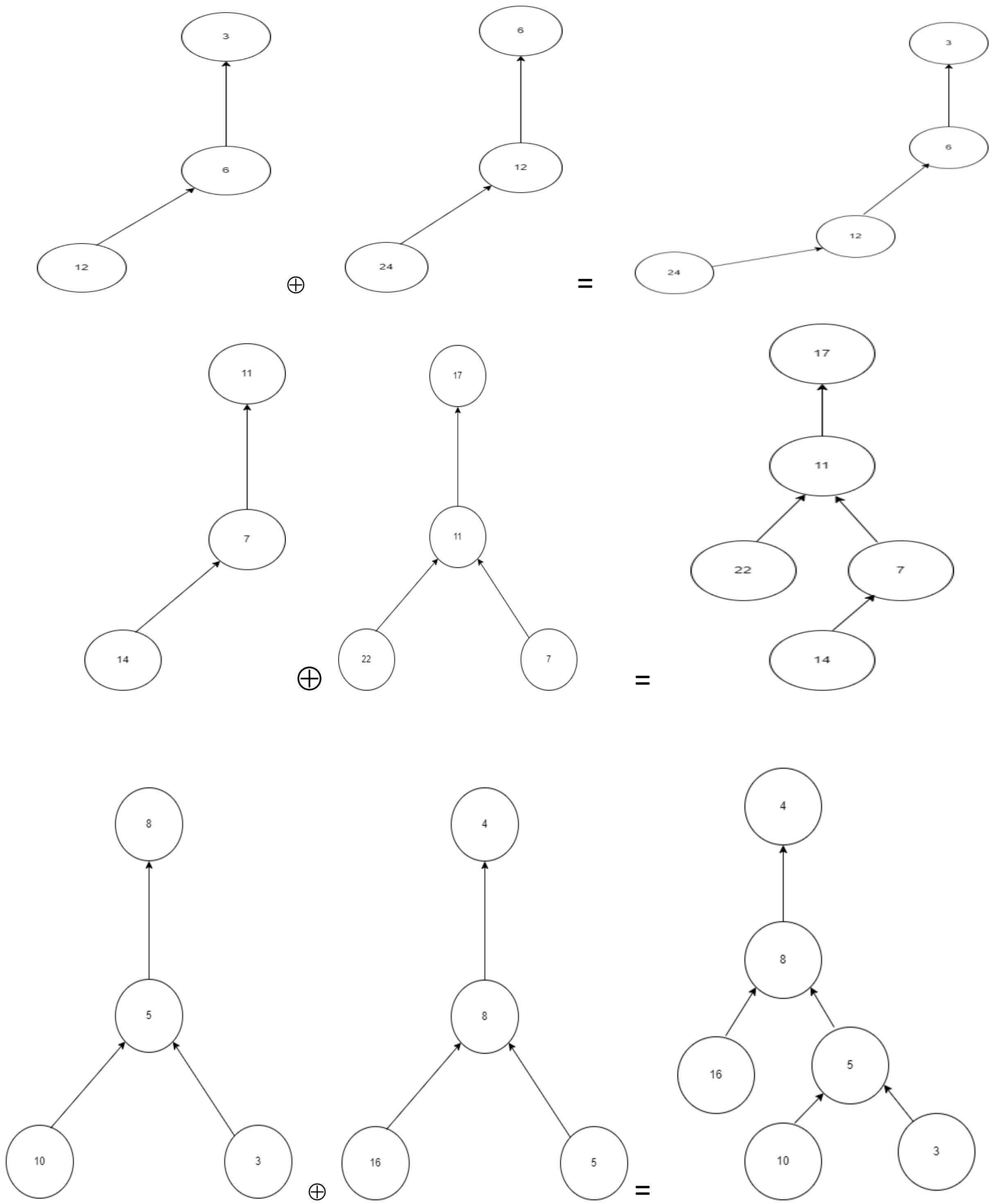


Figure 3. Various cases for the union of two basic Collatz directed graphs

3. **A Complete Collatz directed graph**

Let G represents the union of all basic Collatz directed graphs. Clearly G is connected and can be arranged in levels 0 to ∞ as shown in Figure 4. Node 1 is in level 0. There is no nontrivial cycle or divergence sequence in this graph.

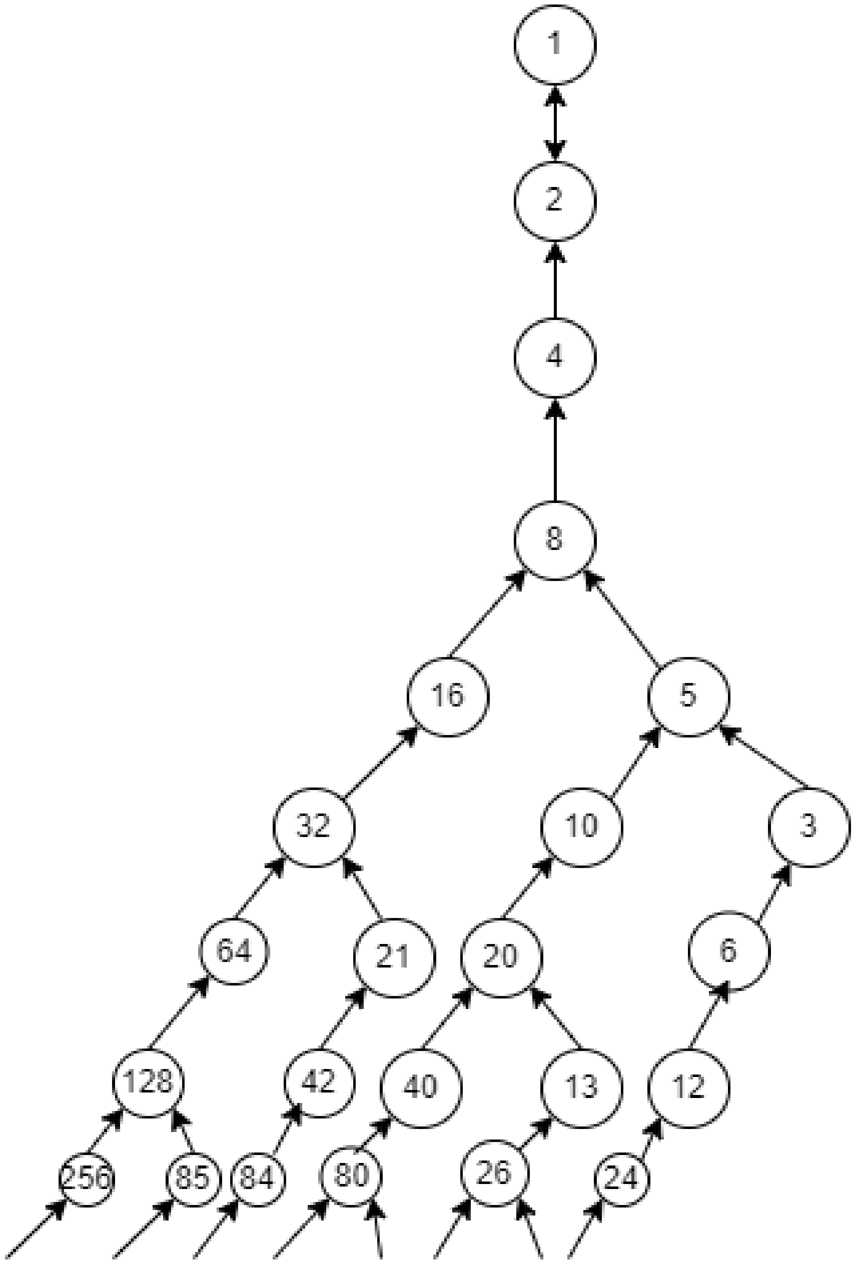


Figure 4. A complete Collatz directed graph

Conclusion

A complete Collatz directed graph covers all positive integers. By starting at any node in this complete Collatz directed graph, there is a unique path from that node to a node 1. Thus, the Collatz conjecture is proved to be valid.

References

[1] R . Terras, (1976). " A stopping time problem on the positive integers". Acta Arithmetica, 30(3), 241-252.

[2] J. C. Lagarias. The 3x+1 Problem: An Overview. https://arxiv.org/abs/2111.02635, 2021.