Symmetries in Goldbach's Conjecture

Timothy W. Jones

August 25, 2024

Abstract

We define a Goldbach table as a table consisting of two rows. The lower row counts from 0 to any n and and the top row counts down from $2n$ to n. All columns will have all numbers that add to $2n$. Using a sieve, all composites are crossed out and only columns with primes are left. We then define a novel prime decimal system: it gives for every n remainders when n is divided by all primes less than n. This suggests linear functions, the divisions used can give another perspective on all the column pairs. The inverses of these functions when put into tabular form give symmetries that suggest Goldbach's conjecture is correct.

Introduction

Hardy and Apostol spend some time on Goldbach's conjecture [1, 2]. The conjecture has it that every even number can be expressed as the sum of two primes.

It is easy to find examples. One can just add any two odd primes and the result will be even. So $3 + 5 = 8$, $5 + 7 = 12$, and so on. This will give lots of even sums fast. If one allows, which the conjecture does, non distinct primes then we can add $3 + 3 = 6$ and $5 + 5 = 10$ and start to sense that, indeed, you might just get all evens.

This article gives a simple scheme that exhaustively shows all sums that give a given even number using both prime and composite numbers. These we call Goldbach tables. We then explore these tables in hopes of proving the conjecture.

A Goldbach table

20	19	18	17	16	15	14	13	12	11	10
0	1	2	3	4	5	6	7	8	9	10

Table 1: A Goldbach table for $2 \cdot 10 = 20$.

Table 1 gives the scheme. One just fills a bottom row of a table from 0 to whatever positive integer one likes and then moving up to the top row counts down from that number. The columns generated will give all sums of two non-negative integers that sum to any given even number. In this table that number is 20.

All four possibilities for the status of bottom and top numbers are represented in this sample. The lower row indexes the columns. So, column 2 consists of 18 on the top, a composite and 2 on the bottom: column 3 consists of the two primes 17 and 3 and shows for this even 20 Goldbach's conjecture is correct. At index 7, we have two primes 13 and 7, another confirmation of Goldbach's conjecture. Is there a way to find all primes and all such prime columns?

A Sieve

We can quickly filter out all composite numbers using a sieve. Scratch off all but the first prime multiples of the first and second rows. You can do this without knowing what numbers are primes: just scratch out non-one multiples of 2, 3, 4, 5, and so on and you will only have primes left. In Table 2 we do this. This procedure is called a sieve. The Greek mathematician Eratosthenes used it to find primes. We are doing the same thing.

~~20~~	19	~~18~~	17	~~16~~	~~15~~	~~14~~	13	~~12~~	11	10
~~0~~	~~1~~	2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~

Table 2: If primes are aligned in the top and bottom rows, Goldbach's conjecture is confirmed.

If columns survive the procedure (meaning that there is no slash in either cell comprising the column), Goldbach's conjecture is confirmed for the particular even number. If we have a way of guaranteeing at least one prime column survives in any such $2n$ table, we will have proven Goldbach's conjecture correct.

Dissecting $n = 20$

Using the division algorithm, we can divide any given number by all the primes less than it and arrive at a sequence of remainders and quotients. So, staying with our 20 and its sieve, we have

$$20 = 0_2^{10} 2_3^6 0_5^4 6_7^2 9_{11}^1 7_{13}^1 3_{17}^1 1_{19}^1, \tag{1}$$

where the *digit* in this *prime base* is of the form

$$R_p^q \text{ where } 2n = p \cdot q + R.$$

For example the fourth digit in the prime base representation of 20 is

$$6_7^2 \text{ meaning } 20 = 7 \cdot 2 + 6.$$

Calling the q superscript an exponent, we notice exponents or powers of 1 emerge in greater valued primes. We see, for example, 7_{13}^1, 3_{17}^1, and 1_{19}^1 in (1) and these translate to winning pairs of primes: $\{(7, 13), (3, 17)\}$ that all sum to 20.

We can use lower valued primes to find all Composites. Consider the primes less than $\sqrt{20}$: $P(20) = \{2, 3\}$. Any number between 2 and 20 that doesn't have a factor from $P(20)$ will be a prime [2] and the converse also holds if a number is divisible by 2 or 3 it is composite. We can form sets of numbers that correspond to our slashing out cells in Goldbach tables. For example, the set

$$GC(2) = \{(f(x), g(x)) | f(x) = 2(10 - x) \text{ and } g(x) = 2x, x \in \mathbb{Z}_0^{10/2}\} \tag{2}$$

consists of $\{(20, 0)_0, (18, 2)_1, (16, 4)_2, (14, 6)_3, (12, 8)_4, (10, 10)_5\}$ and

$$GC(3) = \{(f(x), g(x)) | f(x) = 3(6 - x) \text{ and } g(x) = 3x + 2, x \in \mathbb{Z}_0^6\}$$

of $\{(18, 2)_0, (15, 5)_1, (12, 8)_2, (9, 11)_3, (6, 14)_4, (3, 17)_5, (0, 20)_6\}$. We can now complete the Goldbach table, Table 1 by a different method referencing these sets, Table 3. We do get a spurious confirmation with $(19, 1)$, but, using Bertrand's postulate, we know there will be a prime between $k/2$ and k – the second half of the bottom row. Indeed, for $k = 10$, there is 7 and this aligns with 13 for a confirmation for this case.

	20	19	18	17	16	15	14	13	12	11	10
	0	1	2	3	4	5	6	7	8	9	10
2	x		x		x		x		x		x
3	x		x	x		x	x		x	x	

Table 3: A Goldbach table solved with sets.

Inverse Linear Functions

The functions in GC(2) and GC(3) are linear functions. We can differentiate them with subscripts: $f_2(x) = -2x + 20$ and $g_2(x) = 2x$ are the functions of $GC(2, 20)$and $f_3(x) = -3x + 18$ and $g_3(x) = 3x + 2$ are the functions of $GC(3, 20)$, where we are using the customary $f(x) = mx + b$ slope intercept format and have added the $2n$ parameter: $GC(p, 2n)$ means the prime p for the Goldbach table $2n$.

So far, we are using these functions to find all numbers that have 2 and 3 as factors. Putting in a string of domain values, we get all the multiples of these primes on the bottom row. The inverse of these functions using the domain of all numbers on the bottom row, will have integers returned for multiples of the prime and non-integers for non-multiples. We have a way to find primes on the right half of the bottom row. They are the numbers that have non-integer values for the inverses of the prime functions associated with a given $2n$: when all these functions are non-integer, we have a prime. We'll demonstrate this with our example.

It is easy to crunch the inverse form of these functions. The task is a drill in intermediate algebra books: swap y with x, solve for y, and replace y with the inverse form of the function, a superscript -1. We find

$$f_2(x) = -2x + 20 \text{ gives } f_2^{-1} = -\frac{1}{2}x + 10 \tag{3}$$

and

$$g_2(x) = 2x \text{ gives } g_2^{-1}(x) = \frac{1}{2}x \tag{4}$$

and we can test we got these right with composition $f^{-1}(f(x)) = f(f^{-1}(x)) = x$, we're supposed to get the identity function back and in (3) and (4) we do. We also have

$$f_3(x) = -3x + 18 \text{ gives } f_3^{-1}(x) = -\frac{1}{3}x + 6 \tag{5}$$

and

$$g_3(x) = 3x + 2 \text{ gives } g_3^{-1}(x) = \frac{1}{3}x - \frac{2}{3}. \tag{6}$$

We can show the values for these four inverses for all numbers: Table **??**.

	A	B	C	D	E
1	x	$f^{-1}(x,2)$	$g^{-1}(x,2)$	$f^{-1}(x,3)$	$g^{-1}(x,3)$
2	20	0	10	-0.6666666667	6
3	19	0.5	9.5	-0.3333333333	5.666666667
4	18	1	9	0	5.333333333
5	17	1.5	8.5	0.3333333333	5
6	16	2	8	0.6666666667	4.666666667
7	15	2.5	7.5	1	4.333333333
8	14	3	7	1.333333333	4
9	13	3.5	6.5	1.666666667	3.666666667
10	12	4	6	2	3.333333333
11	11	4.5	5.5	2.333333333	3
12	10	5	5	2.666666667	2.666666667
13	9	5.5	4.5	3	2.333333333
14	8	6	4	3.333333333	2
15	7	6.5	3.5	3.666666667	1.666666667
16	6	7	3	4	1.333333333
17	5	7.5	2.5	4.333333333	1
18	4	8	2	4.666666667	0.6666666667
19	3	8.5	1.5	5	0.3333333333
20	2	9	1	5.333333333	0
21	1	9.5	0.5	5.666666667	-0.3333333333
22	0	10	0	6	-0.6666666667

Figure 1: Non-integer values of the inverse functions indicate primes. The symmetry correlates with aligned primes.

Figure 1 shows that these functions have a symmetry to them. If a row has all non-integer numbers then we can infer that the number is a prime; it is not divisible by 2 or 3. The symmetry seems to indicate that

$f_2^{-1}(20-x) = g_2^{-1}(x)$, for example. That is cells B9 and C9, corresponding to $x = 13$ pairs with cell B15 and C15 corresponding to $x = 7$; that is $f_2^{-1}(20-13) = g_2^{-1}(7)$ and $g_2^{-1}(20-13) = f_2^{-1}(7)$. The importance of this observation is the prime status is maintained in a column. Notice the primes that don't have all decimal numbers such as 17 and 11 are paired with 3 and 9; the first is a prime to the first power and the second is a prime squared. One is a false negative and the other a false positive. As mentioned, using Bertrand's postulate we can predict the existence of a prime between 6 and 10; that's 7. By symmetry this has a prime above it: that's 13. In the next section, we will prove the symmetric nature of these inverse functions.

An attempt

Lemma 1. *Let $f(x) = -m_f x + b_f$ and $g(x) = m_f x + b_g$ be two linear functions with slopes $\pm m_f$ where m_f is a positive integer and $b_f + g_f = 2n$, then*

$$f^{-1}(2n-x) = g^{-1}(x) \tag{7}$$

and

$$g^{-1}(2n-x) = f^{-1}(x). \tag{8}$$

Proof. The inverses for $f(x)$ and $g(x)$ are

$$f^{-1}(x) = -\frac{1}{m_f}x + \frac{b_f}{m_f} \text{ and } g^{-1}(x) = \frac{1}{m_f}x - \frac{b_g}{m_f}.$$

Consider

$$f^{-1}(2n-x) - g_p^{-1}(x) = -\frac{1}{m_f}(2n-x) + \frac{b_f}{m_f} - (\frac{1}{m_f}x - \frac{b_g}{m_f})$$

when multiplied by m_f gives

$$m_f(f^{-1}(2n-x) - g_p^{-1}(x)) = -(2n-x) + b_f - (x - b_g) = b_f + b_g - 2n = 0.$$

As m_f is not zero, this implies (7). By symmetry (8) is also implied. □

We now can prove Goldbach's conjecture is true.

Theorem 1. *Goldbach's conjecture is true.*

Proof. For any even number $2n$, a Goldbach table, $GT(2n)$ can be constructed. There will exist primes p_i less than or equal to $\sqrt{n}$: $P(2n) = \{p_i\}$. These will reside on the bottom row, left half of $GT(2n)$. By Bertrand's postulate there will exist one or more primes, b_i between $\lfloor n/2 \rfloor$ and n. These primes will be different than those between 1 and $\lfloor n/2 \rfloor$; we know this as $\sqrt{n} < n/2$ for $n > 4$.

Using Lemma 1 we can construct inverse linear functions for all p_i. The values of these functions at b_i will all be non-integer, as $b_i \notin P(2n)$ and b_i is a prime (not divisible by any p_i). Using symmetry properties there will exist a prime aligned with each b_i, t_i thus proving that Goldbach's conjecture is correct. □

References

[1] Apostol, T. M. (1976). *Introduction to Analytic Number Theory.* New York: Springer.

[2] Hardy, G. H., Wright, E. M., Heath-Brown, R. , Silverman, J. , Wiles, A. (2008). *An Introduction to the Theory of Numbers*, 6th ed. London: Oxford Univ. Press.