On the equation: $s = \frac{1}{2}\,\Gamma\left(\frac{1}{2}, s^2\right),\ s > 0$

Edgar Valdebenito

16 August 2024

ABSTRACT: We solve the equation: $s = \frac{1}{2}\,\Gamma\left(\frac{1}{2}, s^2\right),\ s > 0$, where $\Gamma(x, y)$ is the incomplete gamma function.

I. Introduction

The incomplete gamma function is defined by

$$\Gamma(x, y) = \int_y^\infty t^{x-1}\, e^{-t}\, dt \tag{1}$$

for details see [1],[2].

we have

$$\frac{1}{2}\,\Gamma\left(\frac{1}{2}, s^2\right) = \int_s^\infty e^{-t^2}\, dt \tag{2}$$

II. The equation: $s = \frac{1}{2}\,\Gamma\left(\frac{1}{2}, s^2\right) = \int_s^\infty e^{-t^2}\, dt,\ s > 0$

$$s = \frac{1}{2}\,\Gamma\left(\frac{1}{2}, s^2\right) \Longrightarrow s = 0.458183538105456969\ldots \tag{3}$$

III. Sequences for s

$$s_1 = 1/2\ ,\ s_{n+1} = \frac{1}{2}\,\Gamma\left(\frac{1}{2}, s_n^2\right) \Longrightarrow s_n \longrightarrow s \tag{4}$$

$$s_1 = 1/2\ ,\ s_{n+1} = \frac{1}{2}\left(s_n + \frac{1}{2}\,\Gamma\left(\frac{1}{2}, s_n^2\right)\right) \Longrightarrow s_n \longrightarrow s \tag{5}$$

$$s_1 = 1/2\ ,\ s_{n+1} = \frac{2\, s_n\, e^{-s_n^2} + \Gamma\left(1/2, s_n^2\right)}{2\left(1 + e^{-s_n^2}\right)} \Longrightarrow s_n \longrightarrow s \tag{6}$$

IV. Integrals

$$s = \int_s^\infty e^{-x^2}\, dx \tag{7}$$

$$s = \int_0^{1/s} \frac{e^{-1/x^2}}{x^2}\, dx \tag{8}$$

$$s = \frac{1}{2}\int_{s^2}^\infty \frac{e^{-x}}{\sqrt{x}}\, dx \tag{9}$$

$$\sqrt{s} = \frac{1}{2}\int_s^\infty \frac{e^{-s x}}{\sqrt{x}}\, dx \tag{10}$$

$$s\, e^{s^2} = \int_0^1 \frac{-\ln(x)}{s + \sqrt{s^2 - \ln(x)}}\, dx \tag{11}$$

$$s\, e^{s^2} = \int_1^\infty \frac{\ln(x)}{\left(s + \sqrt{s^2 + \ln(x)}\right) x^2}\, dx \tag{12}$$

$$s\, e^{s^2} = \int_0^\infty \frac{x\, e^{-x}}{s + \sqrt{s^2 + x}}\, dx \tag{13}$$

$$s\left(1 + e^{-s^2}\right) = \int_0^{e^{-s^2}} \sqrt{\ln\left(\frac{1}{x}\right)}\, dx \tag{14}$$

$$s\left(1 + e^{-s^2}\right) = \int_{s^2}^\infty \sqrt{x}\; e^{-x}\, dx \tag{15}$$

$$s\left(1 + e^{-s^2}\right) = 2\int_s^\infty x^2\, e^{-x^2}\, dx \tag{16}$$

$$s = \int_0^{e^{-s}} x^{-1-\ln(x)}\, dx \tag{17}$$

$$s = \frac{1}{2}\int_0^{e^{-s^2}} \frac{1}{\sqrt{-\ln(x)}}\, dx \tag{18}$$

$$2\, s\, \sqrt{\pi}\; e^{s^2} = \int_0^\infty \frac{e^{-s^2 x}}{\sqrt{x}\;(1 + x)}\, dx \tag{19}$$

$$2\, \sqrt{s\,\pi}\; e^{s^2} = \int_0^\infty \frac{e^{-s x}}{\sqrt{x}\;(s + x)}\, dx \tag{20}$$

$$s\left(2 + e^{-s^2}\right) = \frac{1}{2}\int_0^{e^{-s^2}/s} W\left(\frac{2}{x^2}\right) dx \tag{21}$$

Remark: W (x) is the Lambert W function.

V. Formula for Pi

The number Pi is defined by

$$\pi = 4\sum_{n=0}^{\infty} \frac{(-1)^n}{2\, n + 1} = 3.141592\ldots \tag{22}$$

we have

$$\pi = 16\, s^2 - 4\sum_{n=1}^{\infty} (-1)^{n-1}\, s^{2\,n+2}\, A_n \tag{23}$$

where

$$A_n = \frac{2}{n!\,(2n+1)} + \sum_{k=0}^{n} \frac{1}{(n-k)!\,k!\,(2n-2k+1)(2k+1)} \quad , \; n \geq 1 \tag{24}$$

$$A_n = \frac{2}{n!\,(2n+1)} + \frac{1}{2(2n+1)(n+1)!}\left((2n+1)\,{}_2F_1\left(\frac{1}{2}, -n, \frac{3}{2}, -1\right) + {}_2F_1\left(-\frac{1}{2} - n, -n, \frac{1}{2} - n, -1\right)\right), \; n \geq 1 \tag{25}$$

Remark: ${}_2F_1$ is the Gauss hypergeometric function.

$$\sqrt{\pi} = 2s + e^{-s^2} \sum_{n=0}^{\infty} \frac{(2s)^{2n+1}\,n!}{(2n+1)!} \tag{26}$$

VI. Elementary Estimates

We have

$$\begin{aligned} s > 0 \wedge s = \int_s^{\infty} e^{-x^2}\,dx &\Longrightarrow \sum_{n=1}^{\infty} e^{-(s+n)^2} < s < \sum_{n=0}^{\infty} e^{-(s+n)^2} \\ &\Longrightarrow 0.228 \ldots < s < 0.684 \ldots \end{aligned} \tag{27}$$

$$\begin{aligned} s > 0 \wedge s = \int_s^{\infty} e^{-x^2}\,dx &\Longrightarrow \sum_{n=2}^{\infty} s\,e^{-(s\,n)^2} < s < \sum_{n=1}^{\infty} s\,e^{-(s\,n)^2} \\ &\Longrightarrow \sum_{n=2}^{\infty} e^{-(s\,n)^2} < 1 < \sum_{n=1}^{\infty} e^{-(s\,n)^2} \\ &\Longrightarrow 0.374 \ldots < s < 0.590 \ldots \end{aligned} \tag{28}$$

VII. Integral for s

We have

$$s = \frac{1}{4\pi} \int_0^{2\pi} e^{ix} f\left(\frac{1}{2} + \frac{e^{ix}}{4}\right) dx \quad , \; i = \sqrt{-1} \tag{29}$$

where

$$f(z) = \frac{z\left(1 + e^{-z^2}\right)}{2z - \Gamma\left(1/2, z^2\right)} \tag{30}$$

VIII. References

1. Gradshteyn,I.S. and Ryzhik,I.M.: Table of Integrals,Series, and Products. Seventh Edition. Edited by A. Jeffrey and D. Zwillinger. Academic Press,2007.

2. Olver, F.W.J.,Lozier, D.W.,Boisvert, R.F., and Clark, C.W.: Nist Handbook of Mathematical functions.Cambridge University Press, 2010.

3. Borwein, J., Bailey, D., and Girgensohn, R.: Experimentation in Mathematics. AK Peters, Natick, 2004.