

The Sommerfeld equation with the Uehling potential

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Abstract

The Uehling potential is the one-loop radiative correction to the photon propagator. The correction can be graphically represented by the Feynman diagram of the second order. The physical meaning of this diagram is the process $\gamma \rightarrow (e^- + e^+) \rightarrow \gamma$, where γ is denotation for photon, and e^-, e^+ is the electron-positron pair. It means that photon can exist in the intermediate state with e^+, e^- being virtual particles. Then, the Coulomb potential with radiative corrections can be inserted in the Sommerfeld quantum equation in order to find the Sommerfeld energy.

1 Introduction

The Coulomb's law summarizes experimental data. This law states that two charged bodies with infinitely small dimensions (two point charges) repel each other if they have like charges and attract each other if they have unlike charges. The force of their interaction force is proportional to $\sim \frac{q_1 q_2}{R_{12}^2}$, where q_1 and q_2 are charges of the first and second bodies, respectively and R_{12} is the distance between them.

Let us consider an electrostatic field in a vacuum. A perfect vacuum cannot naturally be achieved in experiments, and a certain amount of air always remains in the vessels being evacuated. This does not at all mean, however, that the laws of an electric field in a vacuum cannot be studied experimentally Tamm (1979).

The force of interaction of charges being inversely proportional to the square of the distance between them can be directly verified experimentally. It can be verified by sequentially measuring the forces of interaction between pairs of charges.

As regards the sign of charges, it is pure convention that the charges which appear on glass when it is rubbed with silk or flannel are positive. Hence, the charges that are repelled by these charges on the glass are also positive.

It is very important that Coulomb's law holds only for the interaction of point charges, i.e. charged particles of infinitely small dimensions.

The expression "infinitely small" should naturally not be understood here in its strictly mathematical sense. In physics, the expression "infinitely small" (or "infinitely great") quantity is always understood in the sense of "sufficiently small" (or "sufficiently great") quantity-sufficiently small with respect to another quite definite physical quantity. (Tamm, 1979).

In the formulation of Coulomb's law, the infinitely small (point) value of the dimensions of charged bodies is understood in the sense that they are sufficiently small relative to the distance between these bodies, sufficiently small in the sense that with the given distance between the bodies the force of their interaction no longer changes within the limits of the preset accuracy of measurements upon a further reduction of their dimensions and an arbitrary change in their shape (Tamm, 1979).

When determining the resultant of electric forces, we must naturally take account of the circumstance that these forces are vectors. \mathbf{R}_{12} stands for a radius-vector drawn from point 1 to point 2, and $R_{12} = |\mathbf{R}_{12}|$ for the numerical value of the distance between points 1 and 2. It is obvious that $\mathbf{R}_{12} = -\mathbf{R}_{21}$.

Coulomb's law, as in general of any law on which the relevant branch of theoretical physics is based, belongs not only to the direct experimental verification of this law. It also belongs, and this is much more significant, to the agreement with experimental data of the entire complex of theoretical conclusions having this law as one of their cornerstones Tamm (1979).

The radiative corrections to the Coulomb potential follows from the quantum electrodynamics and cannot be determined by the classical mathematical procedures of the classical electromagnetism. So, we use in the next section the Green function of photon from which the radiative Uehling corrections to the Coulomb potential follow.

2 The Uehling correction to the Coulomb potential

The Uehling potential describes the interaction potential between two electric charges which, in addition to the classical Coulomb potential, contains an extra term responsible for the electric polarization of the vacuum. This potential was found by Uehling (1935).

Uehling's corrections take into account that the electromagnetic field of a point charge does not act instantaneously at a distance, but rather it is an interaction that takes place via exchange particles, the photons. In quantum field theory, due to the uncertainty principle between energy and time, a single photon can briefly form a virtual particle-antiparticle pair, that influences the point charge. This effect is called vacuum

polarization, because it makes the vacuum appear like a polarizable medium. The dominant contribution comes from the electron. The corrections by Uehling are negligible in everyday practice, but it allows to calculate the spectral lines of hydrogen-like atoms with high precision.

The appropriate modified propagation function implies the change of the interaction between static charges which originally interact by manner of the Coulomb law. Let us first recall the definition of the potential by means of the Green function. This definition of the potential is not involved in the Roche majestic article on the historical development of the potential (Roche, 2003).

So, from the Green function of the massive scalar particle (Schwinger, 1970; 2018)

$$\Delta_+(x - x') = i \int d\omega_p e^{ip(x-x') - ip^0|x^0-x'^0|} \quad (1)$$

we get, as the consequence of it, the massless Green function (Schwinger, 1970; 2018; 2-3.91)

$$D_+(x - x') = \Delta_+(x - x', m = 0) = D_+(\mathbf{x} - \mathbf{x}', \tau) = \frac{i}{4\pi^2} \frac{1}{|\mathbf{x} - \mathbf{x}'|} \int_0^\infty dp^0 \sin(p^0|\mathbf{x} - \mathbf{x}'|) e^{-ip^0|\tau|} \quad (2)$$

with $\tau = x^0 - x'^0$. The potential corresponding to the Green function (2) is then defined by the following way (Schwinger, 1970; 2018; 2-3.92):

$$V(\mathbf{x} - \mathbf{x}') = \int_{-\infty}^\infty d\tau D_+(\mathbf{x} - \mathbf{x}', \tau) = \frac{1}{2\pi^2} \frac{1}{|\mathbf{x} - \mathbf{x}'|} \int_0^\infty dp^0 \frac{\sin(p^0|\mathbf{x} - \mathbf{x}'|)}{p^0} = \frac{1}{4\pi} \frac{1}{|\mathbf{x} - \mathbf{x}'|}. \quad (3)$$

Replacing $D_+(\mathbf{x} - \mathbf{x}', \tau)$ by its modified spin 1/2 version $\tilde{D}_+(\mathbf{x} - \mathbf{x}', \tau)$, we get the modified Coulomb potential (Schwinger, 1970; 2018):

$$\tilde{V}(x) = \frac{1}{4\pi|\mathbf{x}|} + \frac{\alpha}{3\pi} \int_{(2m)^2}^\infty \left(1 + \frac{2m^2}{M^2}\right) \left(1 - \frac{4m^2}{M^2}\right)^{1/2} \frac{e^{-M|\mathbf{x}|}}{4\pi|\mathbf{x}|} = \frac{1}{4\pi|\mathbf{x}|} \left[1 + \frac{\alpha}{\pi} \int_0^1 du \frac{u^2(1-u^2/3)}{1-u^2} \exp\left\{-\frac{2m|\mathbf{x}|}{(1-u^2)^{1/2}}\right\}\right], \quad (4)$$

where

$$u = \left(1 - \frac{4m^2}{M^2}\right)^{1/2}. \quad (5)$$

So, we have seen that the four variable Green function is reduced by time integration to the the tree-variable Green function and the exponential part of it is the Green function

corresponding to operator $-\Delta + M^2$. It can be obtained by the contour integration with the result

$$\int \frac{(d\mathbf{p})}{(2\pi)^3} \frac{e^{i\mathbf{p}\mathbf{x}}}{p^2 + M^2 - i\varepsilon} = \frac{e^{-M|\mathbf{x}|}}{4\pi|\mathbf{x}|}. \quad (6)$$

Now, let us consider two cases in eq. (4):

$$\tilde{V}(\mathbf{x}) \approx V(\mathbf{x}); \quad 2m|\mathbf{x}| \gg 1 \quad (7)$$

$$\tilde{V}(|\mathbf{x}|) \approx \frac{1}{4\pi|\mathbf{x}|} \left[1 + \frac{2\alpha}{3\pi} \left\{ \lg \frac{1}{m|\mathbf{x}|} - C - \frac{5}{6} \right\} \right]; \quad 2m|\mathbf{x}| \ll 1, \quad (8)$$

where

$$C = 0,57721\dots \quad (9)$$

is the Euler constant. The additional logarithmic behavior under these circumstances comes from the interval of M -integration such that $|\mathbf{x}|^{-1} \gg M \gg 2m$. The evaluation of $\tilde{V}(|\mathbf{x}|)$ for $2m|\mathbf{x}| \ll 1$ is obtained by partitioning the integral at some value of M that satisfies the considered inequality.

Let us remark that the used methods of this article can be applied in the area of the general theory of the potential (Gunter, 1953).

3 The Bohr energy

Simple perturbation theory can be applied to the change in interaction energy,

$$\delta V(\mathbf{x}) = -Ze^2 \delta \mathcal{D}(\mathbf{x}), \quad (10)$$

where $\delta \mathcal{D}(\mathbf{x})$ represents the difference between $\tilde{\mathcal{D}}(\mathbf{x})$ and

$$D(\mathbf{x}) = \frac{1}{4\pi} |\mathbf{x}|. \quad (11)$$

In a state with non-relativistic wave function $\psi(\mathbf{x})$, appropriate to the restriction $Z\alpha \ll 1$, we have

$$\delta E = \int d(\mathbf{x}) \delta V(\mathbf{x}) |\psi(\mathbf{x})|^2 \cong -4\pi Z\alpha |\psi(0)|^2 \int d(\mathbf{x}) \delta \mathcal{D}(\mathbf{x}), \quad (12)$$

which uses the fact that the perturbation is significant only over distances that are small compared with atomic dimensions. The integration that appears here is equivalent to evaluating the zero momentum limit of $\delta D_+(k)$, and

$$\int d(\mathbf{x}) \delta \mathcal{D}(\mathbf{x}) = \frac{\alpha}{3\pi} \int_{4m^2}^{\infty} \frac{dM^2}{M^4} \left(1 + \frac{2m^2}{M^2} \right) \left(1 - \frac{4m^2}{M^2} \right)^{1/2} =$$

$$\frac{\alpha}{\pi} \frac{1}{(2m)^2} \int_0^1 dv v^2 \left(1 - \frac{1}{3}v^2\right) = \frac{\alpha}{15\pi} \frac{1}{m^2}. \quad (13)$$

Only s-states need to be considered. For principal quantum number n

$$|\psi_{ns}(0)|^2 = \frac{1}{\pi} \left(\frac{Z\alpha}{n}m\right)^3 \quad (14)$$

and

$$\delta E_{ns} = -\frac{4}{15\pi} \frac{Z^4\alpha^5}{n^3}m, \quad (15)$$

or,

$$\frac{\delta E_{ns}}{\left(\frac{1}{2} \frac{Z^2\alpha^2}{n^2}m\right)} = -\frac{8}{15\pi} \frac{Z^2\alpha^3}{n}, \quad (16)$$

the latter giving a comparison with the Bohr energy values. More details will not be supplied now since, this effect is rather minor compared to another that displaces the s-states in the opposite sense. Let us remark that quantum numbers can take the following values: $n = 1, 2, 3, \dots$ - (principal quantum number), $l = 0, 1, 2, \dots, n - 1$ - (azimuthal quantum number) and $m = -l, \dots, l$, - (magnetic quantum number) (Merzbacher, 1988). In our case, Only s-states are considered being nonzero.

The existence of the vacuum polarization effect must be inferred from the quantitative comparison with experiment; in its absence a small but significant discrepancy with experiment would remain (Schwinger, 1983; 2018).

4 The quantum Sommerfeld equation

Let us find the classical equation of motion and the trajectory of an electron according to relativistic theory. In this case, the r and φ polar coordinates change with different frequencies. Or, the motion is quasi-periodic. We determine the angle through which the perihelion of the electron is shifted during "one" revolution. Then we obtain the Sommerfeld formula for the energy levels and find their splitting. We follow the monograph by Sokolov et al. (1962)

Using the relativistic Lagrangian function

$$\mathcal{L} = -mc^2\sqrt{1 - \beta^2} + \frac{Ze^2}{r}, \quad (17)$$

where

$$\beta^2 = \frac{v^2}{c^2} = \frac{1}{c^2}(\dot{r}^2 + r^2\dot{\varphi}^2), \quad (18)$$

we obtain the equation of motion in the form:

$$\frac{d}{dt} \frac{m\mathbf{v}}{\sqrt{1-\beta^2}} = -\frac{Ze^2}{r^3} \mathbf{r}. \quad (19)$$

For the generalized momenta

$$p_r = \frac{\partial \mathcal{L}}{\partial \dot{r}}, \quad p_\varphi = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}}, \quad (20)$$

we get from eq. (20):

$$p_r = \frac{p_\varphi}{r^2} r', \quad r' = \frac{dr}{d\varphi} \quad (21)$$

Now, it follows, in accordance with the law of conservation of energy, that

$$E = c\sqrt{mc^2 + p_r^2 + \frac{p_\varphi^2}{r^2}} - \frac{Ze^2}{r} = \text{const}, \quad (22)$$

which implies that

$$r' = \frac{r^2}{cp_\varphi} \sqrt{\left(E + \frac{Ze^2}{r}\right)^2 - m^2c^4 - \frac{c^2p_\varphi^2}{r^2}}. \quad (23)$$

The solution of the last equation can be realized by the substitution $r = 1/u$. Then, after some mathematical operations we get the differential equation for u and therefore for r . The solution of the differential equation for u is then the equation of the trajectory r :

$$r = \frac{q}{1 + \varepsilon \cos \gamma\varphi}, \quad (24)$$

where

$$\gamma = \sqrt{1 - \frac{Z^2e^4}{c^2p_\varphi^2}} \quad (25)$$

$$q = \frac{\gamma^2 c^2 p_\varphi^2}{Ze^2 E}. \quad (26)$$

$$\varepsilon = \sqrt{1 + \frac{\gamma^2 \left(1 - \frac{m^2e^4}{E^2}\right)}{(1-\gamma^2)}} \quad (27)$$

It is apparent from Eq. (24) that the motion is quasi-periodic. For the shift $\Delta\varphi$ of the perihelion, we have from (24)

$$\Delta\varphi = \frac{2\pi(1-\gamma)}{\gamma} \approx \frac{\pi Z^2 e^4}{c^2 p_\varphi^2}. \quad (28)$$

With the help of

$$\oint p_i dx_i = I_i, \quad (29)$$

where I_i are so called the Ehrenfest adiabatic invariants, with p_i, x_i being generalized coordinates, we get

$$I_\varphi = 2\pi p_\varphi. \quad (30)$$

$$I_r = 2\pi \left(\frac{B}{\sqrt{A}} - \sqrt{C} \right), \quad (31)$$

where

$$A = m^2 c^2 \left(1 - \frac{E^2}{m^2 c^4} \right). \quad (32)$$

$$B = \frac{Z e^2 E}{c^2}, \quad C = p_\varphi^2 - \frac{Z^2 e^4}{c^2} \quad (33)$$

Then, for the energy E , we obtain the expression

$$E = m^2 c^2 \left\{ 1 + \frac{Z^2 e^4}{c^2 \left[\frac{I_r}{2\pi} + \sqrt{\frac{I_\varphi^2}{4\pi^2} - \frac{Z^2 e^4}{c^2}} \right]^2} \right\}^{-1/2}. \quad (34)$$

We immediately see that the frequencies

$$\omega_\varphi = \frac{\partial E}{\partial I_\varphi}, \quad \omega_r = \frac{\partial E}{\partial I_r} \quad (35)$$

are different.

Using the Bohr-Sommerfeld quantization conditions

$$\oint p_\varphi d\varphi = n_\varphi \hbar, \quad \oint p_r dr = n_r \hbar, \quad (36)$$

where n_φ, n_r are azimuthal and radial quantum numbers, we can transform the energetical formula (34) in the new form:

$$E_{n_r, n_\varphi} = E - mc^2 = mc^2 \left\{ 1 + \frac{Z^2 \alpha^4}{\left[n_r + \sqrt{n_\varphi^2 - Z^2 \alpha^2} \right]^2} \right\}^{-1/2} - mc^2, \quad (37)$$

where $\alpha = e^2/c\hbar \approx 1/137$ is so called the fine structure constant. Expanding the formula (37) into a series in α^2 and restricting ourselves to quantities of the order of α^2 , we have:

$$E_{n, n_\varphi} = -\frac{R\hbar Z^2}{n^2} \left[1 + \frac{\alpha^2 Z^2}{n^2} \left(\frac{n}{n_\varphi} - \frac{3}{4} \right) \right] \quad (38)$$

Since n_φ varies from 1 to n_φ it follows from Eq. (38) that the energy levels, which is determined by the principal quantum number $n = n_r + n_\varphi$ are splitted into n closely sublevels (this close spacing is a consequence of the smallness of α^2).

The fine-structure splitting is

$$\Delta E_{n,n_\varphi} = E_{n,n_\varphi} - E_{n,n_\varphi-1} = \frac{R\hbar Z^4 \alpha^2}{n^2 n_\varphi (n_\varphi - 1)}, \quad (39)$$

where

$$R = \frac{me^4}{2\hbar^3} \quad (40)$$

being the Rydberg constant.

The splitting, or fine structure, of the levels, is a characteristic result of relativistic effects and it is essentially different from the predictions of the nonrelativistic theory.

Let us remark that Dirac formula for H-atom is related to the Sommerfeld formula Or, if we replace the numbers n_φ by $(j + 1/2)$ and n_r by $(j - 1/2)$ in the formula (37), we get:

$$E_{n_r, n_\varphi} = E - mc^2 = mc^2 \left\{ 1 + \frac{Z^2 \alpha^4}{\left[n + (j - 1/2) + \sqrt{(j + 1/2)^2 - Z^2 \alpha^2} \right]^2} \right\}^{-1/2} - mc^2, \quad (41)$$

which is original the Dirac formula for the electron with spin 1/2. The coincidence is miraculous and till this time the explanation of the physical origin of this coincidence was not given. More information of this problem is in the Petrov article (Petrov, 2020)

5 The Sommerfeld equation with the Uehling correction

If we denote the Uehling potential as $\delta V(r)$, being the small radiative contribution to the Coulomb potential, then we can by analogy with the above theory repeat the calculations. We get the following system of mathematical objects.

$$E = c\sqrt{mc^2 + p_r^2 + \frac{p_\varphi^2}{r^2}} - \frac{Ze^2}{r} + \delta V = const, \quad (42)$$

which implies that

$$r' = \frac{r^2}{cp_\varphi} \sqrt{\left(E + \frac{Ze^2}{r} - \delta V \right)^2 - m^2 c^4 - \frac{c^2 p_\varphi^2}{r^2}}. \quad (43)$$

By the substitution $r = 1/u$ we obtain the differential equation for u .

6 Discussion

The effect we are discussing here, is usually named as vacuum polarization. It increases the strength of the Coulomb interaction with diminishing distance. The increase is quite

small, however, at any realizable distance. Thus, with $2m|\mathbf{x}| \sim 10^{-3}$, which represents a distance of roughly 10^{-14} cm when the electron mass is used, it is approximately one percent. In view of the logarithmic dependence on distance, this order of magnitude cannot be changed significantly by any conceivable improvement in experimental process; a ten-percent increase in interaction strength requires dropping to a distance $\sim 10^{-37}$ cm. And long before such distances could be approached, the situation would change qualitatively through the growing importance of particles that are heavier than the electron.

Nevertheless, vacuum polarization effects are measurable at the present level of experimental technique. The most elementary situation is that of hydrogen atoms where the strengthened attraction between electron and nucleus depresses the energy values of zero orbital angular momentum states, these being the ones in which the electron spends appreciable time near the nucleus (Schwinger, 1983; 2018).

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