PROOF OF THE COLLATZ CONJECTURE USING THE REVERSE ALGORITHM

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Abstract

The Collatz conjecture has remained unsolved for a long time. In this paper, a proof of this conjecture will be presented. We know that almost all numbers will eventually reach one through the steps of the Collatz algorithm. Terence Tao has previously proven this. This paper demonstrates that all numbers will reach one by using Tao's proof. If almost all numbers reach one, then the probability that a randomly chosen number from an infinite set of numbers will reach one is one. The probability that a number will not reach one is zero. The probability of selecting the elements of the number sequences associated with a number n that violates the conjecture from an infinite set of numbers is a non-zero value such as c. However, this contradicts the proof that almost all numbers reach one. Therefore, there is no such number n that violates the conjecture, and the conjecture is true for all numbers. In order to prove that the probability of selecting the elements of the number sequences associated with a number n that violates the conjecture from an infinite set of numbers is a non-zero value like c, we examine the sequences associated with one. If the probability of selecting the elements of each branch of these sequences from an infinite set of numbers is a non-zero value, we reach the desired proof.

Introduction

Let m(1), m(2), …, m(k), m, f(k), n, k $\in$ Z^+ and c(1), c(2), …, c(n), c, c(t) $\in$ R where 0 < c(1), c(2), …, c(n), c, c(t) < 1.

s(t[m]): The number of elements in t[m].
t[m]: The set containing the sum of sequences starting with m. For example, if the Collatz conjecture holds, then t[1] = {1, 2, 3, 4, …}. This set includes numbers that can be reached by starting from 1 and applying the reverse Collatz algorithm.
t[16] $\in$ t[1]: t[16] is an element of t[1].
P(t[m]): The probability of selecting elements of the set t[m] from the set of positive integers.
P(t[m]) = s(t[m]) / s(Z^+)
Let n be a number that violates the conjecture. If P(t[n]) = c(n) ≠ 0, then P(t[1]) ≠ 1.
Tao has proven that P(t[1]) = 1.

Collatz Operation:
f(m) = 3m + 1 if m ≡ 1 (mod 2),
f(m) = m / 2 if m ≡ 0 (mod 2).

Reverse Collatz Operation:
f(m) = 2m for m $\in$ Z^+,
f(m) = (m - 1) / 3 if m ≡ 1 (mod 3).

Theorem 1:

When m(1) > m(2), the number of elements in the set of sequences starting with m(1) is less than or equal to the number of elements in the set of sequences starting with m(2), i.e., s(t[m(2)]) ≥ s(t[m(1)]).

Proof:

Let's assume the opposite, i.e., if m(1) > m(2), s(t[m(2)]) < s(t[m(1)]). Then, for m(2) > m(3), s(t[m(3)]) < s(t[m(2)]), and this series continues in the same manner. Eventually, for some number m(k), when m(k) > 1, we would have s(t[1]) < s(t[m(k)]), which is incorrect because this contradicts the result that P(t[1]) = 1. Therefore, if m(1) > m(2), s(t[m(2)]) ≥ s(t[m(1)]). Consequently, if s(t[m(2)]) ≥ s(t[m(1)]), then P(t[m(2)]) ≥ P(t[m(1)]).

The Collatz Conjecture simply states the following: Choose a positive integer. If the number is even, divide it by two; if it's odd, multiply it by three and add one. Repeat the same process for the newly obtained number. Continue this way, and eventually, you will reach the number one. The problem here is to prove that every positive integer will eventually reach one. In this paper, we will demonstrate this proof. There are two possible scenarios that could violate the conjecture: either the numbers increase infinitely, or there are some closed loops.

As a result, Tao has proven that almost all numbers will reach one according to the Collatz conjecture. Now, let's prove that all numbers eventually reach one.

To do this, let's examine a number, n, which is an odd number. Suppose this number does not reach one, i.e., it violates the conjecture. Undoubtedly, we can reach the number n through other numbers. Here, let's reverse the Collatz algorithm and see which numbers we can reach n from. We can reach the number n by dividing 2n by two. Similarly, we can reach the number n by dividing 4n by four. These numbers form a sequence:

n - 2n - 4n - 8n - 16n - …

Thus, if a single number n violates the conjecture, an infinite number of other numbers would also violate it. However, this sequence is not limited to just these numbers. The initial number n can be of the form 3k, 3k+1, or 3k+2. If the number is of the form 3k, we take the next odd number, i.e., (3n+1)/2 or (3n+1)/4. Our goal is to start with a number in the form 3k+1 or 3k+2, because numbers that are exact powers of three do not lead to new sequences in the next steps of the sequence.

Let me explain what I mean with an example. To reach the number 3, the sequence we have is 3 - 6 - 12 - 24 - 48 - 96 … This sequence does not lead to new odd numbers because you cannot reach these even numbers by multiplying an odd number by three and adding one. However, let's look at the sequence of the number 5: 5 - 10 - 20 - 40 - … This sequence does lead to new elements. For example, you can reach the number 10 from the number 3, or you can reach the number 40 from the number 13. In the next step, the number 13 leads to a new sequence 13 - 26 - 52 - 104 - …, and these steps repeat infinitely. We started with a number n, and if n is a multiple of three, it can be proven that the next odd number in the sequence will not be a power of three. If our first number is n, the next odd number will be (3n+1)/2^k, and this number is not a power of three. Consequently, if we have a number n that violates the conjecture, we are faced with a tree-like structure with infinite branches, where each branch leads to infinite new branches.
Example:

21-42-84-…
↑
1- 2- 4- 8- 16- 32- 64 - 128-…
↓
5-10-20-40-…
↓
3-6-12-24-48-…

Our proof aims to assert that the probability of selecting the elements of such a tree, starting from a number n, out of an infinite set of numbers is a probability like c. This would contradict the claim that almost all numbers reach one. Thus, if it is proven that this probability is a number like c, the conclusion is that such elements do not exist within the Collatz sequences. In other words, there are no cases that violate the conjecture, because if the probability of selecting the elements of sequences that violate the conjecture from an infinite set of numbers is a non-zero number like c, this would contradict the proof that almost every number reaches one.

Therefore, it is sufficient to prove that the probability of selecting the elements of the sequences linked to a number n that violates the conjecture from an infinite set of numbers is a number like c. To do this, let's first consider the number one. We know that the branches of the sequence starting from one contain almost all numbers. The probability of selecting the elements of the sequences starting from the number one from an infinite set of numbers is one. The first sequence starting from the number one has infinitely many branches, and each of those branches has infinitely many branches. This process continues infinitely.

Now, we need to show that the probability of selecting the elements of any branch of the sequence starting from one from an infinite set of numbers is c. The total contribution of all the branches sums to one. This can occur in two ways: either each branch contributes a probability like c(1), c(2), c(3), etc., or each branch contributes zero. In other words, $1 = c(1) + c(2) + \ldots$ or $1 = 0 + 0 + \ldots$ The second representation leads to the indeterminate form $0 * \infty$. There is a third possibility, which is: $1 = 1 + 0 + 0 + \ldots$ meaning that only the first branch contributes to the probability. Let's refute this possibility.

Theorem 2: $P(t[16]) \neq 1$

Proof:

If the first branch contributes to the entire probability, then the first sub-branch of this branch would also contribute to the probability. If other branches also contribute a probability greater than zero, the same should be true for the upper branches. However, only the first branch contributes to the probability. As a result, the sub-branch of the first sub-branch would also contribute to the probability. (The result of theorem 1) This process continues, and the numbers grow larger.

At some point, the first element of the branch that contributes to the probability would become greater than the first element of the second upper branch. At this point, it would be illogical for the sub-branch to contribute more to the probability than the second upper branch, because if we compare the number of elements in these branches, the second upper branch would have more elements than the sub-branch of the first upper branch. Thus, it is unreasonable for the first upper branch to contribute to the probability in this way. Let's illustrate this visually.

```
                 85
                 ↑
1-2-4-8-16-32-64-128-256-…
     ↓
     5- 10-  20-  40-…
       ↓          ↓
       3-6-…    13-26-52-…
                    ↓
                   17-34-68-…
                      ↓
                     11-22-44-…
                        ↓
                        7-  14- 28-   56-  112
                                ↓          ↓
                                9-18-…    37-74-148-…
                                               ↓
                                              49-98-196-…
                                                  ↓
                                                  65-130-260-…
                                                     ↓
                                                     43-86-172-344-688...
                                                          ↓        ↓
                                                          57       229
```

In this sequence of branches, the starting number of the last branch we reached is 229. Since 229 > 85, if the probability of selecting elements from an infinite set for the sequence starting with 229 is 1, then the probability for the sequence starting with 85 cannot be 0, because the sequence starting with 85 has either the same number of elements or more than the sequence starting with 229. In other words, if 229 > 85, then $s(t[85]) \geq s(t[229])$, and thus $P(t[85]) \geq P(t[229])$.

In the visual example above, we progressed by taking the first sub-branches of the sequences, but we did not account for sequences starting with powers of three. This is because the probability of selecting elements from an infinite set for sequences starting with powers of three is zero. As a result, the probability of selecting elements from an infinite set for the sequence starting with one cannot be written as $1 = 1 + 0 + 0 + ...$, because the probability of selecting elements for the first sub-branch cannot be one.

Now, let's prove that the sum $1 = 0 + 0 + 0 + \ldots$ is also invalid.

Theorem 3:

The probability of selecting elements from an infinite set for each branch of the sequence starting with the number 1 cannot be zero. In other words, $P(t[1]) = (\infty(1)/\infty + \infty(2)/\infty + \ldots \neq 0 + 0 + \ldots)$.

Proof:

Now, let's examine the number of elements in the branches associated with the number 1. The total contribution for each branch is $P(t[1]) = s(t[1])/s(Z^+) = \infty/s(Z^+) = \infty/\infty = 1$. We know that $t[16] \in t[1]$, and similarly, $t[64] \in t[1]$, and this continues. Therefore, if we represent each branch's contribution with a different symbol, we get $P(t[1]) = (\infty(1) + \infty(2) + \ldots) / \infty$. For example, $s(t[16]) = \infty(1)$, $s(t[64]) = \infty(2)$, and so on. Consequently, $P(t[1]) = (\infty(1)/\infty + \infty(2)/\infty + \ldots) = 0 + 0 + \ldots$ We assumed that for branches related to the number 1, such as $t[f(k)]$, the probabilities $P(t[16]), P(t[64]), \ldots P(t[f(k)]) \ldots = 0$.

Let's examine the sequence:
1 - 2 - 4 - 8 - 16 - …
For each element, the probability of selecting it is $1/\infty = 0$. We previously proved that the probability of selecting an element from this sequence out of an infinite set of numbers, or out of positive integers, is zero. Thus, $0 = 0 + 0 + 0 \ldots = 0 * \infty$. For the branches of the sequence starting with 1, $1 \neq 0 + 0 + 0\ldots = 0 * \infty = 0$ holds because, for the above sequence, the number of zeros in the sum is greater than the number of zeros in the sequence that starts with one and branches out to infinity. Therefore, if $0 * \infty = 0$ for one sequence, then it must also be true for the other.

The point to remember here is that each zero in the sum for the sequence starting with one and branching out represents each first sub-branch of the sequence that starts with one. Let's illustrate this visually.

21-42-84-…
↑
1- 2- 4- 8- 16- 32- 64 - 128-…
↓
5-10-20-40-…
↓
3-6-12-24-48-…

In this sequence, the first sub-branch begins with the number 16, represented by the first zero in the total. The second sub-branch starts with the number 64, represented by the second zero in the total. Since the probability of selecting the elements of numbers like 4, 8, and 32 from an infinite set is zero, excluding these numbers from the sum does not affect the outcome. If we were to include these numbers in the total, the number of zeros for both sequences would be equal, and $0 * \infty$ would still result in zero for both sequences.

This situation is not only valid for the number 1. For any number m, if the probability for the branches associated with m is $P(t[m]) = c(m)$, then for the branches associated with m, we have $c(m) = c^*(1) + c^*(2) + \ldots \neq 0 + 0 + \ldots$ Let's prove this.

Theorem 4: For the branches associated with the number m, $P(t[m]) = c(m) = c^*(1) + c^*(2) + \ldots \neq 0 + 0 + \ldots$

Proof:

The sequence starting with the number m proceeds as $m - 2m - 4m - 8m - \dots$ This distribution matches one-to-one with the sequence starting with the number 1, i.e., $1 - 2 - 4 - 8 - 16 - \dots$ Therefore, if the total probability for one is zero, it must also be zero for the other. Hence, $P(t[m]) = \infty(m1)/\infty + \infty(m2)/\infty + \dots = s(t[m])/s(Z^{+})$.

Here, $\infty(mk) = s(t[mk])$ represents the number of elements in the k-th branch of the sequence starting with m, and $P(t[m]) = P(t[m_1]) + P(t[m_2]) + \dots$ If we assume $P(t[mk]) = 0$, then $P(t[m]) = 0 + 0 + \dots = 0$, which contradicts the assumption that $P(t[m]) = c(m)$. For the sequence $f[1] = \{1, 2, 4, 8, \dots\}$, $P(f[1]) = s(f[1])/s(Z^{+}) = 0 = P(1) + P(2) + P(4) + \dots = 1/\infty + 1/\infty + \dots = 0 + 0 + \dots$

Where:

$s(f[1])$ is the number of elements in $f[1]$,
$s(Z^{+})$ is the number of elements in the set of positive integers,
R:The set of real numbers
Z^{+} :The set of positive integers
$P(k)$ is the probability of selecting the element k from the set of positive integers,
$c^{*}(1), c^{*}(2), \dots, c(m) \in R$ and $0 < c^{*}(1), c^{*}(2), \dots, c(m) < 1$,
$f[m] = \{m, 2m, 4m, \dots\}$

Since the sets $f[m]$ and $f[1]$ match one-to-one, if $P(f[1]) = 0$ and we assume $P(t[mk]) = 0$, then $P(t[m]) = 0$, which is a contradiction. Therefore, we conclude that $P(t[m]) = c(m) = c^{*}(1) + c^{*}(2) + \dots \neq 0 + 0 + \dots$

Conclusion

For a number n that violates the conjecture, there exist infinite branches and sequences associated with n. The probability of selecting the elements of the set of numbers associated with n from the set of positive integers equals a number like c, which is non-zero. However, this is impossible because the probability of selecting numbers that reach 1 from the set of positive integers is 1. Therefore, a number n that violates the conjecture cannot exist, and all numbers must reach 1 according to the Collatz algorithm. To prove that the probability of selecting the elements of the set of numbers associated with n from an infinite set is a non-zero number like c, we look at the sequence that starts with 1. This sequence has branches, and for each branch, we have proven that the probability of selecting the numbers in this branch from the set of positive integers is greater than zero. Therefore, the equation $1 = c(1) + c(2) + c(3) + \dots$ holds true for the branches of the sequence starting with 1. No branch can contribute zero to the probability, and in this case, the total group of numbers associated with a sequence starting with a number t, larger than n, will have a non-zero probability of selecting elements from the set of positive integers, represented as $c(t)$ (The results of theorems 2,3 and 4). We know that the number t does not violate the conjecture, meaning that t is the starting number of one of the sub-branches of the sequence starting from 1. However, the number of elements in the set of numbers associated with n is greater than or equal to the number of elements in the set associated with t, since $t > n$.(The result of theorem 1) Therefore, the probability of selecting the elements of the group of numbers associated with n from the set of positive integers is a number $c(n)$, which is greater than zero. That is, if $t > n$, then $P(t[n]) \geq P(t[t])$. As a result, since it has been proven that almost all numbers reach 1, the probability of selecting the elements of a number block that violates the conjecture from the set of positive integers is zero. This means that if the probability of selecting the elements of a sequence starting with a number n that violates the conjecture is a non-zero number like $c(n)$, then there is no number n that violates the conjecture. The Collatz conjecture is true for all numbers. The conjecture has been proven.

References

Tao,T.(2019).Almost all orbits of the Collatz map attain almost bounded values.

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