Chern-Simons Framework for Particles and Quantum Gravity

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Abstract

A novel particle/quantum gravity correspondence framework is proposed and reviewed. It is a combination of bottom-up and top-down approaches meeting each other at the Chern-Simons action of supersymmetric fields. The former starts from our SM composite particle model with spontaneously broken Chern-Simons binding. The latter approach of other authors incorporates massive spinning fields into the Euclidean path integral of three dimensional quantum gravity via a Chern-Simons formulation. The G_N and the mass renormalization to leading order in perturbation theory are reviewed. On quantum level, all fundamental matter, as defined in this article, and gravity are conjectured to be different limits of a single topological field theory.

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Contents

1 Introduction

We combine and review some recent work of several authors on bottom-up composite particle physics model and top-down quantum gravity proposal, both based on low dimensional spacetime. Particle physics is reviewed in part I. Chern-Simons theory leading to 3d quantum gravity is covered in part II.

Part I Symmetry and Wave Functions

2 Composite Particles

The setup for composite particle scenario is as follows:

- Unbroken supersymmetry is adopted for fundamental particles.[1] Dividing standard model (SM) fermions into three preons a binding mechanism is constructed using spontaneously broken 3d Chern-Simons theory.[2]
- Preons, or chernons, are provided with two unbroken internal gauge symmetries, $U(1)$ for charge and $SU(3)$ for color.
- Gravitation is introduced in the form 3d Chern-Simons theory with single-particle states of massive spin s fields living on dS_3, with de Sitter radius ℓ_{dS}, as representations of $\mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R$. It turns out that the partition function for for Chern-Simons connection $A_{L/R}$ can be calculated. Furthermore, the Chern-Simons path integral can be evaluated to any order in G_N perturbation theory. A detailed review is presented.
- There must be freedom and predictions for dark matter and dark energy.
- The scenario should match the cosmological standard model with preheating observational data and baryon asymmetry of matter.

[1] The Minimal Supersymmetric SM does not fulfill this requirement (it leads rather to quark and lepton kind of "double counting").

[2] Preons, or here chernons, are free particles above the energy scale Λ_{cr}, numerically about $\sim 10^{10} - 10^{16}$ GeV. It is close to reheating scale T_R and the grand unified theory (GUT) scale. At Λ_{cr} chernons make a phase transition by an attractive Chern-Simons model interaction into composite states of standard model quarks and leptons, including gauge interactions. Chernons have undergone "second quarkization".

The above properties make our preon scenario a worthy candidate beyond SM, call it Unbroken Supersymmetric SM (USSM), which includes all four interactions. On the other hand, the generation problem as a composite system excitations and many details remain to be calculated or cannot be done since data are not available. Finally, the scenario should, if possible, indicate the direction to a UV finite theory.

3 Extending the Wess-Zumino action

The divisive point of the chernon model for visible and dark matter is the following: supersymmetry should be unbroken and implemented so that all particles needed to describe nature are written together with their superpartners like in the Lagrangians ((1) - (3)) of this model. Our method was introduced in [1, 2]. The result turned out to have close resemblance to the Wess-Zumino (WZ) model [3], which contains three neutral fields: a spinor m, the real fields s and p with $J^P = \frac{1}{2}^+, 0^+$, and 0^-, respectively. The kinetic WZ Lagrangian is

$$\mathcal{L}_{\mathrm{WZ}} = -\frac{1}{2}\bar{m}\gamma^\mu\partial_\mu m - \frac{1}{2}(\partial s)^2 - \frac{1}{2}(\partial p)^2 \tag{1}$$

where m and s form the chiral supermultiplet. We assume that the pseudoscalar p is the axion [4], and denote it below as a. It has a fermionic superparther, the axino n, a candidate for dark matter but not discussed further here.

To include charged matter we define the following charged chiral field Lagrangian for fermion m^-, complex scalar s^- and the electromagnetic field tensor $F_{\mu\nu}$

$$\mathcal{L}_{\mathrm{WZ}_{\mathrm{Charge}}} = -\frac{1}{2}\bar{m}^-\gamma^\mu\partial_\mu m^- - \frac{1}{2}(\partial s^-)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{2}$$

We set color to the neutral fermion $m \to m_i^0$ ($i = R, G, B$) in (1). The color sector Lagrangian is then

$$\mathcal{L}_{\mathrm{WZ}_{\mathrm{Color}}} = -\frac{1}{2}\sum_{i=R,G,B}\left[\bar{m}_i^0\gamma^\mu\partial_\mu m_i^0 - \frac{1}{2}(\partial g_i)^2\right] \tag{3}$$

We now have the supermultiplets shown in table 1.

Multiplet	Particle, Sparticle
chiral multiplets spins 0, 1/2	s^-, m^-; a, n
vector multiplets spins 1/2, 1	m^0, γ; m_i, g_i

Table 1: The particle s^- is a neutral scalar particle. The particles m^-, m^0 are charged and neutral, respectively, Dirac spinors. The a is axion and n axino. m^0 is color singlet particle and γ is the photon. m_i and g_i (i = R, G, B) are zero charge color triplet fermions and bosons, respectively.

Note that in table 1 there is a zero charge quark triplet m_i but no gluon octet. Instead, supersymmetry demands the gluons to appear only in triplets at this stage of cosmological evolution. The dark sector we get from (3) and the m_i.

The matter-chernon correspondence for the first two flavors (r = 1, 2; i.e. the first generation) is indicated in table 2 for left handed particles.

SM Matter 1st gen.	Chernon state
ν_e	$m^0_R m^0_G m^0_B$
u_R	$m^+ m^+ m^0_R$
u_G	$m^+ m^+ m^0_G$
u_B	$m^+ m^+ m^0_B$
e^-	$m^- m^- m^-$
d_R	$m^- m^0_G m^0_B$
d_G	$m^- m^0_B m^0_R$
d_B	$m^- m^0_R m^0_G$
W-Z Dark Matter	Particle
boson (or BC)	s, axion(s)
e'	axino n
meson, baryon o	$n\bar{n}, 3n$
nuclei (atoms with γ')	multi n
celestial bodies	any dark stuff
black holes	anything (neutral)

Table 2: Visible and Dark Matter with corresponding particles and chernon composites. m^0_i (i = R, G, B) is color triplet, $m^\pm$ are color singlets of charge $\pm 1/3$. e' and γ' refer to dark electron and dark photon, respectively. BC stands for Bose condensate. Chernons obey anyon statistics.

After quarks have been formed by the process described in section 5 the SM octet of gluons will emerge because it is known that fractional charge states have not been observed in nature. To make observable color neutral, integer charge states (baryons and mesons) possible we proceed as follows. The local $SU(3)_{color}$ octet structure is formed by quark-antiquark composite pairs as follows (with only color charge indicated):

$$\text{Gluons}: \mathrm{R\bar{G}}, \mathrm{R\bar{B}}, \mathrm{G\bar{R}}, \mathrm{G\bar{B}}, \mathrm{B\bar{R}}, \mathrm{B\bar{G}}, \frac{1}{\sqrt{2}}(\mathrm{R\bar{R}} - \mathrm{G\bar{G}}), \frac{1}{\sqrt{6}}(\mathrm{R\bar{R}} + \mathrm{G\bar{G}} - 2\mathrm{B\bar{B}}) \tag{4}$$

With the gluon triplet the first hunch is that they form, with octet gluons now available, the $3 \otimes 3 \otimes 3 = 10 \oplus 8 \oplus 8 \oplus 1$ bosonic states with spins 1 and 3. These three gluon coupling states would need a separate investigation.

Finally, we introduce the weak interaction. After the SM quarks, gluons and leptons have been formed at scale Λ_{cr} there is no more observable supersymmetry in nature [5]. To avoid a more complicated vector supermultiplet in table 1, we may append the standard model electroweak interaction in our model as an empirical fact. The standard model has now been heuristically derived.

4 Baryon asymmetry of the Universe

A number of 1+2 dimensional models have properties close to 1+3 dimensional world as can be found in [6–8], see also [9]. Our choice here is 1+2 dimensional Chern-Simons (CS) action is [10, 11]

$$S = \frac{k}{4\pi} \int_M \mathrm{tr}(\mathrm{A} \wedge \mathrm{dA}) + \frac{2}{3}\mathrm{A} \wedge \mathrm{A} \wedge \mathrm{A}) \tag{5}$$

where k is the level of the theory and A the connection. (The compatibility of different dimensions is discussed in section 6.)

The action for a Chern-Simons-QED$_3$ model [12, 13] including two polarization $\pm$ fermionic fields (ψ_+, ψ_-), a gauge field A_μ and a complex scalar field φ with spontaneous breaking of local U(1) symmetry is

$$S_{\mathrm{CS-QED_3}} = \int d^3x\{-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + i\overline{\psi}_+\gamma^\mu D_\mu\psi_+ + i\overline{\psi}_-\gamma^\mu D_\mu\psi_-$$
$$+\frac{1}{2}\theta\epsilon^{\mu v\alpha}A_\mu\partial_v A_\alpha - m_{ch}(\overline{\psi}_+\psi_+ - \overline{\psi}_-\psi_-)$$
$$-y(\overline{\psi}_+\psi_+ - \overline{\psi}_-\psi_-)\varphi^*\varphi + D^\mu\varphi^* D_\mu\varphi - V(\varphi^*\varphi)\}, \quad (6)$$

where the covariant derivatives are $D_\mu\psi_\pm = (\partial_\mu + ie_3A_\mu)\psi_\pm$ and $D_\mu\varphi = (\partial_\mu + ie_3A_\mu)\varphi$. θ is the important topological parameter and e_3 is the coupling constant of the $U(1)$ local gauge symmetry, here with dimension of $(\text{mass})^{1/2}$. $V(\varphi^*\varphi)$ represents the self-interaction potential,

$$V(\varphi^*\varphi) = \mu^2\varphi^*\varphi + \frac{\zeta}{2}(\varphi^*\varphi)^2 + \frac{\lambda}{3}(\varphi^*\varphi)^3 \quad (7)$$

which is the most general sixth power renormalizable potential in 1+2 dimensions [14]. The parameters μ, ζ, λ and y have mass dimensions 1, 1, 0 and 0, respectively. For potential parameters $\lambda > 0, \zeta < 0$ and $\mu^2 \leq 3\zeta^2/(16\lambda)$ the vacua are stable.

In 1+2 dimensions, a fermionic field has its spin polarization fixed up by the sign of mass [15]. The model includes two positive-energy spinors (two spinor families). Both of them obey Dirac equation, each one with one polarization state according to the sign of the mass parameter.

The vacuum expectation value v of the scalar field φ is given by:

$$\langle\varphi^*\varphi\rangle = v^2 = -\zeta/\left(2\lambda\right) + \left[\left(\zeta/\left(2\lambda\right)\right)^2 - \mu^2/\lambda\right]^{1/2} \quad (8)$$

The condition for its minimum is $\mu^2 + \frac{\zeta}{2}v^2 + \lambda v^4 = 0$. After the spontaneous symmetry breaking, the scalar complex field can be parametrized by $\varphi = v + H + i\theta$, where H represents the Higgs scalar field and θ the would-be Goldstone boson. For manifest renormalizability one adopts the 't Hooft gauge by adding the gauge fixing term $S_{R_\xi}^{gt} = \int d^3x[-\frac{1}{2\xi}(\partial^\mu A_\mu - \sqrt{2}\xi M_A\theta)^2]$ to the broken action. Keeping only the bilinear and the Yukawa interaction terms one has the following action

$$
\begin{aligned}
S^{\mathrm{SSB}}_{\mathrm{CS-QED}} = \int d^3x \Big\{ & -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}M_A^2 A^\mu A_\mu \\
& - \frac{1}{2\xi}(\partial^\mu A_\mu)^2 + \overline{\psi}_+(i\gamma^\mu\partial_\mu - m_{eff})\psi_+ \\
& + \overline{\psi}_-(i\gamma^\mu\partial_\mu + m_{eff})\psi_- + \frac{1}{2}\theta\epsilon^{\mu v\alpha}A_\mu\partial_v A_\alpha \\
& + \partial^\mu H\partial_\mu H - M_H^2 H^2 + \partial^\mu\theta\partial_\mu\theta - M_\theta^2\theta^2 \\
& - 2yv(\overline{\psi}_+\psi_+ - \overline{\psi}_-\psi_-)H - e_3\left(\overline{\psi}_+\gamma^\mu A_\mu\psi_+ + \overline{\psi}_-\gamma^\mu A_\mu\psi_-\right) \Big\}
\end{aligned}
\tag{9}
$$

where the mass parameters

$$
M_A^2 = 2v^2e_3^2,\ \ m_{eff} = m_{ch} + yv^2,\ \ M_H^2 = 2v^2(\zeta + 2\lambda v^2),\ \ M_\theta^2 = \xi M_A^2 \tag{10}
$$

depend on the SSB mechanism. The Proca mass, M_A^2originates from the Higgs mechanism. The Higgs mass, M_H^2, is associated with the real scalar field. The Higgs mechanism also contributes to the chernon mass m_{ch}, resulting in an effective mass m_{eff}. There are two photon mass-terms in (9), the Proca and the topological one.

5 Chernon-Chernon interaction

The chernon-chernon scattering amplitude in the non-relativistic approximation is obtained by calculating the t-channel exchange diagrams of the Higgs scalar and the massive gauge field. The propagators of the two exchanged particles and the vertex factors are calculated from the action (9) [12].

The gauge invariant effective potential for the scattering considered is obtained in [16, 17]

$$
V_{\mathrm{CS}}(r) = \frac{e^2}{2\pi}\left[1 - \frac{\theta}{m_{ch}}\right]K_0(\theta r) + \frac{1}{m_{ch}r^2}\left\{l - \frac{e^2}{2\pi\theta}[1 - \theta r K_1(\theta r)]\right\}^2 \tag{11}
$$

where $K_0(x)$ and $K_1(x)$ are the modified Bessel functions and l is the angular momentum ($l = 0$ in this note). In (11) the first term [ ] corresponds to the electromagnetic potential, the second one $\{\ \}^2$ contains the centrifugal barrier $\left(l/mr^2\right)$, the Aharonov-Bohm term and the two photon exchange term.

One sees from (11) the first term may be positive or negative while the second term is always positive. The function $K_0(x)$ diverges as $x \to 0$ and approaches zero for $x \to \infty$ and $K_1(x)$ has qualitatively similar behavior. For our scenario we need negative potential between equal charge chernons. Being embarrassed of having no data points for several parameters in (11) we can give one relation between these parameter values for a negative potential. We must have the condition[3]

$$\theta \gg m_{ch} \tag{12}$$

The potential (11) also depends on v^2, the vacuum expectation value, and on y, the parameter that measures the coupling between fermions and Higgs scalar. Being a free parameter, v^2 indicates the energy scale of the spontaneous breakdown of the $U(1)$ local symmetry.

6 Inflation and Supergravity

We discuss briefly, and in simple terms, the question of different dimensions of CS theory and gravity. We assume that the universe at $t \sim 0$ included a subspace of one dimension less than the manifold of general relativity M_{GR}.[4] A promising example of such a theory is Chern-Simons gauge theory defined in a smooth, compact three-manifold $M_{CS} \subset M_{GR}$, having a gauge group G, which is semi-simple and compact, and an integer parameter k. The Chern-Simons field equations (5) require that A be flat [11]. The curvature

[3] For applications to condensed matter physics, one must require $\theta \ll m_e$, and the scattering potential given by (11) then comes out positive [12].

[4] A line is one dimensional when looked from a distance but by getting very close to it one sees, or rather knows, it consists of zero dimensional points, that is numbers.

tensor may be decomposed, in any spacetime dimension, into a curvature scalar R, a Ricci tensor $R_{\mu\nu}$, and a conformally invariant Weyl tensor $C_{\mu\nu\rho}{}^{\sigma}$. In 1+2 dimensions the Weyl tensor vanishes identically, and the Riemann curvature tensor $R_{\mu\nu\rho\sigma}$ is determined algebraically by the curvature scalar and the Ricci tensor. Therefore any solution of the vacuum Einstein field equations is flat and any solution of the field equations with a cosmological constant $R_{\mu\nu} = 2\Lambda g_{\mu\nu}$ has constant curvature. Physically, a 1+2 dimensional spacetime has no local degrees of freedom. There are no gravitational waves in the classical theory, and no gravitons in the quantum theory

CS theory, defined earlier by the action (5), is a topological, quantizable gauge field theory [11]. The appropriate observables lead to vevs which correspond to topological invariants. The observables have to be gauge invariant. Secondly, they must be independent of the metric. Wilson loops verify these two properties [11], and they are therefore the key to observables to be considered in Chern-Simons theory. Independence of metric gives CS theories the desireable property of background independence. The CS interaction (5) is effective only at energy scales near and above Λ_{cr}. This we interpret as chernons living (mod 3) on surfaces of spheres with diameter of the order of $1/\Lambda_{cr}$. These composite states are quarks and leptons of the standard model in 1+3 dimensions.

In summary, the potential (11) dominates over general relativity, and Coulomb repulsion, at distances below $1/\Lambda_{cr}$ in the 1+2 dimensional manifold M_{CS} while at larger distances gravity is stronger.

At the beginning of inflation, $t = t_i \sim 10^{-36}$ s, the universe is modeled by 1+3 dimensional classical gravity, and Chern-Simons theory as long as $T \geq \Lambda_{cr}$. The Einstein-Hilbert action is

$$S = \int d^4x\sqrt{-g}\Big(\frac{1}{2}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\Big) \tag{13}$$

The E-H action dominates rapidly leading inflation to end at $t_R \approx 10^{-32}$ s. Then the inflaton, which is actually coherently oscillating homogeneous field, a Bose condensate, reaches the minimum of its potential. There it oscillates and decays to SM particles produced from chernons in the earlier phase of inflation. This causes the reheating phase, or the Bang, giving visible matter particles more kinetic energy than dark matter particles have.

The CMB measurements of inflation can be well described by a few simple slow-roll single scalar potentials in (13). One of the best fits to Planck data [18] is obtained by one of the very oldest models, the Starobinsky model [19]. The action is

$$S = \frac{1}{2} \int d^4x \sqrt{-g} \Big(R + \frac{R^2}{6M^2} \Big) \tag{14}$$

where $M \ll M_{\rm Pl}$ is a mass scale. Current CMB measurements indicate scale invariant spectrum with a small tilt in scalar density $n_s = 0.965 \pm 0.004$ and an upper limit for tensor-to-scalar ratio $r < 0.06$. These values are fully consistent with the Starobinsky model (14) which predicts $r \simeq 0.003$.

The model (14) has the virtue of being based on gravity only physics. Furthermore, the Starobinsky model has been shown to correspond to no-scale supergravity coupled to two chiral supermultiplets. Some obstacles have to be sorted out first before reaching supergravity. To do that we follow the review by Ellis et al. [20].

The first problem with generic supergravity models with matter fields is that their effective potentials do not provide slow-roll inflation as needed. Secondly, they may have anti-deSitter vacua instead of deSitter ones. Thirdly, looking into the future, any new model of particles and inflation should preferably be consistent with some string model properties. These problems can be overcome by no-scale supergravity models. No-scale property comes from their effective potentials having flat directions without specific dynamical scale at the tree level. This has been derived from string models, whose low energy effective theory supergravity is.

Other authors have studied other implications of superstring theory to inflationary model building focusing on scalar fields in curved spacetime [21] and the swampland criteria [22–24]. These studies point out the inadequacy of slow roll single field inflation. We find it important to establish first a connection between the Starobinsky model and (two field) supergravity.

The bosonic supergravity Lagrangian includes a Hermitian function of complex chiral scalar fields ϕ_i which is called the Kähler potential $K(\phi^i, \phi^*_j)$. It describes the geometry of the model. In minimal supergravity (mSUGRA) $K = \phi^i \phi^*_i$. Secondly the Lagrangian includes a holomorphic function called the superpotential $W(\phi^i)$. This gives the interactions among the fields ϕ^i and their fermionic partners. K and W can be combined into a function $G \equiv K + \ln|W|^2$. The bosonic Lagrangian is of the form

$$\mathcal{L} = -\frac{1}{2}R + K_i^j \partial_\mu \phi^i \partial^\mu \phi^*_j - V - \frac{1}{4}\mathrm{Re}(\mathrm{f}_{\alpha\beta})\mathrm{F}^\alpha_{\mu\nu}\mathrm{F}^{\beta\mu\nu} - \frac{1}{4}\mathrm{Im}(\mathrm{f}_{\alpha\beta})\mathrm{F}^\alpha_{\mu\nu}\tilde{\mathrm{F}}^{\beta\mu\nu} \tag{15}$$

where $K_i^j \equiv \partial^2 K/\partial\phi^i \partial\phi^*_j$ and $\mathrm{Im}(\mathrm{f}_{\alpha\beta})$ is the gauge kinetic function of the chiral fields ϕ^i. In mSUGRA the effective potential is

$$V(\phi^i, \phi^*_j) = e^K \left[|W_i + \phi^*_i W|^2 - 3|W|^2\right] \tag{16}$$

where $W_i \equiv \partial W/\partial\phi^i$. It is seen in (16) that the last term with negative sign may generate AdS holes with depth $-\mathcal{O}(m^2_{3/2}M^2_{\mathrm{Pl}})$ and cosmological instability. Solution to this and the slow-roll problem is provided by no-scale supergravity models. The simplest such model is the single field case with

$$K = -3\ln(T + T^*) \tag{17}$$

where T is a volume modulus in a string compactification.

The single field (17) model can be generalized to include matter fields ϕ^i with the followng Kähler potential

$$K = -3\ln\left(T + T^* - \frac{1}{3}|\phi_i|^2\right) \tag{18}$$

The no-scale Starobinsky model is now obtained with some extra work from the potential (16) and assuming $\langle T| \, |=\rangle \, \frac{1}{2}$. For the superpotential the Wess-Zumino form is introduced [25]

$$W = \frac{1}{2}M\phi^2 - \frac{1}{3}\lambda\phi^3 \tag{19}$$

which is a function of ϕ only. Then $W_T = 0$ and from $V' = |W_\phi|^2$ the potential becomes as

$$V(\phi) = M^2 \frac{|\phi|^2 |1 - \lambda\phi/M|^2}{(1 - |\phi|^2/3)^2} \tag{20}$$

The kinetic terms in the scalar field Lagrangian can be written now

$$\mathcal{L} = (\partial_\mu \phi^*, \partial_\mu T^*) \Big(\frac{3}{(T + T^* - |\phi|^2/3)^2} \Big) \begin{pmatrix} (T+T^*)/3 & -\phi/3 \\ -\phi^*/3 & 1 \end{pmatrix} \begin{pmatrix} \partial^\mu \phi \\ \partial^\mu T \end{pmatrix} \tag{21}$$

Fixing T to some value one can define the canonically normalized field χ

$$\chi \equiv \sqrt{3} \tanh^{-1} \left(\frac{\phi}{\sqrt{3}} \right) \tag{22}$$

By analyzing the real and imaginary parts of χ one finds that the potential (20) reaches its minimum for $\mathrm{Im}\chi = 0$. $\mathrm{Re}\chi$ is of the same form as the Starobinsky potential in conformally transformed Einstein-Hilbert action [26] with a potential of the form

$$V = \frac{3}{4} M^2 (1 - e^{-\sqrt{2/3}\phi})^2 \tag{23}$$

when

$$\lambda = \frac{M}{\sqrt{3}} \tag{24}$$

Most interestingly, λ/M has to be very accurately $1/\sqrt{3}$, better than one part in 10^{-4}, for the potential to agree with measurements.

This is briefly the basic mechanism behind inflation in the Wess-Zumino mSUGRA model, which foreruns reheating of visible matter. But only the particles containing m chernons, i.e. the visible matter gets reheated. The dark sector is going through reheating unaffected and is distributed smoothly all over space. The quantum fluctuations of the dark fields are enhanced by gravitation and provide a clumpy underlay for visible matter to form objects of various sizes, from stars to large scale structures.

7 Sakharov conditions

Sakharov suggested [27] three necessary conditions that must satisfied to produce matter and antimatter at different rates. They are (i) baryon number B violation, (ii) C-symmetry and CP-symmetry violation and (iii) interactions out of thermal equilibrium.

Baryon number violation is clearly needed to reach baryon asymmetry. This is valid in our model because baryon number is not defined conventionally. C-symmetry violation is needed so that the interactions which produce more baryons than anti-baryons will not be counterbalanced by interactions which produce more anti-baryons than baryons. CP-symmetry violation is required because otherwise equal numbers of left-handed baryons and right-handed anti-baryons would be produced, as well as equal numbers of left-handed anti-baryons and right-handed baryons. The observed pattern of CP-violation [28] remarkably confirms the Cabibbo– Kobayashi–Maskawa (CKM) description of three fermionic generations of particles [29, 30]. CP-violation phenomenology is discussed in detail in [31, 32]. Our present one generation "skeleton" model cannot satisfy this condition but in principle, by completing the model and deriving the low energy limit, it could be explained. In the SM, the CKM model gives an explanation of why the breaking is so small, despite the phase associated to it being of order one. Thirdly, interactions are out of thermal equilibrium in a rapidly expanding universe.

8 Baryon asymmetry

We now examine the potential (11) in the early universe. Consider large number of groups of twelve chernons each group consisting of four m^+, four m^- and four m^0 particles. Any bunch may form only electron and proton (hydrogen atoms H), only positron and antiproton ($\bar{\mathrm{H}}$) or some combination of both H and $\bar{\mathrm{H}}$ atoms [1, 2]. This is achieved by arranging the chernons appropriately (mod 3) using table 1. This way the transition from matter-antimatter symmetric universe to matter-antimatter asymmetric one happens straightforwardly.

Because the Yukawa force (11) is the strongest force the light e^-, e^+ and the neutrinos are expected to form first at the very onset of inflation. To obey condition $B - L = 0$ of baryon-lepton balance and to sustain charge conservation, for one electron made of three chernons, nine other chernons have to be created simultaneously, these form a proton. Accordingly for positrons. One neutrino requires a neutron to be created. The m^0 carries in addition color enhancing neutrino formation. This makes neutrinos different from other leptons and the quarks.

Later, when the protons were formed, because chernons had the freedom to choose whether they are constituents of H or $\bar{\mathrm{H}}$ there are regions of space of various sizes dominated by H or $\bar{\mathrm{H}}$ atoms. Since the universe is the largest statistical system it is expected that there is only a very slight excesses of H atoms (or $\bar{\mathrm{H}}$ atoms which only means a charge sign redefinition) which remain after the equal amounts of H and $\bar{\mathrm{H}}$ atoms have annihilated. The ratio n_B/n_γ is thus predicted to be $\ll 1$. The ratio n_B/n_γ is a multiverse-like concept.

Fermionic dark matter has in this scenario no mechanism to become "baryon" asymmetric like visible matter. Therefore we expect that part of fermionic dark matter has annihilated into bosonic dark matter. Secondly, we predict there should exist both dark matter and anti-dark matter clumps attracting visible matter in the universe. Collisions of anti-dark matter and dark matter celestial bodies would give us a new source for wide spectrum gravitational wave production (the lunar mass alone is $\sim 10^{49}$ GeV).

Part II Subtleties, localization and generalized symmetries

In the rest of this article we follow closely the references. Briefly, Part II material is glued to Part I. A good starting point in references is [33].

9 Chern-Simons gravity

Low-dimensional gravity is an exciting arena to explore and test the gravitational path integral. In two and three spacetime dimensions, there is no propagating graviton and all of the effective degrees of freedom are long-range. A prime example of this phenomenon is the rewriting of pure Einstein gravity with a cosmological constant (of either sign) as a Chern-Simons gauge theory [34] which is the quintessential example of a topological field theory in three-dimensions. A full leveraging of this fact allows the exact evaluation of the gravitational path integral either about a saddle point [35] or as a non-perturbative sum over saddles [36]. While Chern-Simons gravity is not a UV-complete model of quantum gravity [36], its all-loop exactness provides strong tests for potential microscopic models in the spirit of e.g. [37].

One feature that is expected of a UV-complete model of quantum gravity is that it includes matter, in particular massive fields that couple to gravity. The manifest topological invariance that makes Chern-Simons theory so powerful as a description of the gravitational path integral also presents a challenge to incorporating matter. On a practical level, this is simple to illustrate: the action of a massive field theory minimally coupled to a geometry involves both inverse metrics and metric determinants. The rewriting of these terms as Chern-Simons connections is highly non-linear and indicates that integrating out of the massive field will result in a non-local effective action. However, we can take inspiration from the general philosophy that the low-energy avatar of the worldline of a massive degree of freedom is a line-operator of the effective gauge theory [39].

This philosophy was made precise in [40] for massive scalar fields minimally coupled to gravity with a positive cosmological constant. The key result was the expression of the one-loop determinant of a massive scalar field coupled to a background metric, $g_{\mu\nu}$, as a gauge invariant object of the Chern-Simons connections, $A_{L/R}$:

$$Z_{\text{scalar}}[g_{\mu\nu}] = \exp \frac{1}{4}\mathbb{W}[A_L, A_R] \ . \tag{25}$$

The object $\mathbb{W}[A_L, A_R]$, coined the Wilson spool, is a collection of Wilson loop operators wrapped many times around cycles of the base geometry. The equality in (25) is expected to apply to three-dimensional gravity of either sign of cosmological constant, and this was explicitly shown for Euclidean black holes in Anti-de Sitter (i.e., Euclidean BTZ) and Euclidean de Sitter (i.e., the three-sphere S^3) in [40, 42]. It has also been upheld on $T\bar{T}$ deformations of AdS$_3$ [43]. The importance of (25) is not only conceptual, it is practical: it was additionally shown in [40] that certain "exact methods" in Chern-Simons theory (such as Abelianisation [44] and supersymmetric localization [45]) extend to three-dimensional de Sitter (dS$_3$) Chern-Simons gravity with the Wilson spool inserted into the path integral. This allows a precise and efficient calculation of the quantum gravitational corrections to Z_{scalar} at any order of perturbation theory of Newton's constant, G_N.

To be specific, we state our main result, the generalization of (25) for massive spinning fields, as the following. Consider the local path integral,[5] $Z_{\Delta,\mathsf{s}}$, of a spin-s field $\Phi_{\mu_1\mu_2\ldots\mu_s}$ with mass

$$\frac{m^2}{\Lambda} = (\Delta + \mathsf{s} - 2)(\mathsf{s} - \Delta) \ , \tag{26}$$

minimally coupled to a metric geometry, (M_3, g_{M_3}), where M_3 is topologically either Euclidean BTZ or Euclidean dS$_3$. Then, it is proposed that

$$\log Z_{\Delta,\mathsf{s}}[g_{M_3}] = \frac{1}{4}\mathbb{W}_{j_L,j_R}[A_L, A_R] \ , \tag{27}$$

where

$$\mathbb{W}_{j_L,j_R}[A_L, A_R] = \frac{i}{2}\int_{\mathcal{C}} \frac{\mathrm{d}\alpha}{\alpha}\frac{\cos\left(\frac{\alpha}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)}\left(1 + \mathsf{s}^2\sin^2\left(\frac{\alpha}{2}\right)\right) \times$$

[5] Including any additional Stückelberg fields to fix its invariances and associated ghosts.

$$\sum_{\mathsf{R}_L\otimes\mathsf{R}_R} \mathrm{Tr}_{\mathsf{R}_L}\left(\mathcal{P}e^{\frac{\alpha}{2\pi}\oint_\gamma A_L}\right)\mathrm{Tr}_{\mathsf{R}_R}\left(\mathcal{P}e^{-\frac{\alpha}{2\pi}\oint_\gamma A_R}\right) \ . \quad (28)$$

The details of $\mathbb{W}_{j_L,j_R}$ will be made explicit below, however let us briefly summarize the parts appearing in (28). The Chern-Simons connections, $A_{L/R}$, are related to the metric, g_{M_3}, in (27) through the usual Chern-Simons gravity dictionary and they are integrated over a non-trivial cycle, γ, of the base geometry. The representations, $\mathsf{R}_{L/R}$, appearing in the Wilson loops are summed over a set determined by the mass and spin, (Δ, s), of (27) and labeled by weights (j_L, j_R). Lastly the parameter α is integrated along a contour, $\mathcal{C}$, determined by a regularization scheme appropriate for the sign of cosmological constant. The ultimate effect of the α integral is to implement a "winding" of the Wilson loop operators around γ; this occurs through the summing the residues of the poles of its measure (as well as any of representation traces themselves). The above object, (28), is coined the spinning Wilson spool.

10 Non-standard spinning representations of $\mathfrak{su}(2)$

In this section, single-particle states of massive spin-s fields living on dS_3, with de Sitter radius ℓ_{dS}, as representations of $\mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R$, are described. The guiding principle of the construction is to mimic the unitary representations of the Lorentzian dS_3 isometry group, $\mathfrak{so}(1,3)$. In [40] this was done for cases with $\mathsf{s} = 0$, and here, following the same arguments and conventions, that construction is extended to include spin.

Unitary representations of $\mathfrak{so}(1,3)$ are labeled by a conformal dimension Δ (the eigenvalue of the dilatation operator D) and a spin s (the eigenvalue of the spin operator iM) and come in two series:[6]

$$\begin{aligned} \Delta =& 1+\nu \ , \qquad && \nu\in(-1,1) \ , \qquad && \mathsf{s}=0 \ , \\ \Delta =& 1-i\mu \ , \qquad && \mu\in\mathbb{R} \ , \qquad && \mathsf{s}\in\mathbb{Z} \ . \end{aligned} \quad (29)$$

[6] See [40] for a basic introduction on $\mathfrak{so}(1,3)$ representation theory and for our notation here.

The first line is the complementary series describing light scalars with masses $\ell_{\mathrm{dS}}^2 m^2 = 1 - \nu^2$ while the second line is the spinning principal series describing spinning particles with masses $\ell_{\mathrm{dS}}^2 m^2 = (\mathsf{s}-1)^2 + \mu^2$.

To connect this to $\mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R$, let us introduce some basic aspects of $\mathfrak{su}(2)$. The algebra is generated by L_3 and $L_\pm$, where

$$[L_3, L_\pm] = \pm L_\pm \ , \qquad [L_+, L_-] = 2L_3 \ . \tag{30}$$

The Casimir of the representation is $c_2^{\mathfrak{su}(2)} = \frac{1}{2}(L_+L_- + L_-L_+) + L_3^2$. Representations of $\mathfrak{su}(2)$ will be characterized by j, the eigenvalue of L_3. The $\mathfrak{so}(1,3)$ algebra shares a complexification with $\mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R$, such that the dilatation and spin generators can be identified with the Cartan elements of $\mathfrak{su}(2)$ as

$$D = -L_3 - \bar{L}_3 \ , \qquad M = iL_3 - i\bar{L}_3 \ , \tag{31}$$

respectively. Similar relations follow for the remaining generators, which can be found in [40]. In (31) we have distinguished the generators of $\mathfrak{su}(2)_R$ from those of $\mathfrak{su}(2)_L$ by an overbar. Within this complexification, the quadratic Casimir of $\mathfrak{so}(1,3)$ is equal to the sum of the $\mathfrak{su}(2)$ Casimirs:

$$c_2^{\mathfrak{so}(1,3)} = -2c_2^{\mathfrak{su}(2)_L} - 2c_2^{\mathfrak{su}(2)_R} \ . \tag{32}$$

Equation (31) suggests that we need to look for $\mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R$ representations with

$$\Delta = -j_R - j_L \ , \qquad \mathsf{s} = j_R - j_L \ , \tag{33}$$

where j_L is the eigenvalue of L_3 in $\mathfrak{su}(2)_L$, and similarly for j_R.

The relations (33), together with (29), indicate that one has to construct representations with continuous and complex eigenvalues (j_L, j_R). These do not fall into the standard finite dimensional representation theory of $\mathfrak{su}(2)$. However, in [40] it was shown how to construct an alternative inner product in $\mathfrak{su}(2)$ (or equivalently, an alternative notion of Hermitian conjugation) allowing for highest weight representations with a continuous and complex weight while preserving norm-positivity. Such representations were coined non-standard representations in the latter reference. These representations are built from a highest weight state, $|j, 0\rangle$, satisfying

$$L_3|j,0\rangle = j|j,0\rangle \ , \qquad\qquad L_+|j,0\rangle = 0 \ , \tag{34}$$

and $j \in \mathbb{C}$ is the weight of the representation. Acting with lowering operators defines

$$|j,p\rangle = N_{j,p}\,(L_-)^p\,|j,0\rangle \ , \tag{35}$$

for some normalizations $N_{j,p}$ that we will determine shortly.

The key ingredient distinguishing the non-standard representations is a map, $\mathcal{S}$, between highest weight representations

$$\mathcal{S}|j,p\rangle = |j^*,p\rangle \ , \tag{36}$$

where j^* is the complex conjugate of j. This map is utilized in Hermitian conjugation in the following way:

$$L_3^\dagger := \mathcal{S}^{-1}L_3\mathcal{S} \ , \qquad L_\pm^\dagger := -\mathcal{S}^{-1}L_\mp\mathcal{S} \ . \tag{37}$$

Note that $j \xrightarrow{\mathcal{S}} j^*$ is consistent with the action of $L_3^\dagger$.

In [40], norm-positive inner products were constructed for non-standard representations with $j = -\frac{1}{2}(1+\nu)$ or $j = -\frac{1}{2}(1-i\mu)$ (and $\mu, \nu \in \mathbb{R}$) which relate to the scalar complementary and principal series with $\mathsf{s} = 0$. Here we will relax this condition and show that a norm-positive inner product can be constructed for any complex j.

We will first determine the normalizations $N_{j,p}$. We note that

$$\langle j,p-1|L_+|j,p\rangle = \frac{N_{j,p}}{N_{j,p-1}}p(2j+1-p)\langle j,p-1|j,p-1\rangle \ , \tag{38}$$

which can be seen from replacing L_+L_- with $c_2^{\mathfrak{su}(2)} + L_3 - L_3^2$ in the matrix element. Alternatively, utilizing (37) and (36), this same matrix element is

$$\langle j,p-1|L_+|j,p\rangle = \langle j,p|L_+^\dagger|j,p-1\rangle^* = -\left(\frac{N_{j^*,p-1}}{N_{j^*,p}}\right)^*\langle j,p|j,p\rangle \ . \tag{39}$$

If one assumes that we can set $\langle j, p-1|j, p-1\rangle = 1$ and also $\langle j, p|j, p\rangle = 1$, then (38) and (39) imply

$$\left(\frac{N_{j^*,p-1}}{N_{j^*,p}}\right)^* = p(p-2j-1)\left(\frac{N_{j,p}}{N_{j,p-1}}\right) \ . \tag{40}$$

To solve this constraint, we write

$$\alpha_{j,p} = \mathrm{Arg}\left(N_{j,p}\right) \ , \qquad \phi_{j,p} = \mathrm{Arg}\left(p-2j-1\right) \ , \tag{41}$$

leading to recurrence relations

$$\begin{aligned} \frac{|N_{j^*,p-1}|}{|N_{j^*,p}|} =& p|p-2j-1|\frac{|N_{j,p}|}{|N_{j,p-1}|} \ , \\ \alpha_{j^*,p} - \alpha_{j^*,p-1} =& \alpha_{j,p} - \alpha_{j,p-1} + \phi_{j,p} \mod(2\pi) \ . \end{aligned} \tag{42}$$

There is a lot of freedom in solving these recurrence relations. We will choose a particular solution by fixing

$$|N_{j,p}| = |N_{j^*,p}| \, , \qquad \alpha_{j,0} = \alpha_{j^*,0} = 0 \, , \qquad \alpha_{j,p} = -\alpha_{j^*,p} \ \ \forall\, p \, . \tag{43}$$

This sets

$$\frac{|N_{j,p-1}|}{|N_{j,p}|} = \frac{|N_{j^*,p-1}|}{|N_{j^*,p}|} = \sqrt{p|p-2j-1|} \tag{44}$$

and[7]

$$\alpha_{j,p} = -\frac{1}{2}\sum_{p'=1}^{p} \phi_{j,p'} \ . \tag{45}$$

For concreteness, we summarize the generator actions on this representation as

$$\begin{aligned} L_3|j,p\rangle &= (j-p)|j,p\rangle \ , \\ L_-|j,p\rangle &= e^{i\phi_{j,p}/2}\sqrt{(p+1)|p-2j|}|j,p+1\rangle \ , \\ L_+|j,p\rangle &= -e^{i\phi_{j,p}/2}\sqrt{p|p-2j-1|}|j,p-1\rangle \ , \end{aligned} \tag{46}$$

with the action on $|j^*,p\rangle$ obtained by simply replacing $j \to j^*$ in the above formulas. We can now show norm-positivity by induction

[7] Note that $\phi_{j^*,p} = -\phi_{j,p}$.

starting with $\langle j,0|j,0\rangle = 1$. Let us investigate the first descendant state:

$$\begin{aligned}
|L_-|j,0\rangle|^2 =& - \langle j,0|\mathcal{S}^{-1}L_+\mathcal{S}L_-|j,0\rangle \\
=& e^{i\phi_{j,1}/2+i\phi_{j^*,1}/2}\sqrt{|2j|}\sqrt{|2j|}\langle j,0|j,0\rangle \\
=& |2j| > 0 \ .
\end{aligned} \tag{47}$$

Similarly norm-positivity of $|j,p\rangle$ induces norm-positivity of $|j,p+1\rangle$[8]:

$$\begin{aligned}
|L_-|j,p\rangle|^2 =& - \langle j,p|\mathcal{S}^{-1}L_+\mathcal{S}L_-|j,p\rangle \\
=& e^{i\phi_{j,p+1}/2+i\phi_{j^*,p+1}/2}\sqrt{(p+1)|p-2j|}\sqrt{(p+1)|p-2j|}\langle j,p|j,p\rangle \\
=& (p+1)|p-2j|\langle j,p|j,p\rangle > 0 \ .
\end{aligned} \tag{48}$$

This establishes norm positivity for all states in the representation. We additionally note that these representations have well defined characters

$$\chi_j(z) = \mathrm{Tr}_j\left(e^{i2\pi z L_3}\right) = \sum_{p=0}^{\infty} e^{i2\pi z(j-p)} = \frac{e^{i\pi(2j+1)z}}{2i\sin(\pi z)} \ . \tag{49}$$

In [40] the restrictions of the highest-weights, $j = -\frac{1}{2}(1+\nu)$ or $j = -\frac{1}{2}(1-i\mu)$, followed from Hermiticity of the $\mathfrak{su}(2)$ Casimirs, separately. Here we take a more Lorentzian perspective, and impose reality of the $\mathfrak{so}(1,3)$ Casimir at the level of (32); doing so ties the $\mathfrak{su}(2)_{L/R}$ representations together and the highest weights take the generic form

$$j_L = -\frac{1}{2}(1+\mathsf{s}-i\mu) \ , \qquad j_R = -\frac{1}{2}(1-\mathsf{s}-i\mu) \ . \tag{50}$$

Let us briefly show this now. Assuming that j_L and j_R are generically complex the condition for reality of the $\mathfrak{so}(1,3)$ Casimir is that

$$\mathrm{Im}\left(j_L(j_L+1)\right) = -\mathrm{Im}\left(j_R(j_R+1)\right) \ , \tag{51}$$

[8] We are assuming j is a generic complex number and ignoring the potential for possible null states when $j \in \frac{1}{2}\mathbb{N}_0$. Of course, in these cases the representation simply terminates and we recover the standard, finite-dimensional, representations of $\mathfrak{su}(2)$.

or equivalently

$$\mathrm{Im}(j_L)\,(1+2\mathrm{Re}(j_L)) = -\mathrm{Im}(j_R)\,(1+2\mathrm{Re}(j_R)) \ . \tag{52}$$

If both sides are simultaneously zero then we must have either $j \in \mathbb{R}$ or $j \in -\frac{1}{2} + i\mathbb{R}$ (for either j_L or j_R) which lead to the complimentary and principal-type representations discussed in [40]. More generally there will be a family of highest weight solutions satisfying (52). However, if we further insist[9] that $j_L - j_R \in \mathbb{R}$ then it must be the case that

$$\mathrm{Im}(j_L) = \mathrm{Im}(j_R) \ , \qquad\qquad \mathrm{Re}(j_L) + \mathrm{Re}(j_R) = -1 \ , \tag{53}$$

which lead to the highest weights appropriate for the spinning principal series, (50). We will refer to such representations as the spinning principal-type representations.

As noted already in [40], our non-standard characters can be massaged into the suggestive Lorentzian form via

$$\chi_{j_L}(z_L)\chi_{j_R}(z_R) + \chi_{j_L^*}(z_L)\chi_{j_R^*}(z_R) = \frac{w^{\mathsf{s}}q^{\Delta} + w^{-\mathsf{s}}q^{\bar{\Delta}}}{(1-w^{-1}q)(1-wq)} \ , \tag{54}$$

where $q = e^{-i\pi(z_L+z_R)}$ and $w = e^{i\pi(z_L-z_R)}$ and we have identified (33) as well as $\bar{\Delta} = 2-\Delta$. This matches the Harish-Chandra character, $\mathrm{Tr}\left(q^D w^{iM}\right)$, for the $\mathfrak{so}(1,3)$ spinning principal series.

11 Spinning spool on S^3

We now present the construction of the Wilson spool for massive spin-s fields on S^3. That is, we will derive an expression for the one-loop determinant of these fields on S^3 in terms of the representations of $\mathfrak{su}(2)_L \otimes \mathfrak{su}(2)_R$ constructed in the previous section.

To start, let us describe the path integral of a single massive spin-s field, with no self-interactions. The local partition function for this theory contains a symmetric spin-s tensor, $\Phi_{\mu_1\mu_2\cdots\mu_{\mathsf{s}}}$, as well as a

[9] At this point we will impose this by hand; we will see later that in the quantum theory only representations with $j_L - j_R \in \mathbb{Z}$ contribute to the matter one-loop determinant.

tower of Stückelberg fields which enforce that $\Phi_{\mu_1\mu_2\ldots\mu_{\mathsf{s}}}$ is transverse and traceless [41]:

$$\nabla^\nu \Phi_{\nu\mu_2\ldots\mu_{\mathsf{s}}} = \Phi^\nu{}_{\nu\mu_3\ldots\mu_{\mathsf{s}}} = 0 \ . \tag{55}$$

As emphasized in [37], on a compact manifold the path integral over symmetric, transverse, traceless (STT) tensors leads to non-local divergences which cannot be canceled by local counterterms. This path integral must be compensated by the path integral over the Stückelberg fields and ghosts which leave behind a finite product from integrating over normalizable zero modes. To that end, we write

$$Z_{\Delta,\mathsf{s}} = Z_{\text{zero}} Z_{\text{STT}} \ , \tag{56}$$

where

$$Z_{\text{STT}} = \int [\mathcal{D}\Phi_{\mu_1\mu_2\ldots\mu_{\mathsf{s}}}]_{\text{STT}} e^{-\frac{1}{2}\int \Phi\left(-\nabla^2_{(\mathsf{s})} + \ell^2_{\text{dS}}\bar{m}^2_{\mathsf{s}}\right)\Phi} \ . \tag{57}$$

Above $\nabla^2_{(\mathsf{s})}$ is the Laplace-Beltrami operator

$$\left[\nabla^2_{(\mathsf{s})}\Phi\right]_{\mu_1\mu_2\ldots\mu_{\mathsf{s}}} = \nabla_\nu\nabla^\nu \Phi_{\mu_1\mu_2\ldots\mu_{\mathsf{s}}} \ , \tag{58}$$

and $\bar{m}^2_{\mathsf{s}}$ is an effective mass

$$\ell^2_{\text{dS}}\bar{m}^2_{\mathsf{s}} = \ell^2_{\text{dS}} m^2 + 3\mathsf{s} - \mathsf{s}^2 \ , \tag{59}$$

where we recall that m^2 is the standard mass parameter in dS_3 [37], and related to the representation theory of section 10 via

$$\ell^2_{\text{dS}} m^2 = (\Delta + \mathsf{s} - 2)(\mathsf{s} - \Delta) \ . \tag{60}$$

The zero mode contribution in (56) follows from counting conformal Killing tensor modes on S^3 and is given by [37][10]

$$Z_{\text{zero}} = \left[(\Delta - 1)(\bar{\Delta} - 1)\right]^{\frac{\mathsf{s}^2}{2}} \prod_{n=0}^{\mathsf{s}-2} \left[(\Delta + n)(\bar{\Delta} + n)\right]^{-(n+1+\mathsf{s})(n+1-\mathsf{s})} \ , \tag{61}$$

where we recall that $\bar{\Delta} = 2 - \Delta$.

[10] This is true for $\mathsf{s} \geq 2$ while for $\mathsf{s} = 0, 1$ the product over n is replaced by 1.

In the following, we will show that

$$\log Z_{\Delta,\mathsf{s}} = \frac{1}{4}\mathbb{W}_{j_L,j_R} \ , \tag{62}$$

for fields on S^3. That is, we will express $Z_{\Delta,\mathsf{s}}$ as a function of the Chern-Simons connections which we will explicitly construct utilizing the non-standard representation theory of section 10. While our construction takes place on a fixed classical background, we will see that $\mathbb{W}_{j_L,j_R}$ is an integral over gauge invariant Wilson loop operators and naturally generalizes into an off-shell operator that can be inserted into the Chern-Simons path integral, which we will discuss in section 14.

The roadmap to derive (62) is as follows. We will first focus on Z_{STT}. This determinant can be evaluated via the method of quasinormal modes pioneered in [38]. We will adapt this method such that each component has a group theoretic interpretation: we will show how defining properties of quasinormal modes can be translated to conditions on the representation of the fields. This follows [40], however, for spinning fields, we will take particular care with the role of global conditions (i.e., Euclidean solutions are regular and single-valued) in isolating physical contributions to the quasinormal mode product. The additional contribution of Z_{zero}, which is not part of the quasinormal mode product but crucial for maintaining locality of $Z_{\Delta,\mathsf{s}}$, will permit a Schwinger parameterization of $\log Z_{\Delta,\mathsf{s}}$, regularized by an $i\varepsilon$ prescription. This will organize the quasinormal mode sum into an integral over representation traces of the background holonomies. From this follows our main result.

11.1 A group theory perspective on S^3 quasinormal modes

As the first step in our construction, we will recast the functional determinant

$$Z_{\mathrm{STT}} = \det\left(-\nabla^2_{(\mathsf{s})} + \ell^2_{\mathrm{dS}}\bar{m}^2_{\mathsf{s}}\right)^{-\frac{1}{2}} \tag{63}$$

in $\mathfrak{su}(2)$ representation theoretic language. We recall that the DHS method [38] instructs us to treat $Z^2_{\rm STT}$ as meromorphic function of Δ. Then, up to a holomorphic function, $Z^2_{\rm STT}$ is equal to the product containing the same zeros and poles. Here $Z_{\rm STT}$ only has poles on states satisfying $(-\nabla^2_{(\mathsf{s})}+\bar{m}^2_{\mathsf{s}})\Phi_{\mu_1\mu_2\ldots\mu_{\mathsf{s}}}=0$. These are precisely the spin-s quasinormal modes. We have explicitly computed these modes and their product to obtain $Z_{\rm STT}$ directly. Here we will reinterpret these modes in terms of $\mathfrak{su}(2)$ representation theory to obtain an expression natural to the Chern-Simons theory formulation of gravity.

We note that the isometry algebra of the three-sphere is generated by two mutually commuting sets of $\mathfrak{su}(2)$ vector fields $\{\zeta_a\}$ and $\{\bar{\zeta}_b\}$ which are the infinitesimal left and right group actions acting on $S^3 \simeq SU(2)$. On spin-s STT tensors the Casimirs of their Lie derivatives, $\{\mathcal{L}_{\zeta_a}\}$ and $\{\mathcal{L}_{\bar{\zeta}_b}\}$, act as the Laplace-Beltrami operator:

$$-2\delta^{ab}\left(\mathcal{L}_{\zeta_a}\mathcal{L}_{\zeta_b}+\mathcal{L}_{\bar{\zeta}_a}\mathcal{L}_{\bar{\zeta}_b}\right)\Phi_{\mu_1\mu_2\ldots\mu_{\mathsf{s}}}=\left[\nabla^2_{(\mathsf{s})}-\mathsf{s}(\mathsf{s}+1)\right]\Phi_{\mu_1\mu_2\ldots\mu_{\mathsf{s}}}\ . \tag{64}$$

Hence we can write (63) suggestively as

$$Z_{\rm STT}=\det\left(2c_2^{\mathfrak{su}(2)_L}+2c_2^{\mathfrak{su}(2)_R}+\Delta(2-\Delta)-\mathsf{s}^2\right)^{-\frac{1}{2}}\ . \tag{65}$$

Following the DHS methodology, we then expect $Z^2_{\rm STT}$ to have pole contributions from states in $\mathfrak{su}(2)_L\oplus\mathfrak{su}(2)_R$ representations satisfying

$$\left(-2c_2^{\mathfrak{su}(2)_L}-2c_2^{\mathfrak{su}(2)_R}\right)|\psi\rangle=\left[\Delta(2-\Delta)-\mathsf{s}^2\right]|\psi\rangle\ . \tag{66}$$

This is precisely the condition satisfied by the non-standard representations constructed in section 10 with highest weights $(j_L,j_R)=(-\frac{\Delta+\mathsf{s}}{2},-\frac{\Delta-\mathsf{s}}{2})$. We are interested in the poles in $Z^2_{\rm STT}$ arising from weights of the representations $\mathsf{R}_{j_L}\otimes\mathsf{R}_{j_R}$ as we continue Δ in the complex plane.[11] In principle we should consider all representations that satisfy (66), so for a given (Δ,s), we also encounter

[11] It will be important as we progress to take special care of the cases when Δ is such that $j_{L/R}\in\frac{1}{2}\mathbb{N}$; in these cases the weight spaces terminate discontinuously to finite-dimensional representations.

poles associated to highest weight representations arrived at by sending $\Delta \to \bar{\Delta} = 2 - \Delta$ as well as $\mathsf{s} \to -\mathsf{s}$.[12] If we define $(j_L, j_R) = (-\frac{\Delta+\mathsf{s}}{2}, -\frac{\Delta-\mathsf{s}}{2})$ then we will denote

$$(\bar{j}_L, \bar{j}_R) = \left(-\frac{\bar{\Delta}+\mathsf{s}}{2}, -\frac{\bar{\Delta}-\mathsf{s}}{2}\right) , \tag{67}$$

while $\mathsf{s} \to -\mathsf{s}$ is equivalent to $j_L \leftrightarrow j_R$. We will then have pole contributions from any of the representations appearing in

$$\mathcal{R}_{\Delta,\mathsf{s}} = \left\{\mathsf{R}_{j_L} \otimes \mathsf{R}_{j_R}, \mathsf{R}_{\bar{j}_L} \otimes \mathsf{R}_{\bar{j}_R}, \mathsf{R}_{j_R} \otimes \mathsf{R}_{j_L}, \mathsf{R}_{\bar{j}_R} \otimes \mathsf{R}_{\bar{j}_L}\right\} . \tag{68}$$

We make a special note that for scalar representations ($j_L = j_R = j$) it is sufficient to consider the smaller set

$$\mathcal{R}_{\Delta,\text{scalar}} = \left\{\mathsf{R}_j \otimes \mathsf{R}_j, \mathsf{R}_{\bar{j}} \otimes \mathsf{R}_{\bar{j}}\right\} \tag{69}$$

as in [40].

The "mass-shell condition" (66) is only a necessary condition to contribute a physical pole to Z^2_{STT}. Functional determinants come with boundary and regularity conditions on their functional domain and we must impose these on weight spaces satisfying (66) to obtain a physical answer. We will state these up-front in an $\mathfrak{su}(2)$ natural language as the following:

Condition I. Single-valued solutions: A configuration constructed from a representation $\mathsf{R}_L \otimes \mathsf{R}_R \in \mathcal{R}$ must return to itself under parallel transport around any closed cycle in the Euclidean manifold.

Condition II. Globally regular solutions: A configuration constructed from a representation $\mathsf{R}_L \otimes \mathsf{R}_R \in \mathcal{R}$ must be globally regular on the Euclidean manifold. When the base space is homogeneous this means $\mathsf{R}_L \otimes \mathsf{R}_R$ lifts from a representation of the isometry algebra to a representation of the isometry group.

[12] This is consistent with the Lorentzian picture: a spin-s field is built out of $\mathfrak{so}(1,3)$ representations labelled by both $(\Delta, \pm\mathsf{s})$ while the (Δ, s) representation is isomorphic to that labelled by $(\bar{\Delta}, -\mathsf{s})$ through the $\mathfrak{so}(1,3)$ shadow map [54].

Let us first expand upon **Condition I** for spin-s fields on S^3. A field Φ living in a representation $\mathsf{R}_L \otimes \mathsf{R}_R$ is parallel transported around a cycle, γ, through the background connections:

$$\Phi_f = \mathsf{R}_L \left[\mathcal{P}\exp\left(\oint_\gamma a_L \right) \right] \, \Phi_i \, \mathsf{R}_R \left[\mathcal{P}\exp\left(-\oint_\gamma a_R \right) \right] \, . \tag{70}$$

When $a_{L/R}$ are flat this conjugation is trivial. However for the backgrounds appropriate for describing the S^3 metric geometry, the background connections take non-trivial holonomies

$$\mathcal{P}\exp\left(\oint_\gamma a_L \right) = u_L^{-1} \, e^{i2\pi L_3 \mathsf{h}_L^{(\gamma)}} \, u_L^{-1} \, , \qquad \mathcal{P}\exp\left(\oint_\gamma a_R \right) = u_R^{-1} \, e^{i2\pi \bar{L}_3 \mathsf{h}_R^{(\gamma)}} \, u_R^{-1} \, , \tag{71}$$

when γ wraps one of two lines on the base S^3 [40]. These lines are Hopf linked and Wick rotate to the coordinate positions of the static patch origin and horizon. They yield respective holonomies

$$\gamma_{\text{orig.}}: \quad (\mathsf{h}_L, \mathsf{h}_R) = (1, 1) \, , \qquad \gamma_{\text{hor.}}: \quad (\mathsf{h}_L, \mathsf{h}_R) = (1, -1) \, . \tag{72}$$

The salient point is that a single-valued field will satisfy

$$\lambda_L \mathsf{h}_L - \lambda_R \mathsf{h}_R \in \mathbb{Z} \, , \tag{73}$$

for each of the two sets of holonomies in (72) and for all weights (λ_L, λ_R) in the representation $\mathsf{R}_L \otimes \mathsf{R}_R$.

From cycles wrapping the origin weights must satisfy

$$\lambda_L - \lambda_R \in \mathbb{Z} \tag{74}$$

to contribute a pole to Z^2_{STT}. Weights of a highest-weight representation $\mathsf{R}_{j_{L/R}}$ necessarily take the form

$$\lambda_{L/R} = j_{L/R} - p_{L/R} \, , \qquad p_{L/R} \in \mathbb{N}_0 \, , \tag{75}$$

simply via the structure of the $\mathfrak{su}(2)$ algebra. Single-valuedness around the static patch origin then requires

$$j_L - j_R \in \mathbb{Z} \qquad \Leftrightarrow \qquad \mathsf{s} \in \mathbb{Z} \, . \tag{76}$$

This condition is the same for all other representations in $\mathcal{R}_{\Delta,\mathsf{s}}$. We pause here to note that while the representation theory in section 10 only relies on $j_L - j_R \in \mathbb{R}$, we now see that only fields with quantized spin can contribute physical poles to $Z^2_{\rm STT}$. We will thus fix $\mathsf{s} \in \mathbb{Z}$ and consider the analytic structure of $Z^2_{\rm STT}$ as a function of Δ. This analytic structure is constrained by single-valuedness around the static patch horizon, which requires

$$\lambda_L + \lambda_R \in \mathbb{Z} \ . \tag{77}$$

We will return to this condition shortly.

We now address **Condition II**, that configurations contributing to $Z^2_{\rm STT}$ are globally regular. Without loss of generality we will state this for $\mathsf{R}_{j_L} \otimes \mathsf{R}_{j_R} \in \mathcal{R}_{\Delta,\mathsf{s}}$. For the S^3 background in question the isometry group acts transitively. Thus regularity at a point guarantees global regularity as long as the isometry group, $SU(2)_L \times SU(2)_R$, acts faithfully on the field in question: that is, $\mathsf{R}_{j_L} \otimes \mathsf{R}_{j_R}$ lifts to a representation of the isometry group. The Peter-Weyl theorem states that these must be finite-dimensional representations of $\mathfrak{su}(2)_L \otimes \mathfrak{su}(2)_R$, where such representations have weights (75) satisfying

$$\lambda_{L/R} = j_{L/R} - p_{L/R} \ , \qquad 0 \leq p_{L/R} \leq 2 j_{L/R} \ , \qquad j_{L/R} \in \frac{1}{2}\mathbb{N}_0 \ . \tag{78}$$

To be clear about interpretation: the DHS method instructs us to consider the structure of $Z^2_{\rm STT}$ as Δ continues to the complex plane. The mass-shell condition, (66), then instructs us to consider representations with generically complex highest weights, $j_{L/R} \in \mathbb{C}$. Such representations are non-standard and infinite-dimensional. However **Condition II** simply tells us that the poles of $Z^2_{\rm STT}$ are located at $\Delta \in \mathbb{Z}_{\leq -\mathsf{s}}$ and the orders of these poles are correctly counted not by weights of an infinite dimensional representation but instead by (78). In this counting we notice that weights of finite dimensional representations of $SU(2)$ are centered about zero and so for any weight satisfying $\lambda_L + \lambda_R = N > 0$ there is a corresponding weight with $\lambda_L + \lambda_R = -N$. Thus for the purposes of counting the number of weights contributing to a particular pole, we can restate (77) as

$$\lambda_L + \lambda_R = |N| \ , \qquad N \in \mathbb{Z} \tag{79}$$

as a necessary condition for incorporating **Condition II**.

We pause to note that for minimally coupled scalar fields (79) is also sufficient to imply **Condition II**. Thus the scalar one-loop determinant can be written as

$$Z_{\text{scalar}} = \prod_{\substack{(\lambda_L,\lambda_R) \\ \in \mathsf{R}_j \otimes \mathsf{R}_j}} \prod_{N\in\mathbb{Z}} \left(|N| - \lambda_L - \lambda_R\right)^{-1/2} \times \prod_{\substack{(\bar\lambda_L,\bar\lambda_R) \\ \in \mathsf{R}_{\bar j} \otimes \mathsf{R}_{\bar j}}} \prod_{\bar N\in\mathbb{Z}} \left(|\bar N| - \bar\lambda_L - \bar\lambda_R\right)^{-1/2} \ , \tag{80}$$

where we have written explicitly the product over the two representations appearing in $\mathcal{R}_{\Delta,\text{scalar}}$. From here the expression of Z_{scalar} as a Wilson spool follows the procedure in [40].

For massive spin-s fields, (79) is no longer sufficient and we must impose additional constraints to reproduce Z^2_{STT}. For a given $|N|$ in (79), **Condition II** additionally implies the weights $\lambda_{L/R} = j_{L/R} - p_{L/R}$ must satisfy

$$p_L \geq -|N| - \mathsf{s} \ , \qquad\qquad p_R \geq -|N| + \mathsf{s} \ . \tag{81}$$

While the first of these is always satisfied (for positive s) the second is an additional constraint on counting the order of the poles appearing in Z^2_{STT} and is only non-trivial when $|N| \leq \mathsf{s}$. For these $2\mathsf{s}+1$ cases we observe that $\tilde p_R = p_R + |N| - \mathsf{s} \geq 0$ and (79) is equivalently written

$$j_L + j_R - p_L - \tilde p_R = \mathsf{s} \ , \qquad p_L, \tilde p_R \geq 0 \ . \tag{82}$$

We can thus treat this as a condition on the weights $(\lambda_L, \tilde\lambda_R) = (j_L - p_L, j_R - \tilde p_R)$ of highest-weight representations $\mathsf{R}_{j_L} \otimes \mathsf{R}_{j_R}$ and for each pole arising from this condition being satisfied, it arises $2\mathsf{s}+1$ times.

Applying this same procedure to all representations appearing in

$\mathcal{R}_{\Delta,\mathsf{s}}$ we arrive at

$$Z_{\mathrm{STT}} = \prod_{\mathcal{R}_{\Delta,\mathsf{s}}} \prod_{(\lambda_L,\lambda_R)} \left((\mathsf{s} - \lambda_L - \lambda_R)^{-\frac{2\mathsf{s}+1}{2}} \prod_{\substack{N\in\mathbb{Z},\\ |N|>\mathsf{s}}} (|N| - \lambda_L - \lambda_R)^{-\frac{1}{2}} \right) , \tag{83}$$

where the first product is understood to take the product over all pairs $\mathsf{R}_L \otimes \mathsf{R}_R \in \mathcal{R}_{\Delta,\mathsf{s}}$ and the second product is taken over all weights $(\lambda_L, \lambda_R) \in \mathsf{R}_L \otimes \mathsf{R}_R$ of a particular pair in $\mathcal{R}_{\Delta,\mathsf{s}}$. Where it does not cause confusion we will maintain this shorthand (in both products and sums) for compactness of notation.

As mentioned at the beginning to this section, the local spin-s partition function, $Z_{\Delta,\mathsf{s}}$, includes, in addition to this quasinormal product, the product from integrating over normalizable zero modes, (61):

$$Z_{\Delta,\mathsf{s}} = Z_{\mathrm{zero}} Z_{\mathrm{STT}} \ . \tag{84}$$

In the next section we will show how this combination, with the expression of Z_{STT} as a product over representation weights, (83), will lead to the Wilson spool.

11.2 Constructing the spool

The procedure to cast $\log Z_{\Delta,\mathsf{s}}$ as an integral over Wilson loop operators starts by rearranging (61) and (83). We first make use of the Schwinger parameterization of the logarithm

$$\log M = -\int_{\times}^{\infty} \frac{\mathrm{d}\alpha}{\alpha} e^{-\alpha M} \ , \tag{85}$$

with a regularization of the divergence at $\alpha \to 0$ that we will leave unspecified for now. We will address this regularization through a suitable $i\varepsilon$ prescription below. Applying (85) to (83), we first see that the sum over weights in $\log Z_{\mathrm{STT}}$ can then be organized into representation traces

$$\sum_{(\lambda_L,\lambda_R)} e^{\alpha(\lambda_L+\lambda_R)} = \mathrm{Tr}_{\mathsf{R}_L}\left(e^{\alpha L_3}\right) \mathrm{Tr}_{\mathsf{R}_R}\left(e^{\alpha \bar{L}_3}\right) \ , \tag{86}$$

which are the characters of the non-standard representation in (49). This then leads to

$$\log(Z_{\mathrm{STT}}) = \frac{1}{2}\int_{\times}^{\infty}\frac{\mathrm{d}\alpha}{\alpha}\left(\sum_{\substack{N\in\mathbb{Z}\\|N|>\mathsf{s}}} e^{-|N|\alpha} + (2\mathsf{s}+1)e^{-\mathsf{s}\alpha}\right)\times \sum_{\mathcal{R}_{\Delta,\mathsf{s}}}\mathrm{Tr}_{\mathsf{R}_L}\left(e^{\alpha L_3}\right)\mathrm{Tr}_{\mathsf{R}_R}\left(e^{\alpha\bar{L}_3}\right)\,. \tag{87}$$

Similarly, we can introduce a Schwinger parameter to $\log Z_{\mathrm{zero}}$, where now (61) reads

$$\begin{aligned}\log Z_{\mathrm{zero}} &= \int_{\times}^{\infty}\frac{\mathrm{d}\alpha}{\alpha}\left(\sum_{n=0}^{\mathsf{s}-2}((n+1)^2-\mathsf{s}^2)e^{(j_L+j_R-n)\alpha} - \frac{\mathsf{s}^2}{2}e^{(j_L+j_R+1)\alpha}\right) + (j_{L/R}\to\bar{j}_{L/R})\\ &= \frac{1}{2}\int_{\times}^{\infty}\frac{\mathrm{d}\alpha}{\alpha}(e^{\alpha}-1)^2e^{-2\alpha}\left(\sum_{n=0}^{\mathsf{s}-2}((n+1)^2-\mathsf{s}^2)e^{-n\alpha} - \frac{\mathsf{s}^2}{2}e^{\alpha}\right)\times\\ &\qquad \sum_{\mathcal{R}_{\Delta,\mathsf{s}}}\mathrm{Tr}_{\mathsf{R}_L}\left(e^{\alpha L_3}\right)\mathrm{Tr}_{\mathsf{R}_R}\left(e^{\alpha\bar{L}_3}\right)\,.\end{aligned} \tag{88}$$

In the first line, we used $\Delta = -j_L - j_R$, and in the second, the characters (49) to cast this as a trace. The zero mode contribution (88) combines nicely with $\log(Z_{\mathrm{STT}})$ to give

$$\log Z_{\Delta,\mathsf{s}} = \frac{1}{2}\int_{\times}^{\infty}\frac{\mathrm{d}\alpha}{\alpha}\left(\frac{\cosh\left(\frac{\alpha}{2}\right)}{\sinh\left(\frac{\alpha}{2}\right)} - \mathsf{s}^2\sinh(\alpha)\right)\sum_{\mathcal{R}_{\Delta,\mathsf{s}}}\mathrm{Tr}_{\mathsf{R}_L}\left(e^{\alpha L_3}\right)\mathrm{Tr}_{\mathsf{R}_R}\left(e^{\alpha\bar{L}_3}\right)\,, \tag{89}$$

where we used

$$\sum_{n\in\mathbb{Z}} e^{-|n|\alpha} = \frac{\cosh\left(\frac{\alpha}{2}\right)}{\sinh\left(\frac{\alpha}{2}\right)}\,. \tag{90}$$

At this point we use the even parity of the integrand to regulate the divergence through the following $i\varepsilon$ prescription:

$$\int_{\times}^{\infty}\frac{\mathrm{d}\alpha}{\alpha}f(\alpha) := \lim_{\varepsilon\to 0}\frac{1}{4}\sum_{\pm}\int_{-\infty}^{\infty}\frac{\mathrm{d}\alpha}{\alpha\pm i\varepsilon}f(\alpha\pm i\varepsilon)\,. \tag{91}$$

This is a choice of regularization scheme for the one-loop determinant. Finally, under a change of integration variables $\alpha \to -i\alpha$ we can write the partition function as

$$\log Z_{\Delta,\mathsf{s}} = \frac{i}{8}\int_{\mathcal{C}} \frac{\mathrm{d}\alpha}{\alpha}\left(\frac{\cos\left(\frac{\alpha}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)} + 2\mathsf{s}^2\cos\left(\frac{\alpha}{2}\right)\sin\left(\frac{\alpha}{2}\right)\right)\sum_{\mathcal{R}_{\Delta,\mathsf{s}}} \mathrm{Tr}_{\mathsf{R}_L}\left(e^{i\alpha L_3}\right)\mathrm{Tr}_{\mathsf{R}_R}\left(e^{i\alpha \bar{L}_3}\right) \,, \tag{92}$$

where the contour $\mathcal{C}$ runs upwards along the imaginary α axis to the left and right of the divergence at the origin, as depicted in figure 1.

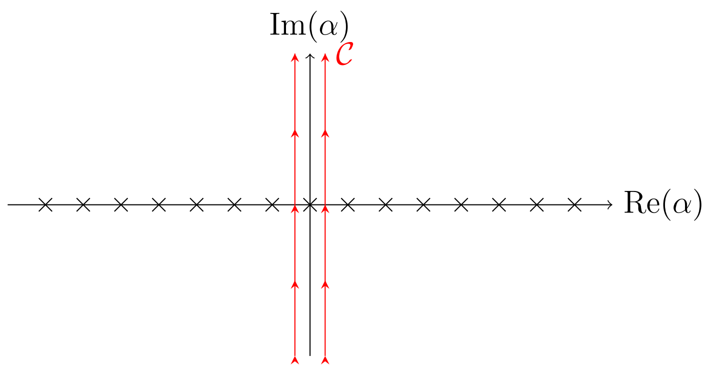

Figure 1: The integration contour regulating the $\alpha \to 0$ divergence.

As a final step we now rewrite the holonomies inside the traces to restore the background connections, arriving at (62) with $\mathbb{W}_{j_L,j_R}$ the spinning Wilson spool:

$$\begin{aligned}\mathbb{W}_{j_L,j_R}[a_L,a_R] := \frac{i}{2}\int_{\mathcal{C}} \frac{\mathrm{d}\alpha}{\alpha}\frac{\cos\left(\frac{\alpha}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)}\left(1+2\mathsf{s}^2\sin^2\left(\frac{\alpha}{2}\right)\right)\times \\ \sum_{\mathcal{R}_{\Delta,\mathsf{s}}} \mathrm{Tr}_{\mathsf{R}_L}\left(\mathcal{P}e^{\frac{\alpha}{2\pi}\oint_\gamma a_L}\right)\mathrm{Tr}_{\mathsf{R}_R}\left(\mathcal{P}e^{-\frac{\alpha}{2\pi}\oint_\gamma a_R}\right) \,,\end{aligned} \tag{93}$$

where above $\gamma = \gamma_{\text{hor.}}$ is a cycle wrapping the singular point corresponding to the horizon.

At this point let us make several comments:

- The spinning Wilson spool takes a form similar to that of the scalar spool found in [40,42]; importantly the "operator pieces" of the expression (93) have been organized into gauge invariant Wilson loop operators. The only modification the spinning spool brings is in the integration measure.

- The modification to the integration measure, proportional to s^2, will have the effect of lowering the degree of each pole at $\alpha \in 2\pi\mathbb{Z}$ by two. As we will shortly see, this effect reproduces the "edge partition function" of [37].
- Mathematically the holonomies corresponding to $\gamma_{\text{hor.}}$ appear in the one-loop determinant because they are sensitive to Δ on which Z_{STT} is treated as an meromorphic function. The physics behind this is clear: we are reproducing a one-loop determinant of massive fields. In the worldline quantum mechanics framework this corresponds to averaging over worldlines of a massive particle in the static patch. Such wordlines are timelike and rotate to a contour gauge equivalent to $\gamma_{\text{hor.}}$.

11.3 Testing the Wilson spool

We now uphold equations (62) and (93) by verifying that it indeed reproduces the correct path integral of a massive spinning field on S^3. There are several ways how to evaluate this path integral, and here we will focus on two approaches to use as a comparison. The first is to implement the DHS method traditionally. We evaluate this path integral by explicitly listing the quasinormal modes and applying DHS. The second approach we can compare to are the expressions found in [37], which cast the results in terms of $\mathfrak{so}(1,3)$ characters.

Since we are not turning on gravity ($G_N \to 0$), we evaluate the Wilson loop operators in (93) as characters with the appropriate holonomies in (72). Using the form of our non-standard representation character, (49), we then write

$$\log Z_{\Delta,\mathsf{s}} = -\frac{i}{8}\int_{\mathcal{C}} \frac{\mathrm{d}\alpha}{\alpha}\left(\frac{\cos\left(\frac{\alpha}{2}\right)}{\sin^3\left(\frac{\alpha}{2}\right)} + 2\mathsf{s}^2\frac{\cos\left(\frac{\alpha}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)}\right)e^{i\alpha(1-\Delta)} \,, \tag{94}$$

where we note that the sum over representations in $\mathcal{R}_{\Delta,\mathsf{s}}$ is already neatly packaged into our two contours. We recognize the first term in the parentheses of (94) as twice the on-shell scalar Wilson spool found in [40]. We evaluate both terms by deforming the α contours to run above and below the real α axis to pick up the residues at the poles lying at $2\pi\mathbb{Z}_{\neq 0}$. This deformation is depicted in figure 2.

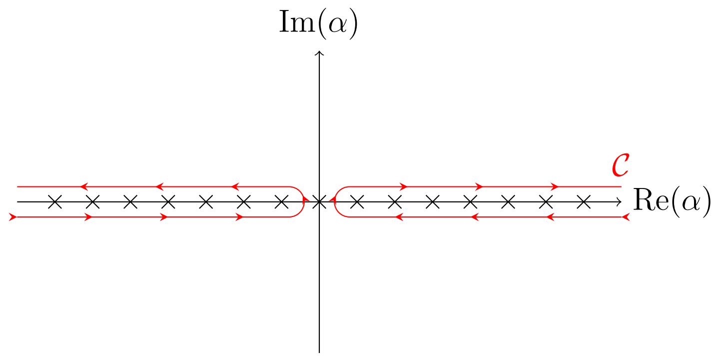

Figure 2: We deform the α integration contour to wrap the poles lying along the real axis.

Summing the towers of poles and expressing the $\mathfrak{su}(2)_{L/R}$ highest-weight labels in terms of μ and s as in (50), we write this as

$$\begin{aligned}\log Z_{\Delta,\mathsf{s}} &= \sum_{\pm}\left(-\frac{1}{4\pi^2}\mathrm{Li}_3\left(e^{\mp 2\pi\mu}\right) \mp \frac{\mu}{2\pi}\mathrm{Li}_2\left(e^{\mp 2\pi\mu}\right) - \frac{\mu^2+\mathsf{s}^2}{2}\mathrm{Li}_1\left(e^{\mp 2\pi\mu}\right)\right)\\ &= 2\log Z_{\mathrm{scalar}} - \frac{\mathsf{s}^2}{2}\sum_{\pm}\mathrm{Li}_1\left(e^{\mp 2\pi\mu}\right)\ ,\end{aligned} \tag{95}$$

where

$$\mathrm{Li}_p(z) = \sum_{n=1}^{\infty}\frac{z^n}{n^p} \tag{96}$$

is the polylogarithm. As mentioned above, $\log Z_{\mathrm{scalar}}$ is the one-loop determinant of a massive scalar field on S^3 with a mass set by $\ell_{\mathrm{dS}}^2 m^2 = \Delta(2-\Delta) = 1+\mu^2$. Up to an overall phase, unfixed by the quasinormal mode method, (95) matches the path integral of a massive spin-s field on S^3 via DHS in and the results reported in [37].

12 dS_3 gravity and Chern-Simons theory

In this section we will review the relation between Chern-Simons theory and three-dimensional general relativity with a positive cosmological constant, i.e., dS_3 gravity [11]. Our presentation follows the work of [36,37], which discusses the tree-level and loop relation between the two theories.

In the first half of this section we review geometrical properties of dS_3, and then classical (tree-level) aspects of the theories at hand. The second half is devoted to quantum aspects of dS_3. Our aim is to capture perturbative corrections to all orders in G_N via the Chern-Simons formulation. An important and novel portion of our analysis is to alter known methods to quantize Chern-Simons theory such that we meet the basic features that give Chern-Simons theory a gravitational interpretation. These alterations are to incorporate a non-trivial background connection and a complex level in the path integral. This is done in section 12.3, where we show how to adapt the derivation of the exact Chern-Simons path integral on S^3 via two different methods commonly used in the literature: Abelianisation and supersymmetric localization. In section 12.4 we discuss how these modifications are in perfect agreement with perturbative results in the metric formulation of the gravitational theory.

12.1 A primer on dS_3 spacetime

Three-dimensional Lorentzian de Sitter space can be realised as the hypersurface in $\mathbb{R}^{1,3}$ (what we will call embedding space) given by

$$\eta_{AB}X^A X^B = \ell^2 \ , \qquad \eta = \mathrm{diag}(-1,1,1,1) \ , \tag{97}$$

where $A, B \in \{0,1,2,3\}$ and ℓ is the dS_3 radius. Embedding space makes it manifest that the isometry group of dS_3 is $SO(1,3)$ which is generated by Killing vectors preserving this hypersurface

$$L_{AB} = X_A \frac{\partial}{\partial X^B} - X_B \frac{\partial}{\partial X^A} \ . \tag{98}$$

Different parametrizations of (97) give different coordinate patches of global de Sitter. A particular coordinate patch of interest to us in this paper is the coordinate patch available to an observer moving along a timelike geodesic, called the static patch. Due to the accelerated expansion of the spacetime, individual observers lose causal contact with increasing portions of space which become hidden behind a causal horizon. Thus the static patch covers a finite causal diamond, depicted as the blue region of the Penrose diagram found in figure 3. The parametrization for this static patch is given by

$$\begin{aligned} X^0 =& \ell \cos(\rho) \sinh(t/\ell) \ , \\ X^1 =& \ell \cos(\rho) \cosh(t/\ell) \ , \\ X^2 =& \ell \sin(\rho) \cos(\varphi) \ , \\ X^3 =& \ell \sin(\rho) \sin(\varphi) \ , \end{aligned} \tag{99}$$

for which the metric takes the following form

$$ds^2 = \eta_{AB} \mathrm{d}X^A \mathrm{d}X^B = -\cos^2\rho \, \mathrm{d}t^2 + \ell^2 \mathrm{d}\rho^2 + \ell^2 \sin^2\rho \, \mathrm{d}\varphi^2 \ . \tag{100}$$

The coordinates range over $t \in (-\infty, \infty)$, $\rho \in [0, \pi/2)$, $\varphi \in [0, 2\pi)$ which covers the right-wedge ("north-pole") of the static patch. The point $\rho = 0$ corresponds to the worldline of the observer defining the static patch, while $\rho = \pi/2$ corresponds to this observer's causal horizon. The metric (100) has an obvious time-like Killing vector, $\zeta = \partial_t$. This Killing vector is in fact the same as the "dilatation" Killing vector, $D = L_{03}$, of $\mathfrak{so}(1,3)$. As depicted in figure 3, this Killing vector is not globally time-like, however.

Euclidean de Sitter can be defined through the Wick rotation $X^0 = -iX^0_E$, which at the level of the static patch coordinates can be achieved through $t = -i\ell\tau$. The defining equation (97) then defines a three-sphere and indeed the Lorentzian static patch metric rotates to

$$\frac{ds^2}{\ell^2} = \cos^2\rho \, \mathrm{d}\tau^2 + \mathrm{d}\rho^2 + \sin^2\rho \, \mathrm{d}\varphi^2 \ , \tag{101}$$

which is the metric for S^3 in torus coordinates. Regularity at the horizon, $\rho = \pi/2$, requires the identification $\tau \sim \tau + 2\pi$, consistent

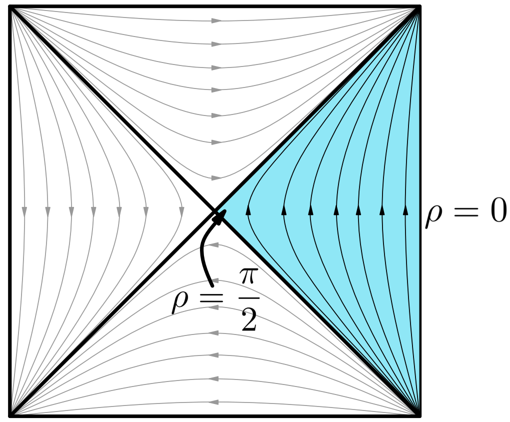

Figure 3: The Penrose diagram of dS_3. The blue region is the static patch covered by the coordinates (99). The observer defining the patch lies at $\rho = 0$. Their causal horizon lies at $\rho = \frac{\pi}{2}$. Also depicted are the flow lines of D which are time-like in this patch.

with it being a coordinate for S^3. The isometry group of Euclidean de Sitter is easily seen to be $SO(4) \simeq SU(2) \times SU(2)/\mathbb{Z}_2$. The $SU(2)$'s are the left and right group actions acting on S^3 which itself is diffeomorphic to $SU(2)$. As such we will label these two groups by subscripts L and R.

12.2 Chern-Simons theory and dS_3 gravity: tree-level

Now let us briefly review the Chern-Simons formulation of three-dimensional gravity [11]; see [36] for more details and complementary aspects. Much like the previous subsection, this portion is intended to lay out the necessary ingredients and to establish our notation.

As we noted in section 12.1, the splitting $\mathfrak{so}(4) \simeq \mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R$ of the isometry algebra of Euclidean dS_3 indicates that we will be interested in quantizing a pair of $SU(2)$ Chern-Simons theories

$$S = k_L S_{\text{CS}}[A_L] + k_R S_{\text{CS}}[A_R] \ , \tag{102}$$

where

$$S_{\text{CS}}[A] = \frac{1}{4\pi} \operatorname{Tr} \int_M \left(A \wedge \mathrm{d}A + \frac{2}{3} A^3 \right) \ , \tag{103}$$

and the trace is taken in the fundamental representation. The levels, $k_{L/R}$, will be non-integer and ultimately related to G_N^{-1}. Following [48], the correct framework for approaching this theory is through its complexification $\mathfrak{sl}(2,\mathbb{C})$ with $\mathfrak{su}(2)$ taken as a real form. As emphasized in that paper a decomposition of levels consistent with reality of the action and with Euclidean gravity with positive cosmological constant is given by[13]

$$k_L = \delta + is \ , \qquad k_R = \delta - is \ , \tag{104}$$

where $\delta \in \mathbb{Z}$ and $s \in \mathbb{R}$. As further discussed in [48], quantum effects lead to a finite renormalization of the levels

$$k_L \to r_L = k_L + 2 \ , \qquad k_R \to r_R = k_R + 2 \ . \tag{105}$$

Importantly these are renormalized in the same way and can be regarded as a renormalization of δ to $\hat{\delta} = \delta + 2$. For the rest of this section we will work with the renormalized levels.

To see that indeed this can be related to a theory of gravity, we can decompose the connections as

$$A_L = i\left(\omega^a + \frac{1}{\ell}e^a\right)L_a \ , \qquad A_R = i\left(\omega^a - \frac{1}{\ell}e^a\right)\bar{L}_a \ , \tag{106}$$

where $\{L_a\}$ and $\{\bar{L}_a\}$ generate $\mathfrak{su}(2)_L$ and $\mathfrak{su}(2)_R$, respectively.[14] It is natural to interpret e^a as the dreibein and $\omega^a = \frac{1}{2}\varepsilon^{abc}\omega_{bc}$ is the (dual) spin-connection. Indeed the action (102) is equivalent to

$$iS = -I_{\text{EH}} - i\hat{\delta} I_{\text{GCS}} \ , \tag{108}$$

where I_{EH} is the Einstein-Hilbert action written in first-order (or Palatini),

$$I_{\text{EH}} = -\frac{s}{4\pi\ell}\int \varepsilon_{abc}e^a \wedge \left(R^{bc} - \frac{1}{3\ell^2}e^b \wedge e^c\right) \ . \tag{109}$$

[13] Strictly speaking, $\mathfrak{sl}(2,\mathbb{C})$ Chern-Simons theory parameterized in this way describes Lorentzian gravity with positive cosmological constant and with $\mathfrak{sl}(2,\mathbb{R})$ as its real form. We obtain the Euclidean theory from the Wick rotation: i.e. (supposing the negative sign of the metric is associated with e^3) $e^3 \to ie^3$, $L_3 \to iL_3$.

[14] With respect to this basis, we have

$$\mathrm{Tr}(L_aL_b) = \mathrm{Tr}\big(\bar{L}_a\bar{L}_b\big) = \frac{1}{2}\delta_{ab} \ . \tag{107}$$

Here $R^{ab} = \varepsilon^{ab}{}_c \left(\mathrm{d}\omega^c - \frac{1}{2}\varepsilon^c{}_{de}\omega^d \wedge \omega^e\right)$ is the Riemann two-form, and we have a positive cosmological constant, $\Lambda = \ell^{-2}$. This identifies the imaginary part of the levels with Newton's constant

$$s = \frac{\ell}{4G_N} \ , \tag{110}$$

which establishes that the semi-classical regime of this theory is the large s limit. The second part of this action, once restricted to torsion-free spin connections, is the gravitational Chern-Simons action:

$$I_{\text{GCS}} = \frac{1}{2\pi}\int \mathrm{Tr}\left(\omega \wedge d\omega + \frac{2}{3}\omega^3\right) + \frac{1}{2\pi\ell^2}\int \mathrm{Tr}\left(e \wedge T\right) \ , \tag{111}$$

where $T^a = \mathrm{d}e^a - \varepsilon^a{}_{bc}\omega^b \wedge e^c$ is the torsion two-form.

It is also simple to establish a relation at the level of the equations of motion. The classical equations of motion of the Chern-Simons theories (102) are

$$\mathrm{d}A_L + A_L \wedge A_L = 0 \ , \qquad \mathrm{d}A_R + A_R \wedge A_R = 0 \ . \tag{112}$$

The sum and difference of these equations translate, in terms of e^a and ω^a, to the vacuum Einstein equation (with positive cosmological constant) and the vanishing of the torsion two-form:

$$R^{ab} = \frac{1}{\ell^2} e^a \wedge e^b \ , \qquad T^a = 0 \ . \tag{113}$$

These derivations establish a correspondence between classical solutions in the metric formulation of dS_3 gravity and classical solutions in the Chern-Simons theory.

Background configuration It will be important to make explicit how to cast Euclidean dS_3 space, i.e., the three-sphere, in the language of Chern-Simons theory. We start by constructing the appropriate flat connections on S^3, which we will coin (a_L, a_R). Given the metric (101), a convenient choice of dreibein is

$$e^1 = \ell \mathrm{d}\rho \ , \qquad e^2 = \ell \sin\rho \, \mathrm{d}\varphi \ , \qquad e^3 = \ell \cos\rho \, \mathrm{d}\tau \ , \tag{114}$$

with associated torsion-free spin connection

$$\omega^1 = 0 \ , \qquad \omega^2 = -\sin\rho\, \mathrm{d}\tau \ , \qquad \omega^3 = -\cos\rho\, \mathrm{d}\varphi \ . \tag{115}$$

From these expressions, we find

$$\begin{aligned} a_L =& iL_1 \mathrm{d}\rho + i\left(\sin\rho\, L_2 - \cos\rho\, L_3\right)\left(\mathrm{d}\varphi - \mathrm{d}\tau\right) = g_\rho^{-1} g_-^{-1} \mathrm{d}\left(g_- g_\rho\right) \ , \\ a_R =& - i\bar{L}_1 \mathrm{d}\rho - i\left(\sin\rho\, \bar{L}_2 + \cos\rho\, \bar{L}_3\right)\left(\mathrm{d}\varphi + \mathrm{d}\tau\right) = -\mathrm{d}\left(g_\rho g_+\right) g_+^{-1} g_\rho^{-1} \ , \end{aligned} \tag{116}$$

where we used (106). The second equality of each line above emphasizes that a_L and a_R are pure gauge with

$$g_\rho = e^{iL_1\rho} \ , \qquad g_\pm = e^{-iL_3(\tau\pm\varphi)} \ . \tag{117}$$

The connections (116) are locally flat, however they possess a point-like singularity. These are singularities for $\mathrm{d}\tau$ and $\mathrm{d}\varphi$ at $\rho = \pi/2$ and $\rho = 0$, respectively. These will be treated, as distributions, by

$$\mathrm{d}(\mathrm{d}\varphi) = \delta(\rho)\mathrm{d}\rho \wedge \mathrm{d}\varphi \ , \qquad \mathrm{d}(\mathrm{d}\tau) = -\delta(\rho - \pi/2)\mathrm{d}\rho \wedge \mathrm{d}\tau \ . \tag{118}$$

It is simple to extract the holonomies of a_L and a_R, which are important to record for later use. For any cycle γ wrapping the singular points of the connections, the connections possess holonomies

$$\mathcal{P}\exp\oint_\gamma a_L = g_\rho^{-1} e^{i2\pi L_3 \mathsf{h}_L} g_\rho \ , \qquad \mathcal{P}\exp\oint_\gamma a_R = g_\rho e^{i2\pi \bar{L}_3 \mathsf{h}_R} g_\rho^{-1} \ . \tag{119}$$

Requiring that the above group elements' action on $S^3 \simeq SU(2)$ itself is single-valued implies that $\mathsf{h}_L, \mathsf{h}_R \in \mathbb{Z}$ with either both even or both odd.[15] In particular, for cycles wrapping the causal horizon at $\rho = \frac{\pi}{2}$, we have

$$\mathsf{h}_L = 1 \ , \qquad \mathsf{h}_R = -1 \ . \tag{120}$$

Finally, we report on the value of the on-shell action for this background. A short calculation, which uses (118), shows that they

[15] Namely, this geometric action is in the fundamental representation. In that case $e^{i2\pi L_3 \mathsf{h}_{L/R}}$ is obviously the identity if $\mathsf{h}_{L/R}$ is even. If h_L and h_R are both odd this yields the group element $(-1,-1) \in SU(2)_L \times SU(2)_R$ which is also the identity inside the $\mathbb{Z}_2$ quotient.

have non-trivial action

$$r_L S_{\rm CS}[a_L] = -\pi r_L = -\pi\hat{\delta} - i\pi s \ , \qquad r_R S_{\rm CS}[a_R] = \pi r_R = \pi\hat{\delta} - i\pi s \ , \tag{121}$$

and thus

$$\begin{aligned} iS|_{\rm tree-level} &= i r_L S_{\rm CS}[a_L] + i r_R S_{\rm CS}[a_R] \\ &= \frac{\pi\ell}{2G_N} \ , \end{aligned} \tag{122}$$

where we used (110). This is the correct on-shell action for dS_3 [36]. Note that the gravitational Chern-Simons term of S^3 vanishes identically.

12.3 $SU(2)$ Chern-Simons theory: the partition function

We now turn to quantum aspects of dS_3 gravity. Our aim is to perform the gravitational path-integral about a fixed background S^3 saddle. This will be done in the Chern-Simons formulation of the theory which we introduced in section 12.2.

It is well-known that many observables in Chern-Simons theory can be evaluated exactly, i.e., to all orders in perturbation theory and also including non-perturbative effects. However, for our gravitational purpose, some caution is needed since these results are not always applicable due to the subtle relation between Chern-Simons and gravity. Here we will address these subtleties at the level of evaluating the path integral on S^3. In a nutshell, we will re-derive $Z_k[S^3]$ for $SU(2)$ Chern-Simons theory with level k, while allowing the level to be complex and also allowing non-trivial background connections. These are two key features that are persistent in the relation among the two theories, as we reviewed in the previous subsection.

Let us therefore begin by reviewing some basic facts and definitions. The Chern-Simons partition function over a three-manifold, M, is the path-integral

$$Z_k[M] = \int \frac{\mathcal{D}A}{\mathcal{V}}\, e^{ikS_{\rm CS}[A]} \tag{123}$$

over the action (103). Here $A = A^a L_a$ is to be regarded as a connection one-form of a principal $SU(2)$ bundle over M, where $\{L_a\}$ generates the $\mathfrak{su}(2)$ Lie algebra.[16] In the measure we indicate, schematically, a division by the gauge group as $1/\mathcal{V}$.

There are three remarks that will be important in what follows. First, the action (103) is clearly topological and the quantum theory itself is almost topological: its sole geometric input is a choice of framing which arises from regularizing the phase of Z_k. While there is no "rule" for establishing the framing, partition functions differing by choices of framing are related by well-established phases [34]. In this paper we will be careful to work with a fixed convention for the phase of Z_k.[17]

Second, our evaluation of (123) will cover complex values of the level k. In particular, our derivations will hold for a decomposition as in done in (104)-(105).

Third, we will incorporate a flat background connection to the path integral (123). To that end we will write

$$A = a + B \ . \tag{124}$$

Here a is a flat background connection on M—for most of our purposes $M = S^3$. It is important to emphasize at this point that, unlike what is typical for Chern-Simons theory quantized on S^3, we will not take the trivial background $a = 0$: such a background leads to a degenerate metric which is an unnatural saddle for a theory of gravity. Instead we want connections corresponding to a round S^3 metric, i.e., they will be (116), with holonomies (119)-(120), for each copy of the $SU(2)$ theory. The field B captures the quantum fluctuations that we will integrate over in the path integral shortly afterward.

Adapting exact results We now turn to the tools we will use for evaluating $Z_k[S^3]$. There are several ways to obtain $Z_k[S^3]$, and

[16] Note that given the form of (103), we are working with the convention that A is anti-Hermitian in the fundamental representation, i.e. $A^a_\mu \in i\mathbb{R}$. We will use this convention consistently throughout.

[17] Which is ultimately related to two-units away from so-called "canonical framing".

we do not attempt to describe them all. We selected methods for which the choice of background connections and background topology (and later, expectation values for Wilson loops) are tractable in the path integral of $SU(2)_k$ Chern-Simons theory. The two methods we will discuss in detail are:

Abelianisation. The process of Abelianisation was developed in [44, 47]. In a nutshell, it demonstrates how the non-Abelian Chern-Simons path integral can be reduced to a two-dimensional Abelian theory, under suitable conditions present on the manifold M.

Supersymmetric localization. As a complementary method, we will show how one obtains $Z_k[S^3]$ via supersymmetric localization techniques [46] (see also [45]). The biggest penalty here is the introduction of fermions in the path integral. Still, the outcome is robust and completely agrees with Abelianisation.

Both methods will be capable of successfully accommodating the features necessary for dS_3 gravity, and we stress that they report the same result (up to a trivial normalization). This subsection will summarize the main steps of both methods, highlighting in particular the features that need to be altered to accommodate gravity.

12.3.1 Abelianisation

Abelianisation is a powerful method for evaluating the Chern-Simons path integral for compact, connected and simply connected Lie groups with Lie algebra $\mathfrak{g}$ on particular types of three manifolds [44]. In particular, Abelianisation is useful when it is possible to choose the background M to be a circle fibration over a two dimensional base, Σ_g, i.e. $M = M_{(g,p)}$ can seen as a principal $U(1)$ bundle: $U(1) \to M_{(g,p)} \xrightarrow{\pi} \Sigma_g$ with monopole degree p. This is obviously relevant for us by considering $M = S^3$ as a Hopf fibration.

The approach in [44] reduces computations from non-Abelian Chern-Simons theory in three dimensions to computations in a two-dimensional Abelian q-deformed Yang-Mills theory on Σ_g in the following way. Using the geometry of M we decompose. the Chern-Simons connection $A \in \Omega^1(M, \mathfrak{g})$ into "vertical" and "horizontal" parts,

$$A = \boldsymbol{\sigma}\kappa + A_H \ , \tag{125}$$

with respect to a globally-defined real-valued one-form κ on M, and where $\boldsymbol{\sigma}$ is a $\mathfrak{g}$-valued scalar.[18] Abelianisation works by adding BRST-exact terms to the action to fix the gauge so that $\boldsymbol{\sigma}$ is a $U(1)$-invariant section of $M \times \mathfrak{g}$. This allows us to "push" $\boldsymbol{\sigma}$ down to the base, Σ_g, where it can be diagonalised, setting $\boldsymbol{\sigma} \in \mathfrak{t}$ (where $\mathfrak{t} \subset \mathfrak{g}$ is a Cartan subalgebra). The result of the gauge-fixing and the Abelianisation is that $\boldsymbol{\sigma}$ is $\mathfrak{t}$-valued and constant along the $U(1)$ fibers of M. The remaining fields can then be easily integrated out.

With an eye towards applying Abelianisation [47] to a background saddle relevant for gravity, we will expand the Chern-Simons action, (103), about a flat background connection, a, which is generically non-zero:

$$A = a + B \ , \qquad \mathrm{d}a + a \wedge a = 0 \ . \tag{126}$$

The difference between the Chern-Simons action for A and that of the background connection is then

$$S_{\mathrm{CS}}[A] - S_{\mathrm{CS}}[a] = \frac{k}{4\pi} \int \mathrm{Tr} \left(B \wedge \mathrm{d}_a B + \frac{2}{3} B^3 \right) \ , \tag{127}$$

where we have imposed flatness for a and dropped a total derivative. We have defined above a "background exterior derivative" acting on p-forms as

$$\mathrm{d}_a \omega_p = \mathrm{d}\omega_p + a \wedge \omega_p - (-1)^p \omega_p \wedge a \ . \tag{128}$$

We now will try to adapt Abelianisation to B, however we need to address the non-canonical kinetic term in (127). We will write the background connection in terms of a group element, g, as

$$a := g^{-1} \mathrm{d}g \ . \tag{129}$$

[18] We will differ in notation slightly from previous literature [44] where $\boldsymbol{\sigma}$ is called ϕ. This is to keep notation throughout the paper uniform and to make comparison of results clearer.

In writing above, it might be the case that g is not single-valued on M. This fact manifests itself in the possible existence of holonomies of a around the closed curve, γ, along a $U(1)$ fibre of M:

$$\mathcal{P}\exp\left(\oint_\gamma a\right) = g_f^{-1} g_i = \exp 2\pi\mathfrak{m} \ , \tag{130}$$

where $\mathfrak{m} \in \mathfrak{g}$. Performing the field redefinition

$$\tilde{B} := gBg^{-1} \ , \tag{131}$$

we can recover a canonical kinetic form for $\tilde{B}$:

$$\int \mathrm{Tr}\left(B\wedge \mathrm{d}_a B + \frac{2}{3}B^3\right) = \int \mathrm{Tr}\left(\tilde{B}\wedge \mathrm{d}\tilde{B} + \frac{2}{3}\tilde{B}^3\right) \ . \tag{132}$$

The cost of this, however, is that $\tilde{B}$ now possesses twisted boundary conditions: going around the cycle γ defining (130) gives

$$\tilde{B}_f = g_f B g_f^{-1} = e^{-2\pi\tilde{\mathfrak{m}}}\,\tilde{B}_i\, e^{2\pi\tilde{\mathfrak{m}}} \ , \tag{133}$$

where $e^{2\pi\tilde{\mathfrak{m}}} = g_i^{-1} e^{2\pi\mathfrak{m}} g_i$. We can state these boundary conditions more clearly by decomposing B into a root space compatible with $\tilde{\mathfrak{m}}$. That is writing $\tilde{B} = \tilde{B}^{(i)}T_i + \tilde{B}^{(\alpha)}T_\alpha$ where T_i is a basis of a Cartan subalgebra containing $\tilde{\mathfrak{m}}$ and T_α is a basis of the root space for this Cartan, then

$$\tilde{B}_f^{(i)} = \tilde{B}_i^{(i)} \ , \qquad \tilde{B}_f^{(\alpha)} = e^{-2\pi\alpha\cdot\tilde{\mathfrak{m}}}\tilde{B}_i^{(\alpha)} \ , \tag{134}$$

that is the fields aligned with the Cartan defined by $\tilde{\mathfrak{m}}$ retain their periodicity along γ while fields aligned with roots transform by phases. In terms of $\mathfrak{g} = \mathfrak{su}(2)$ we can write $\tilde{\mathfrak{m}} = i\mathsf{h}\, L_3$ in which case

$$\tilde{B}_f^{(3)} = \tilde{B}_i^{(3)} \ , \qquad \tilde{B}_f^{(\pm)} = e^{\mp i 2\pi\mathsf{h}}\tilde{B}_i^{(\pm)} \ . \tag{135}$$

At this point, we will procede to adapt the Abelianisation procedure [47] to $\tilde{B}$. To be explicit, we will specialize to the case where $M = S^3$.

Abelianisation on $\tilde{B}$

From here many of the steps mirror those in [47]. Namely, the connection $\tilde{B}$ is split into

$$\tilde{B} = B_\kappa + B_H := \boldsymbol{\sigma}\kappa + B_H \ , \tag{136}$$

and similarly the exterior derivative on M is split into a "horizontal" piece (that is, along the base, Σ) and an action along the fibre

$$\mathrm{d} = (\pi^*\mathrm{d}_\Sigma) + \kappa \wedge \mathscr{L}_\xi \ , \tag{137}$$

where $\mathscr{L}_\xi = \{\mathrm{d}, \iota_\xi\}$ is the Lie derivative along the fundamental vector field generating the $U(1)$ action. The action (132) can then be massaged to the form

$$\frac{k}{4\pi}\int \mathrm{Tr}\left(\tilde{B}\wedge \mathrm{d}\tilde{B} + \frac{2}{3}\tilde{B}^3\right) = \frac{k}{4\pi}\int \mathrm{Tr}\left(\boldsymbol{\sigma}^2\kappa\wedge\mathrm{d}\kappa + 2\boldsymbol{\sigma}\kappa\wedge\mathrm{d}B_H + B_H\wedge\kappa\wedge\mathcal{L}_\sigma B_{H}\right.$$

up to total derivative. Above we have also defined

$$\mathcal{L}_\sigma = \mathscr{L}_\xi + [\boldsymbol{\sigma}, \cdot] \ . \tag{139}$$

Fixing a gauge

The choice of gauge that allows the Abelianisation procedure to be applied is [19]

$$\mathscr{L}_\xi(\boldsymbol{\sigma}\kappa) = 0 \quad \Leftrightarrow \quad \mathscr{L}_\xi\boldsymbol{\sigma} = \iota_\xi \mathrm{d}\boldsymbol{\sigma} = 0 \ , \tag{140}$$

which states that $\boldsymbol{\sigma}$ is $U(1)$-invariant. We additionally gauge-fix that $\boldsymbol{\sigma}$ is valued in the Cartan, $\mathfrak{t}$:

$$\boldsymbol{\sigma}^{\mathfrak{l}} = 0 \ , \tag{141}$$

where $\mathfrak{g} = \mathfrak{t}\oplus\mathfrak{l}$. Without loss of generality we will choose this Cartan to align with that defined by the holonomy of the background

[19] Note that $\iota_\xi\kappa = 1$ and $\iota_\xi\mathrm{d}\kappa = 0$, so $\mathcal{L}_\xi\kappa = 0$.

connection, a, (130) so that $\boldsymbol{\sigma}$ remains single-valued on M. This gauge is fixed [47] by adding the following BRST-exact action

$$\int_M \mathrm{Tr}\left(E \star \boldsymbol{\sigma} + \bar{c} \star \mathcal{L}_\sigma c\right) = \int_M \mathrm{Tr}\left(E \star \boldsymbol{\sigma} + \kappa \wedge \mathrm{d}\kappa\ \bar{c}\mathcal{L}_\sigma c\right) \ , \quad (142)$$

where E is a Lagrange-multiplier, and c and $\bar{c}$ are ghosts. It is understood that the $U(1)$ invariant modes of these fields (i.e. those satisfying $\mathscr{L}_\xi E^{\mathfrak{t}} = \mathscr{L}_\xi c^{\mathfrak{t}} = \mathscr{L}_\xi \bar{c}^{\mathfrak{t}} = 0$) are not path-integrated [47]. We can now describe integrating out modes.

Effect of integrating over fields

- The part of B_H valued in the Cartan sub-algebra, denoted $B_H^{\mathfrak{t}}$, contains the $U(1)$-invariant modes $\hat{B}_H^{\mathfrak{t}}$ obeying $\mathscr{L}_\xi \hat{B}_H^{\mathfrak{t}} = 0$. Because they are $\mathfrak{t}$-valued, and since $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathfrak{t}}$ as a result of the gauge-fixing, (141), the term $\mathcal{L}_\sigma \hat{B}_H^{\mathfrak{t}}$ vanishes from (138). Thus the only term of (138) in which the fields $\hat{B}_H^{\mathfrak{t}}$ enter is

$$2\boldsymbol{\sigma}\kappa \wedge \mathrm{d}\hat{B}_H^{\mathfrak{t}} \ , \quad (143)$$

  and integrating over these fields imposes the constraint that $\boldsymbol{\sigma} =$ constant on M.

- The Gaussian integrals over the fields $B_H^{\mathfrak{l}}$, $B_H^{\mathfrak{t}}$, (where $B_H^{\mathfrak{t}}$ are not $U(1)$-invariant), and over the ghost fields $c^{\mathfrak{l}}$, $c^{\mathfrak{t}}$ give ratios of determinants [47]:

$$\frac{\mathrm{Det}\left(i\mathcal{L}_\sigma\right)_{\Omega^0_H(S^3,\mathfrak{l})}\mathrm{Det}'\left(i\mathscr{L}_\xi\right)_{\Omega^0_H(S^3,\mathfrak{t})}}{\mathrm{Det}^{1/2}\left(\star\kappa\wedge i\mathcal{L}_\sigma\right)_{\Omega^1_H(S^3,\mathfrak{l})}\mathrm{Det}'^{1/2}\left(\star\kappa\wedge i\mathscr{L}_\xi\right)_{\Omega^1_H(S^3,\mathfrak{t})}} \ . \quad (144)$$

  At this point we need to pause to emphasize that these determinants are, in principle, to be taken over fields with twisted boundary conditions defined by (134) along the $U(1)$ fibre. These boundary conditions do not affect the determinants over the Cartan-valued fields. For fields living in the α root space, given the form of $\mathcal{L}_\sigma$, (139), the effect of the twisted boundary conditions, (134), is to shift the eigenvalues of $i\mathcal{L}_\sigma$

$$2\pi n + i\alpha\cdot\boldsymbol{\sigma} \quad \to \quad 2\pi n + i\alpha\cdot\boldsymbol{\sigma} - i2\pi\alpha\cdot\tilde{\mathfrak{m}} \ , \qquad n \in \mathbb{Z} \ . \quad (145)$$

The absolute value of the ratio of these determinants can then be evaluated in manner similar to [47] to give

$$\mathrm{Abs}\left[\frac{\mathrm{Det}\,(i\mathcal{L}_\sigma)_{\Omega^0_H(S^3,\mathfrak{l})}\,\mathrm{Det}'\,(i\mathscr{L}_\xi)_{\Omega^0_H(S^3,\mathfrak{t})}}{\mathrm{Det}^{1/2}\,(\star\kappa\wedge i\mathcal{L}_\sigma)_{\Omega^1_H(S^3,\mathfrak{l})}\,\mathrm{Det}'^{1/2}\,(\star\kappa\wedge i\mathscr{L}_\xi)_{\Omega^1_H(S^3,\mathfrak{t})}}\right]=T_{S^1}(\boldsymbol{\sigma}-2\pi\tilde{\mathfrak{m}})\ , \tag{146}$$

where

$$T_{S^1}(\boldsymbol{x})=\det{}_{\mathfrak{l}}\,(1-\mathrm{Ad}e^{\boldsymbol{x}})=\prod_{\alpha>0}4\sin^2(i\alpha\cdot\boldsymbol{x}/2) \tag{147}$$

is the Ray-Singer torsion of the connection along the fibre. For $\mathfrak{g}=\mathfrak{su}(2)$ we can set $\boldsymbol{\sigma}=-i2\pi\sigma L_3$ (where σ is now a real constant) and $\tilde{\mathfrak{m}}=i\mathsf{h}L_3$; for this choice we have

$$T_{S^1}(\boldsymbol{\sigma}-2\pi\tilde{\mathfrak{m}})=4\sin^2\left(\pi(\sigma+\mathsf{h})\right)\ . \tag{148}$$

We have stated this result somewhat generally, however it is useful to keep in mind that for the backgrounds of interest for this paper, h will always be integer valued (as discussed at the end of section 12.2) and so (146) reduces to $T_{S^1}=4\sin^2(\pi\sigma)$. This is of course consistent with (134) reducing to single-valued boundary conditions when $\mathsf{h}\in\mathbb{Z}$. The phase of the determinants (144) can be defined through a regularized eta invariant and is responsible for the renormalization of level $k\to r=k+2$ for $\mathfrak{g}=\mathfrak{su}(2)$ [47].

The end result of this is the expression[20] of the Chern-Simons partition function as a simple integral over a Cartan-valued field $\boldsymbol{\sigma}$

$$Z_k[S^3]=e^{irS_{\mathrm{CS}}[a]}\int_{\mathfrak{t}}\mathrm{d}\boldsymbol{\sigma}\,T_{S^1}(\boldsymbol{\sigma}-2\pi\tilde{\mathfrak{m}})\exp\left[-i\frac{r}{4\pi}\mathrm{Tr}\boldsymbol{\sigma}^2\right]\ , \tag{149}$$

up to inessential overall normalization and constant (r-independent) phase. We fix this normalization/phase by fiat. More explicitly setting $\boldsymbol{\sigma}=-i2\pi\sigma L_3$ and $\tilde{\mathfrak{m}}=i\mathsf{h}\,L_3$, we normalize Z_k as

$$Z_k[S^3]=e^{irS_{\mathrm{CS}}[a]}\int_{\mathbb{R}}\mathrm{d}\sigma\sin^2\left(\pi(\sigma+\mathsf{h})\right)e^{i\frac{\pi}{2}r\,\sigma^2}\ . \tag{150}$$

[20] We shift the level of the classical action trivially when $S_{\mathrm{CS}}[a]\in\mathbb{Z}$.

While the integral over σ begins life along the real axis, this integral is Gaussian and we can formally define it through appropriate contour deformations depending on the phase of r. It is simple to perform the integral (letting $\mathsf{h} \in \mathbb{Z}$)

$$Z_k[S^3] = e^{irS_{\text{CS}}[a]}\, e^{i\phi} \sqrt{\frac{2}{r}} \sin\frac{\pi}{r} \ , \tag{151}$$

where the phase of Z_k

$$\phi = \frac{3\pi}{4} - \frac{\pi}{r} = \frac{\pi}{6}\mathfrak{c} + \frac{\pi}{4} \ , \qquad \mathfrak{c} \equiv \frac{3(r-2)}{r} \ , \tag{152}$$

can be identified with a framing phase [34] (two-units away from canonical framing) plus a phase stemming from a σ contour rotation.[21]

12.3.2 $\mathcal{N} = 2$ supersymmetric localization

We now describe an alternative route to the exact calculation of the Chern-Simons partition through localization techniques. We will focus particularly on $\mathcal{N} = 2$ supersymmetric localization [46]. One benefit of this approach is that much of the basic machinery has been established with a non-trivial background connection, a, in mind allowing a fairly straightforward incorporation of $a \neq 0$. However: the situations with non-trivial background connections have historically arisen on manifolds with interesting topology (e.g. Lens spaces) and many of the explicit results for S^3 have been established with $a = 0$. Below we collect and synthesize these results in a way that is useful for dS_3 gravity.

Before jumping in, let us also make the following brief comments. Supersymmetry in the context of de Sitter is a contentious subject, with much of the difficulty arising from realizing unitary representations of the supersymmetry algebra in Lorentzian signature. In this paper we will take a somewhat agnostic stance on this topic:

[21] This latter phase is entirely a by-product of our conventions and does not occur in usual Chern-Simons formulas. However we will give it a gravitational interpretation below.

by working directly in Euclidean signature, we are ultimately discussing $SU(2)_k$ Chern-Simons theory on S^3 whose $\mathcal{N}=2$ supersymmetric extension is well-established. We use the existence of this symmetry to our advantage to localize the path integral all while verifying that the extension to $\mathcal{N}=2$ does not alter essential features of the original partition function. Ultimately, however, this localization will simply verify the results of section 12.3.1.

Let us set the stage and collect the necessary background. Much of what follows mirrors the friendly review [45]. The vector multiplet of three dimensional $\mathcal{N}=2$ gauge theory is given by fields

$$\{A_\mu,\ \boldsymbol{\sigma},\ \mathfrak{D},\ \lambda,\ \bar{\lambda}\}\ , \tag{153}$$

where A is a $\mathfrak{g}=\mathfrak{su}(2)$ connection, $\boldsymbol{\sigma}$, $\mathfrak{D}$ are scalars,[22] and λ, $\bar{\lambda}$ are Dirac spinors. Note the same field content as in (2). All fields are $\mathfrak{g}$-valued and by convention we will take them all to be anti-Hermitian,[23] with supersymmetry variations parameterized by two Grassmann variables $\bar{\epsilon}$ and ϵ as specified in [45]. The supersymmetric Chern-Simons action is

$$S_{\text{SCS}}=\frac{1}{4\pi}\int \operatorname{Tr}\left(A\wedge \mathrm{d}A+\frac{2}{3}A^3\right)-\frac{1}{4\pi}\int d^3x\sqrt{g}\,\operatorname{Tr}\left(\bar{\lambda}\lambda-2\mathfrak{D}\boldsymbol{\sigma}\right)\ , \tag{154}$$

and enters the path integral multiplied by the level k

$$Z_k^{\text{SCS}}[S^3]=\int\frac{\mathcal{D}A}{\mathcal{V}_G}\mathcal{D}\bar{\lambda}\mathcal{D}\lambda\mathcal{D}\mathfrak{D}\mathcal{D}\boldsymbol{\sigma}\,e^{ikS_{\text{SCS}}}\ . \tag{155}$$

To make subsequent notation less cumbersome, we will drop the "$[S^3]$" above with it understood that we are always working on the three sphere. Note that on a formal level, as far as the function dependence on k is concerned, the addition of the auxiliary fields in the multiplet does not alter Z_k^{SCS} with respect to the non-supersymmetric path integral, Z_k [45].

The deformation that allows us to localize the path integral Z_k^{SCS} is the super Yang-Mills action

---

[22] The $\boldsymbol{\sigma}$ appearing here is a priori a different field than what appeared in section 12.3.1. We give it the same name because, ultimately, it will play the same role in the final result.

[23] In comparison to the notation of [45], a field here is related to a field there by $\Phi_{\text{here}}=i\Phi_{\text{there}}$.

$$
\begin{aligned}
S_{\text{SYM}} =& -\int \operatorname{Tr}\left(\frac{1}{2}F \wedge \star F + D\boldsymbol{\sigma} \wedge \star D\boldsymbol{\sigma}\right) \\
& - \int d^3x\sqrt{g}\operatorname{Tr}\left(\frac{1}{2}\left(\mathfrak{D}+\boldsymbol{\sigma}\right)^2 + \frac{i}{2}\bar{\lambda}\gamma^{\mu}D_{\mu}\lambda - \frac{1}{2}\bar{\lambda}[\boldsymbol{\sigma},\lambda] - \frac{1}{4}\bar{\lambda}\lambda\right) \ ,
\end{aligned}
\tag{156}
$$

where D_μ is the gauge covariant derivative and γ_μ can be taken to be the Pauli-matrices acting on spinor indices. S_{SYM} is itself a super-derivative and therefore Q-exact. Adding this to the path integral with coefficient $\mathtt{t}$, i.e.,

$$
Z_k^{\text{SCS+SYM}}(\mathtt{t}) = \int \frac{\mathcal{D}A}{\mathcal{V}_G}\mathcal{D}\bar{\lambda}\mathcal{D}\lambda\mathcal{D}D\mathcal{D}\boldsymbol{\sigma}\, e^{ikS_{\text{SCS}}-\mathtt{t}S_{\text{SYM}}} \ , \tag{157}
$$

is then innocuous: $Z_k^{\text{SCS+SYM}}(\mathtt{t}) = Z_k^{\text{SCS}}$ for any $\mathtt{t}$, including in the limit $\mathtt{t}\to\infty$ where the path-integral localizes on the saddle of S_{SYM}.

Localization locus

In the $\mathtt{t}\to\infty$ limit, the path integral localizes on the following equations of motion

$$
F = 0 \ , \qquad D\boldsymbol{\sigma} = \mathrm{d}\boldsymbol{\sigma} + [A,\boldsymbol{\sigma}] = 0 \ , \qquad \mathfrak{D} + \boldsymbol{\sigma} = 0 \ . \tag{158}
$$

We expand the solutions around a flat connection $a = g^{-1}\mathrm{d}g$, for some group element g. Again, g may not be single-valued and a may possess a holonomy, à la (130),

$$
\mathcal{P}\exp\left(\oint_\gamma a\right) = \exp(2\pi\mathfrak{m}) \ , \tag{159}
$$

for some curve γ. The other fields that have saddle solutions to (158) are given by

$$
\boldsymbol{\sigma}_0^{(g)} = g^{-1}\boldsymbol{\sigma}_0 g \ , \qquad \mathfrak{D}_0 = -\boldsymbol{\sigma}_0^{(g)} \ , \qquad \lambda_0 = 0 \ , \qquad \bar{\lambda}_0 = 0 \ , \tag{160}
$$

for $\boldsymbol{\sigma}_0$ a constant element of $\mathfrak{g}$. We require $\boldsymbol{\sigma}_0^{(g)}$ to be single-valued and so the constant element defining the saddle must obey

$$
[\mathfrak{m}, \boldsymbol{\sigma}_0] = 0 \ . \tag{161}
$$

With this we can take $\boldsymbol{\sigma}_0$ to be in a Cartan subalgebra containing $\mathfrak{m}$. We will scale fluctuations as

$$\begin{gathered}A = a + \frac{1}{\sqrt{\mathsf{t}}}B \;, \qquad \boldsymbol{\sigma} = \boldsymbol{\sigma}_0^{(g)} + \frac{1}{\sqrt{\mathsf{t}}}\hat{\sigma} \;, \qquad \mathfrak{D} = -\boldsymbol{\sigma}_0^{(g)} + \frac{1}{\sqrt{\mathsf{t}}}\hat{\mathfrak{D}} \;, \\ \lambda = \frac{1}{\sqrt{\mathsf{t}}}\hat{\lambda} \;, \qquad \bar{\lambda} = \frac{1}{\sqrt{\mathsf{t}}}\hat{\bar{\lambda}} \;,\end{gathered} \tag{162}$$

and perturb the action (154) around the saddle (160) as $\mathsf{t} \to \infty$. The leading contribution to S_{SCS} is

$$\lim_{\mathsf{t}\to\infty} S_{\mathrm{SCS}} = S_{\mathrm{CS}}[a] - \frac{\mathrm{Vol}(S^3)}{2\pi}\mathrm{Tr}\boldsymbol{\sigma}_0^2 \;. \tag{163}$$

Meanwhile the leading contribution to $\mathsf{t}\,S_{\mathrm{SYM}}$ is

$$\begin{aligned}\mathsf{t}\,S_{\mathrm{SYM}} = &-\int \mathrm{Tr}\left(\frac{1}{2}\mathrm{d}_a B \wedge \star \mathrm{d}_a B + (\mathrm{d}_a\hat{\sigma} + [B, \boldsymbol{\sigma}_0^{(g)}]) \wedge \star(\mathrm{d}_a\hat{\sigma} + [B, \boldsymbol{\sigma}_0^{(g)}])\right) \\ &- \int d^3x\sqrt{g}\mathrm{Tr}\left(\frac{1}{2}\left(\hat{\mathfrak{D}} + \hat{\sigma}\right)^2 + \frac{i}{2}\hat{\bar{\lambda}}\gamma^\mu D_\mu^{(a)}\hat{\lambda} - \frac{1}{2}\hat{\bar{\lambda}}[\boldsymbol{\sigma}_0^{(g)}, \hat{\lambda}] - \frac{1}{4}\hat{\bar{\lambda}}\hat{\lambda}\right) + \ldots\end{aligned} \tag{164}$$

where d_a is the background exterior derivative (128), and $D_\mu^{(a)}$ is the spinor covariant derivative with fixed connection, a. This action can be made Gaussian under a suitable gauge fixing and then path integrated in standard fashion. We briefly highlight the main points of that procedure below, but many details can be found [45] and references therein.

Gauge choice

We will choose the gauge[24]

$$\mathcal{G}_a[B] = \mathrm{d}_a^\dagger B \equiv - \star \mathrm{d}_a \star B = 0 \;, \tag{165}$$

whose Fadeev-Popov determinant, $\Delta_a[B]$, can be enacted through adding ghosts $\bar{c}, c$:

[24] This gauge fixing is only consistent when a is a flat-connection, implying that $\mathrm{d}_a^2 = 0$ defines an equivariant cohomology.

$$Z_k^{\text{SCS+SYM}} = e^{ikS_{\text{CS}}[a]} \int \mathrm{d}\boldsymbol{\sigma}_0\, e^{-i\frac{k}{2\pi}\text{vol}(M_3)\text{Tr}\boldsymbol{\sigma}_0^2} \times \int \frac{\mathcal{D}B}{\mathcal{V}} \mathcal{D}\hat{\bar{\lambda}}\mathcal{D}\hat{\lambda}\mathcal{D}\hat{\mathfrak{D}}\mathcal{D}\hat{\sigma}\mathcal{D}\bar{c}\mathcal{D}c\,\delta[\mathrm{d}_a^\dagger B]\, e^{-\mathtt{t}S_{\text{SYM}}-S_{\text{ghost}}}\ , \tag{166}$$

with action

$$S_{\text{ghost}} = \int \text{Tr}\left(\bar{c}\wedge\star\mathrm{d}_a^\dagger\mathrm{d}_{a+\mathtt{t}^{-1/2}B}\, c\right) = \int d^3x\sqrt{g}\,\text{Tr}\left(\bar{c}\wedge\star\Delta_a^0\, c\right) + O(\mathtt{t}^{-1/2})\ , \tag{167}$$

where $\Delta_a^0 = \mathrm{d}_a^\dagger\mathrm{d}_a$ is the a-deformed Laplacian acting on $\mathfrak{g}$-valued zero-forms.[25] The ghost determinants simply cancel the determinants from $\hat{\mathfrak{D}}$ and $\hat{\sigma}$ (as well as a Jacobian from $\delta[\mathrm{d}_a^\dagger B]$) and so we arrive at the promised Gaussian path-integral:

$$Z_k^{\text{SCS+SYM}} = e^{ikS_{\text{CS}}[a]} \int \mathrm{d}\boldsymbol{\sigma}_0\, e^{-i\frac{k}{2\pi}\text{vol}M_3\text{Tr}\boldsymbol{\sigma}_0^2}\, Z_{\text{Gauss}}[\boldsymbol{\sigma}_0]\ , \tag{168}$$

with

$$Z_{\text{Gauss}}[\boldsymbol{\sigma}_0] := \int [\mathcal{D}B]_{\text{kerd}_a^\dagger}\mathcal{D}\hat{\bar{\lambda}}\mathcal{D}\hat{\lambda}\, e^{\frac{1}{2}\int\text{Tr}(\mathrm{d}_aB)^2 + \int\text{Tr}[B,\boldsymbol{\sigma}_0^{(g)}]^2 - \int\text{Tr}\left(\frac{i}{2}\hat{\bar{\lambda}}\gamma_\mu D_\mu^{(a)}\hat{\lambda} - \frac{1}{2}\hat{\bar{\lambda}}[\boldsymbol{\sigma}_0^{(g)},\hat{\lambda}] - \frac{1}{4}\hat{\bar{\lambda}}\hat{\lambda}\right)}\ . \tag{169}$$

One loop determinants

The remaining task is now to compute the one loop determinants from integrating out $\{B,\hat{\bar{\lambda}},\hat{\lambda}\}$. Recalling the procedure from section 12.3.1, the first step is to "canonicalize" the kinetic terms by redefining the fluctuating fields $\{B,\hat{\bar{\lambda}},\hat{\lambda}\}\to\{\tilde{B},\tilde{\bar{\lambda}},\tilde{\lambda}\}$ via

$$\Phi = g^{-1}\tilde{\Phi}g\ , \qquad \Phi\in\{B,\hat{\bar{\lambda}},\hat{\lambda}\}\ . \tag{170}$$

As a result the one loop integration becomes ostensibly simpler

$$Z_{\text{Gauss}}[\boldsymbol{\sigma}_0] = \int [\mathcal{D}\tilde{B}]_{\text{kerd}^\dagger}[\mathcal{D}\tilde{\bar{\lambda}}\mathcal{D}\tilde{\lambda}]e^{\frac{1}{2}\int\text{Tr}\left(\mathrm{d}\tilde{B}\right)^2 + \int\text{Tr}[\tilde{B},\boldsymbol{\sigma}_0]^2 - \int\text{Tr}\left(\frac{i}{2}\tilde{\bar{\lambda}}\gamma^\mu\nabla_\mu\tilde{\lambda} - \frac{1}{2}\tilde{\bar{\lambda}}[\boldsymbol{\sigma}_0,\tilde{\lambda}] - \frac{1}{4}\tilde{\bar{\lambda}}\tilde{\lambda}\right)}\ , \tag{171}$$

[25] It is tacit in (166) that the zero modes of $\bar{c}, c$ under Δ_a^0 are not to be integrated over.

however, as we saw earlier, this is at the cost of twisting the fields along the curve γ:

$$\tilde{\Phi}_f = \exp(-2\pi\tilde{\mathfrak{m}})\tilde{\Phi}_i \exp(2\pi\tilde{\mathfrak{m}}) \ . \tag{172}$$

In terms of a root space decomposition $\tilde{\Phi} = \tilde{\Phi}^{(i)}T_i + \tilde{\Phi}^{(\alpha)}T_\alpha$, then (172) reads

$$\tilde{\Phi}_f^{(i)} = \tilde{\Phi}_i^{(i)} \ , \qquad \tilde{\Phi}_i^{(\alpha)} = \exp(-2\pi\alpha\cdot\tilde{\mathfrak{m}})\tilde{\Phi}_i^{(\alpha)} \ , \tag{173}$$

where T_i is a basis of the Cartan subalgebra containing $\boldsymbol{\sigma}_0$ and $\mathfrak{m}$, and T_α are a basis of the α root space; $\{\tilde{\Phi}^{(i)}, \tilde{\Phi}^{(\alpha)}\}$ are honest fields and not elements of $\mathfrak{g}$. The one loop determinants of the $\mathcal{N}=2$ vector multiplet with twisted boundary conditions, (173), turns out to be

$$Z_{\text{Gauss}}[\boldsymbol{\sigma}_0] = \prod_{\alpha>0} \frac{\sin\left(i\alpha\cdot(\boldsymbol{\sigma}_0 - 2\pi\tilde{\mathfrak{m}})\right)}{\pi^2} \ . \tag{174}$$

Again, we have written Z_{Gauss} rather generally, but for the purposes of this paper, we can let $\tilde{\mathfrak{m}} = i\mathsf{h}\,L_3$ with $\mathsf{h}\in\mathbb{Z}$ in which case it reduces to the usual expression for the Ray-Singer torsion in terms of $\boldsymbol{\sigma}_0 = -i\sigma L_3$, i.e. $Z_{\text{Gauss}} = \sin^2(\pi\sigma)/\pi^2$. The phase of Z_{Gauss} again is responsible for the renormalization $k\to r = k+2$ as explained in [45]. Gathering these results and fixing the normalization, we find again the familiar integral, (150),

$$Z_k^{\text{SCS}} = e^{irS_{\text{CS}}[a]} \int_{\mathbb{R}} \mathrm{d}\sigma\, \sin^2(\pi(\sigma+\mathsf{h}))\, e^{i\frac{\pi}{2}r\sigma^2} \ , \tag{175}$$

where again the integration contour over σ should be deformed depending on the phase of r.

12.4 Chern-Simons theory and dS$_3$ gravity: all loop path integral

Having assembled these exact results, we now address the gravity path integral about the S^3 saddle which, given the discussion in section 12.2, we path integral quantize as the product of Chern-Simons theories

$$\mathcal{Z}_{\text{grav}}[S^3] = Z_{k_L}[S^3] Z_{k_R}[S^3] \ . \tag{176}$$

Utilizing the exact partition function in the form of (150) and (175), the gravity path integral can be written as

$$\mathcal{Z}_{\text{grav}}[S^3] = e^{ir_L S_{\text{CS}}[a_L] + ir_R S_{\text{CS}}[a_R]}$$
$$\times \int \mathrm{d}\sigma_L \mathrm{d}\sigma_R \, e^{i\frac{\pi}{2}r_L\sigma_L^2 + i\frac{\pi}{2}r_R\sigma_R^2} \sin^2\left(\pi(\sigma_L + \mathsf{h}_L)\right) \sin^2\left(\pi(\sigma_R + \mathsf{h}_R)\right) \ . \tag{177}$$

The background holonomies, $\mathsf{h}_{L,R}$, being ± 1, decouple from this integral which is Gaussian and can be performed exactly:

$$\mathcal{Z}_{\text{grav}}[S^3] = e^{2\pi s} \left(i \, e^{-i\frac{\pi}{r_L} - i\frac{\pi}{r_R}} \right) \frac{2}{\sqrt{r_L r_R}} \sin\left(\frac{\pi}{r_L}\right) \sin\left(\frac{\pi}{r_R}\right) \ . \tag{178}$$

Let us briefly dissect the phase in the parenthesis: the overall i stems from integration contour deformations. Given the identifications (104) and (105), the σ_L integral (177) is already damped, however the σ_R integral is anti-damped. Deforming the σ_R integration contour to a damped region accounts for this i; this is wholly analogous to "Polchinski's phase" [49] arising from deforming the integration contour of the conformal mode in the gravitational path-integral. The exponents are a combined framing phase.

At this point, this result, (178), is not surprising. Up to a total phase, our expression for $\mathcal{Z}_{\text{grav}}$ has been arrived at before through analytic continuation of the celebrated $SU(2)$ Chern-Simons partition function [36, 37] . Here we have simply justified those analytic continuations, and incorporated the background contributions of $a_{L/R}$, through Abelianisation and supersymmetric localization.

It is instructive to cast (178) in gravitational terms. We recall from section 12.2 that

$$r_{L/R} = \hat{\delta} \pm i\, s \ , \qquad s = \frac{\ell}{4G_N} \ . \tag{179}$$

First, let's inspect the case when $\hat{\delta} = 0$, i.e., in the absence of the gravitational Chern-Simons term (111). The path integral reads

$$\begin{aligned}\log\left(\mathcal{Z}_{\text{grav}}[S^3]_{\hat{\delta}=0}\right) &= \log\left(\frac{8G_N}{i\ell}\exp\left(\frac{\pi}{2}\frac{\ell}{G_N}\right)\sinh^2\left(4\pi\frac{G_N}{\ell}\right)\right)\\ &= \frac{\pi}{2}\frac{\ell}{G_N} + 3\log\left(\frac{4G_N}{\ell}\right) + \log\left(2\pi^2 i\right) + \frac{16\pi^2}{3}\frac{G_N^2}{\ell^2} + \cdots\ .\end{aligned} \tag{180}$$

The first line should be viewed as an exact expression in G_N for the fixed background manifold S^3. The second line is the loop expansion as $G_N \to 0$, where rather curiously the two-loop correction vanishes. In [37] the real part of $\log \mathcal{Z}_{\text{grav}}$ was shown to match the graviton one loop determinant on an S^3 background at one-loop order (that is, $O(G_N^0)$ and $O(\log G_N)$). See [36] for analogous results and matching on Lens spaces.

Next, when $\hat{\delta} \neq 0$, the structure of the results is slightly different. Casting (178) in terms of the gravitational variables, we find that the perturbative expansion is

$$\begin{aligned}\log \mathcal{Z}_{\text{grav}}[S^3] =&\log\left(e^{2\pi s}\left(i\, e^{-2\pi i\frac{\hat{\delta}}{\hat{\delta}^2+s^2}}\right)\frac{2}{\sqrt{\hat{\delta}^2+s^2}}\left|\sin\left(\frac{\pi}{\hat{\delta}+is}\right)\right|^2\right)\\ =&2\pi s + \log\left(\frac{2\pi^2 i}{s^3}\right) + \frac{\pi^2}{3}\frac{1}{s^2} - 2\pi i\frac{\hat{\delta}}{s^2} - \frac{3}{2}\frac{\hat{\delta}^2}{s^2} + \cdots\ .\end{aligned} \tag{181}$$

Here we kept s as the coupling for clarity, instead of replacing G_N. Again, the first line is an exact result, and in the second line we are doing an expansion in G_N (or equivalently large s). It is important to mention that in this expansion $\hat{\delta}$ is kept fixed in the limit $G_N \to 0$. The additional purely imaginary term in the perturbative expansion (as compared to (180)) comes from the framing phase, and it only starts to contribute to $\mathcal{Z}_{\text{grav}}$ at $O(G_N^2)$. Furthermore, this framing term vanishes if the coefficient of I_{GCS} renormalizes to $\hat{\delta} = 0$.

13 Looping matter in

In this section we address our central question: how to couple matter to gravity in the Chern-Simons formulation, and how to quantify this coupling beyond leading order in the gravity path integral. We will provide a concise and precise answer to these questions.

To that end, we will first cover the unitary representation theory of $\mathfrak{so}(1,3)$, the isometry algebra of Lorentzian dS_3. We will pay particular attention to representations corresponding to massive scalar particles. Afterwards we will turn our attention towards the Euclidean rotation, $\mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R$, and see how to define non-standard representations of $\mathfrak{su}(2)$ mimicking the single-particle representations of $\mathfrak{so}(1,3)$. This is presented in section 13.1.

The end result of these analyses is to then propose a gauge invariant observable built from non-standard representations, an object we will call the Wilson spool (a nomenclature that will become duly clear below). This is an object that incorporates quantum gravity effects to a free massive scalar field minimally coupled to dS_3 gravity. This object can be intuitively motivated from the world line quantum mechanics of a single particle moving on S^3, however we will construct the spool bottom-up through a formula for one loop determinants as a product over quasi-normal modes. Lastly, enjoying the fruit of our labors from retooling Abelianisation and localization, we show how the spool can be evaluated order-by-order in G_N perturbation theory to give controlled and finite quantum gravity corrections to scalar one-loop determinants.

13.1 Single-particle representation theory

Unitary representations of the $SO(1,3)$ de Sitter isometry group describe single-particle states propagating on dS_3 spacetime [50–53] (see [54] for multi-particle states). For the purposes of a Chern-Simons description of Euclidean dS_3 it will be useful to cast quantities in terms of the Euclidean isometry algebra $\mathfrak{so}(4) \simeq \mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R$ and make use of its split structure. Recently, it was shown how to mimic the essential features of light scalars ($\mathsf{m}^2\ell^2 < 1$) with novel representations of the $\mathfrak{so}(4)$ algebra [55].

It is important to note that although $\mathfrak{so}(1,3)$ and $\mathfrak{so}(4)$ share a common complexification, the representations constructed in [55] do not analytically continue to standard representations of $\mathfrak{so}(1,3)$. Instead they furnish a representation of quasi-normal modes of Lorentzian dS_3, as opposed to single-particle states.

As noted by several authors [37, 38, 55–57], quasi-normal modes provide a rather useful basis for computing a number of physical quantities. Particularly, in the context of Chern-Simons gravity, classical Wilson lines carrying these representations have been shown to describe Green's functions and other gravitational probes in dS_3 [55]. Already in [56] it was emphasized that the quasi-normal mode spectrum of four dimensional de Sitter is unitarily realized in a non-standard way. We will connect to (and extend) these ideas further below.

In this subsection we will briefly review both the unitary representation theory of $SO(1,3)$ as well as the non-standard representations of $SO(4)$, emphasizing important differences in how they are realized. In doing so, we will also extend the construction of non-standard representations constructed in [55] to incorporate heavy scalar fields ($\mathsf{m}^2\ell^2 > 1$) in a unified way.

14 Coupling quantum matter to $\mathbf{dS}_3$ quantum gravity

Having constructed the spinning Wilson spool on S^3 in section 11, in this section we will promote it to an off-shell object with the aim of incorporating quantum effects from gravity around S^3. More concretely, we posit the following: let A_L and A_R be $\mathfrak{su}(2)_L$ and $\mathfrak{su}(2)_R$ connections respectively, yielding a non-degenerate dreibein $e^a = -i\,(A_L^a - A_R^a)$ with an associated metric geometry (M_3, g_{M_3}) that is topologically equivalent to the round S^3. Then the partition function of massive spin-s fields minimally coupled to g_{M_3} is determined by the spinning Wilson spool given by (93) with $a_{L/R}$ replaced with $A_{L/R}$:

$$\log Z_{\Delta,\mathsf{s}}[g_{M_3}] = \frac{1}{4}\mathbb{W}_{j_L,j_R}[A_L, A_R] \ . \tag{182}$$

For scalar fields it was argued in [40] that the manipulations leading to the Wilson spool remain valid for off-shell geometries. This argumentation relied on the expression of the Laplacian $-\nabla^2_{g_{M_3}}$ as a Casimir of local $\mathfrak{su}(2)_{L/R}$ action (see appendix D of [40]). While this remains true when acting on spin-s fields (via the transverse-traceless condition), various manipulations leading to (93) make the leap to (182) less rigorous than for scalar fields. This includes both manipulations relying on the integer nature of the holonomies, $\mathsf{h}_{L/R}$, as well as the explicit form of Z_{zero}. Barring a detailed analysis of normalizable zero modes for the ghost and Stückelberg fields appearing in $Z_{\Delta,\mathsf{s}}$ on generic three-geometries, at this point we take (182) as a proposal. This proposal is upheld on the grounds that it utilizes and generalizes naturally the gauge-invariant observables in Chern-Simons theory (namely its Wilson loops); these observables appear in a form that reduces to twice[26] the scalar path integral when $\mathsf{s} \to 0$ and, as we will see in the following section, continues to work for negative cosmological constant.

With this proposal in hand, the Wilson spool gives us a concrete route for calculating finite G_N effects[27] to $\log Z_{\Delta,\mathsf{s}}$. That is we can consider

$$\begin{aligned}\big\langle \log Z_{\Delta,\mathsf{s}}\big\rangle_{\text{grav}} &:= \int [\mathcal{D}g_{M_3}]_{S^3}\ e^{-I_{\text{EH}}[g_{M_3}]} \log Z_{\Delta,\mathsf{s}}[g_{M_3}] \\ &\equiv \int [\mathcal{D}A_L\mathcal{D}A_R]_{S^3}\, e^{iS[A_L,A_R]}\left(\frac{1}{4}\mathbb{W}_{j_L,j_R}[A_L, A_R]\right) \ ,\end{aligned} \tag{183}$$

[26] This stems simply from the fact that a STT field contains roughly two scalars corresponding to the polarizations with s and $-\mathsf{s}$ and has nothing to do with the ambiguities of a compact space and possible zero modes. This counting is obviously not continuous as $\mathsf{s} \to 0$.

[27] To be clear on scope: here we mean to all orders in G_N perturbation theory about the S^3 saddle. We do not consider topology change or other non-perturbative effects here.

where the second line follows from the rewriting of the gravitational variables and action (in a first-order formalism) as two Chern-Simons path integrals. Additionally, in the first line, $[\mathcal{D}g_{M_3}]_{S^3}$ indicates that we integrate over metric geometries topologically equivalent to S^3, and in the second line, $[\mathcal{D}A_L\mathcal{D}A_R]_{S^3}$ indicates we perform the Chern-Simons path integral on the base S^3 topology.

The power of the second line of (183) lies in the breadth of techniques for evaluating Wilson loop observables for $SU(2)$ Chern-Simons theories on the three sphere [44–47,62]. In [40] it was shown how to adapt two such techniques, Abelianisation [44,47] and localization through a $\mathcal{N}=2$ supersymmetric extension [45,46], for the evaluation of Wilson loop expectation values in light of the features unique to Chern-Simons gravity: non-trivial background connections, complex levels, and non-standard representations appearing in Wilson loop operators. While prima facie these are two very different techniques, they both lead to the expression of a Wilson loop expectation value as a deformation of its on-shell (i.e. flat) background value integrated over a single modulus:

$$\int [\mathcal{D}A] e^{ik\, S_{\mathrm{CS}}[A]}\, \mathrm{Tr}_{\mathsf{R}}\, \mathcal{P} e^{\oint_\gamma A} = e^{irS_{\mathrm{CS}}[a]} \int_{\mathbb{R}} \mathrm{d}\sigma\, \sin^2(\pi\sigma) e^{ir\,\frac{\pi}{2}\sigma^2} \chi_{\mathsf{R}}\left(\sigma + \mathsf{h}\right)\,, \tag{184}$$

where $r = k+2$ is the renormalized level, a is the on-shell background connection, h is its holonomy about the contour γ, and $\chi_{\mathsf{R}}(z) = \mathrm{Tr}_{\mathsf{R}}\, e^{i2\pi z\, L_3}$ is the $\mathfrak{su}(2)$ representation character. Equation (184) holds for complex levels and non-standard $\mathfrak{su}(2)$ representations, including those found in section 10. Applying this to (183), we can write

$$\begin{aligned} \left\langle \log Z_{\Delta,\mathsf{s}} \right\rangle_{\mathrm{grav}} &= \frac{i}{8} e^{ir_L S_{\mathrm{CS}}[a_L] + ir_R S_{\mathrm{CS}}[a_R]} \\ &\int \mathrm{d}\sigma_L \mathrm{d}\sigma_R\, e^{ir_L \frac{\pi}{2}\sigma_L^2 + ir_R\frac{\pi}{2}\sigma_R^2} \sin^2(\pi\sigma_L)\sin^2(\pi\sigma_R)\times \\ &\sum_{\mathcal{R}_{\Delta,\mathsf{s}}} \int_{\mathcal{C}} \frac{\mathrm{d}\alpha}{\alpha} \frac{\cos\left(\frac{\alpha}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)} \left(1 + 2\mathsf{s}^2 \sin^2\left(\frac{\alpha}{2}\right)\right)\times \\ &\chi_{\mathsf{R}_L}\left(\frac{\alpha}{2\pi}(1+\sigma_L)\right) \chi_{\mathsf{R}_R}\left(\frac{\alpha}{2\pi}(1-\sigma_R)\right)\,, \end{aligned} \tag{185}$$

where we used the off-shell version of the spool in (93). Upon writing

$$r_L = \hat{\delta} + is \ , \qquad r_R = \hat{\delta} - is \ , \qquad s = \frac{\ell_{\rm dS}}{4G_N} \ , \qquad \hat{\delta} \in \mathbb{Z} \ , \quad (186)$$

we posit that (185) is exact (that is, holding to all orders) in G_N perturbation theory about the S^3 saddle.

While this claim has power in principle, in practice (185) is complicated as an integral. By rescaling $\sigma_{L/R} \to r_{L/R}^{-1/2}\sigma_{L/R}$, we can proceed systematically in $\ell_{\rm dS}^{-1}G_N$ perturbation theory which amounts to a Taylor expansion in $\sigma_{L/R}$ of non-Gaussian pieces of the integrand in (185). At any order in this expansion the Gaussian integrals over $\sigma_{L/R}$ can be performed. This leaves the contour integral over α which can be deformed to pick up its poles (which remain at $2\pi\mathbb{Z}_{\neq 0}$ at each order of perturbation theory) in a similar spirit as our computation in section 11.3. This procedure completely mirrors that outlined for the scalar Wilson spool in [40] and can be efficiently implemented on a computer algebra system.

To illustrate this concretely, we evaluate $\left\langle \log Z_{\Delta,\mathsf{s}} \right\rangle_{\rm grav}$ normalized by the gravitational path integral to the first non-zero order of $\ell_{\rm dS}^{-1}G_N$ perturbation theory. The gravitational path integral on S^3 has a close form given by

$$\begin{aligned}\mathcal{Z}_{\rm grav} =& e^{ir_L S_{\rm CS}[a_L] + ir_R S_{\rm CS}[a_R]} \int {\rm d}\sigma_L {\rm d}\sigma_R \, e^{ir_L \frac{\pi}{2}\sigma_L^2 + ir_R\frac{\pi}{2}\sigma_R^2} \sin^2(\pi\sigma_L)\sin^2(\pi\sigma_R) \\ =& e^{2\pi s}\left(ie^{-i2\pi\frac{\hat{\delta}}{\hat{\delta}^2+s^2}}\right)\frac{2}{\sqrt{\hat{\delta}^2+s^2}}\left|\sin\left(\frac{\pi}{\hat{\delta}+is}\right)\right|^2 \ , \qquad (187)\end{aligned}$$

as outlined in [40] (see also [36,37,63]). By implementing the rescaling $\sigma_{L/R} \to r_{L/R}^{-1/2}\sigma_{L/R}$ in (185), to leading order in the coupling we find

$$\frac{\left\langle \log Z_{\Delta,\mathsf{s}} \right\rangle_{\rm grav}}{\mathcal{Z}_{\rm grav}} = \log Z_{\Delta,\mathsf{s}}[S^3] + \left(\frac{G_N}{\ell_{\rm dS}}\right)^2 [\log Z_{\Delta,\mathsf{s}}]_{(2)} + \dots \ , \quad (188)$$

where the dots correspond to subleading corrections in $\ell_{\rm dS}^{-1}G_N$, and the first non-trivial correction is

$$[\log Z_{\Delta,\mathsf{s}}]_{(2)} = \sum_{\pm}\sum_{i=0}^{3} z_i^{(2)}[\pm\mu,\mathsf{s}]\mathrm{Li}_{-i}\left(e^{\mp 2\pi\mu}\right) \ , \tag{189}$$

with

$$\begin{aligned}
z_0^{(2)}[\mu,\mathsf{s}] =& -\left(\frac{16\pi}{3} - \frac{8}{\pi} - 8i\hat{\delta}\right)\mu^3 - \left(16\pi - \frac{144}{\pi} - 24i\hat{\delta} + 16\pi\mathsf{s}^2 - \frac{216}{\pi}\mathsf{s}^2 - 24i\hat{\delta}\mathsf{s}^2\right)\\
z_1^{(2)}[\mu,\mathsf{s}] =& \left(\frac{16\pi^2}{3} + 12 - i8\pi\hat{\delta}\right)\mu^4 + \left(16\pi^2 - 276 - i24\pi\hat{\delta} - 432\mathsf{s}^2\right)\mu^2\\
& - (16\pi^2 - 372 - i24\pi\hat{\delta})\mathsf{s}^4 + (16\pi^2 - 588 - i24\pi\hat{\delta})\mathsf{s}^2 - 48\,,\\
z_2^{(2)}[\mu,\mathsf{s}] =& -\frac{24\pi}{5}\mu^5 + \left(104\pi + 96\pi\mathsf{s}^2\right)\mu^3 + \left(96\pi - 168\pi\mathsf{s}^4 + 792\pi\mathsf{s}^2\right)\mu\,,\\
z_3^{(2)}[\mu,\mathsf{s}] =& -\left(8\pi^2 - 16\pi^2\mathsf{s}^2\right)\mu^4 - \left(32\pi^2 + 40\pi^2\mathsf{s}^4 + 160\pi^2\mathsf{s}^2\right)\mu^2\\
& + 8\pi^2\mathsf{s}^6 + 24\pi^2\mathsf{s}^4 - 32\pi^2\mathsf{s}^2\ .
\end{aligned} \tag{190}$$

We note that taking $\mathsf{s} \to 0$ doubles the corrections to the scalar one-loop determinant [40] as expected. At leading order in a large μ expansion (which amounts to a large mass expansion while holding s fixed)

$$[\log Z_{\Delta,\mathsf{s}}]_{(2)} = -\frac{48\pi\,\mu^5}{5}e^{-2\pi\mu} + \left(24 - \frac{16\pi^2}{3} + 32\pi^2\mathsf{s}^2 - i16\pi\hat{\delta}\right)\mu^4\,e^{-2\pi\mu} + \dots\ , \tag{191}$$

where we have kept to next-to-leading order in the large mass expansion where the first contribution from spin appears.

It is natural at this stage to interpret this as a renormalization of the mass[28] of the spin-s field. Writing an expansion for the renormalized mass

$$\mu_R = \mu + \left(\frac{G_N}{\ell_{\mathrm{dS}}}\right)^2 \delta_\mu^{(2)} + \dots\ , \tag{192}$$

we note to $O(G_N^2\ell_{\mathrm{dS}}^{-2})$

$$\log Z_{\Delta,\mathsf{s}} = \log Z_{\Delta_R,\mathsf{s}} - \pi\frac{\cosh(\pi\mu_R)}{\sinh(\pi\mu_R)}(\mu_R^2 + \mathsf{s}^2)\left(\frac{G_N^2}{\ell_{\mathrm{dS}}^2}\right)\delta_\mu^{(2)} + \dots\ , \tag{193}$$

[28] Importantly the spin of the field does not get renormalized.

where $\Delta_R = 1 - i\mu_R$. Interpreting the corrections due to quantum gravity as a mass renormalization then sets as a renormalization condition

$$\frac{\left\langle \log Z_{\Delta,\mathsf{s}} \right\rangle_{\text{grav}}}{\mathcal{Z}_{\text{grav}}} \stackrel{!}{=} \log Z_{\Delta_R,\mathsf{s}} \ , \tag{194}$$

with G_N held fixed. This then determines the renormalized mass as

$$\begin{aligned} \delta_\mu^{(2)} =& \frac{1}{\pi}\frac{\tanh(\pi\mu_R)}{\mu_R^2+\mathsf{s}^2}\left[\log Z_{\Delta,\mathsf{s}}\right]_{(2)} \\ =& -\frac{48}{5}\mu_R^3\, e^{-2\pi\mu_R} + \left(\frac{24}{\pi} - \frac{16\pi}{3} + 32\pi\mathsf{s}^2 - i16\hat{\delta}\right)\mu_R^2\, e^{-2\pi\mu_R} + \dots \ , \end{aligned} \tag{195}$$

where in the second line we have written the leading and next to leading terms in a large mass expansion. Note that the magnitude of the leading term is s independent and consistent[29]with the leading mass renormalization for scalar fields [40].

We note that this is not the only renormalization condition that one may choose to set. We could instead to renormalize Newton's constant, $G_N \to G_{N,R}$, while holding the mass of the spinning field fixed. To illustrate this, we will set $\hat{\delta} = 0$ and consider the following renormalization condition:

$$\begin{aligned} \frac{\int [\mathcal{D}g] e^{-I_{\text{EH}}[g]} Z_{\Delta,\mathsf{s}}[g]}{Z_{\Delta,\mathsf{s}}[g_{S^3}]}\Big|_{G_N} =& \mathcal{Z}_{\text{grav}}\left(1 + \frac{\left\langle \log Z_{\Delta,\mathsf{s}} \right\rangle_{\text{grav}}}{\mathcal{Z}_{\text{grav}}} - \log Z_{\Delta,\mathsf{s}}[g_{S^3}] + \dots\right)\Big|_{G_N} \\ \stackrel{!}{=}& \ \mathcal{Z}_{\text{grav}}\Big|_{G_{N,R}} \ , \end{aligned} \tag{197}$$

or equivalently

$$\frac{\mathcal{Z}_{\text{grav}}\big|_{G_{N,R}}}{\mathcal{Z}_{\text{grav}}\big|_{G_N}} = 1 + \left(\frac{G_{N,R}}{\ell_{\text{dS}}}\right)^2 \left[\log Z_{\Delta,\mathsf{s}}\right]_{(2)} + \dots \ . \tag{198}$$

[29] In [40] the renormalization condition was taken (tacitly) as

$$\frac{\left\langle \log Z_{\Delta_R,\text{scalar}} \right\rangle_{\text{grav}}}{\mathcal{Z}_{\text{grav}}} \stackrel{!}{=} \log Z_{\Delta,\text{scalar}} \ , \tag{196}$$

which leads to an overall minus sign with respect to (195).

Note we have normalized by $Z_{\Delta,\mathsf{s}}[g_{S^3}]$ in (197) as this leading term decouples from metric fluctuations. It would be responsible for a renormalization of the cosmological constant, but does not mediate the gravitational self-interactions relevant for renormalizing the coupling, G_N. Writing

$$G_N = G_{N,R} \left(1 + \delta^{(2)}_{G_N} \left(\frac{G_{N,R}}{\ell_{\text{dS}}} \right)^2 + \dots \right) , \tag{199}$$

we find from expanding (187) and comparing to the right-hand side of (198)

$$\begin{aligned} \delta^{(2)}_{G_N} =& -\frac{1}{3} \left[\log Z_{\Delta,\mathsf{s}} \right]_{(2)} \\ =& \frac{16\pi}{5} \mu^5 \, e^{-2\pi\mu} - \left(8 - \frac{16\pi^2}{9} + \frac{32\pi^2}{3} \mathsf{s}^2 \right) \mu^4 \, e^{-2\pi\mu} + \dots , \end{aligned} \tag{200}$$

where again we have written the leading and next-to-leading terms in a large mass expansion.

It is worth noting that the results of this section, both the renormalization of the field mass, (195), and the renormalization of G_N, (198), are novel. To our knowledge, corresponding computations have not been carried out in the metric formulation of dS_3.[30] To this end, the results of this section provide concrete and testable predictions of the Chern-Simons formulation of gravity.

15 Spinning spool on AdS_3

To illustrate the general utility of the spinning Wilson spool constructed in section 11.2, as well as bolster the **Conditions I & II** that lead to it in section 11.1, we can repeat this construction in an AdS_3 background. An expression for the one-loop determinant of massive spinning fields on a BTZ black hole as a classical Wilson spool was conjectured in [42] based on an extension of the result

[30] In the absence of a cosmological constant, a three loop computation involving gravitons was done in [64]. It seems feasible that one could apply that approach to dS_3 and verify (195) and (198).

for a massive scalar field. In this section we show how to derive the spinning spool on AdS_3 from the principles set out in section 11.1 and demonstrate that the final expression accords with the general result (28).

We wish to compute

$$Z_{\Delta,\mathsf{s}} = \det\left(-\nabla^2_{(\mathsf{s})} + \ell^2_{\mathrm{AdS}}\bar{m}^2_{\mathsf{s}}\right)^{-\frac{1}{2}} \tag{201}$$

with $\nabla^2_{(\mathsf{s})}$ being the Laplace-Beltrami operator acting on spin-s STT tensors. The background geometry entering in (201) will be the rotating BTZ black hole with AdS_3 radius ℓ_{AdS}. The effective mass $\ell^2_{\mathrm{AdS}}\bar{m}^2_{\mathsf{s}}$ is related to the standard mass, m^2, via [65]

$$\ell^2_{\mathrm{AdS}}\bar{m}^2_{\mathsf{s}} = \ell^2_{\mathrm{AdS}}m^2 + \mathsf{s}(\mathsf{s}-3) \ . \tag{202}$$

We recall that the standard mass is related to the conformal dimension of a dual primary through [66]

$$\ell^2_{\mathrm{AdS}}m^2 = (\Delta+\mathsf{s}-2)(\Delta-\mathsf{s}) \ . \tag{203}$$

$Z_{\Delta,\mathsf{s}}$ is completely captured by this one-loop determinant over STT tensors: in contrast to the previous section, there is no additional zero mode product as AdS_3 is non-compact.

The isometry group of Lorentzian AdS_3 is $SO(2,2)$ with an algebra isomorphic to $\mathfrak{sl}(2,\mathbb{R})_L \oplus \mathfrak{sl}(2,\mathbb{R})_R$ which we will take to be generated by $\{L_0, L_\pm\}$ and $\{\bar{L}_0, \bar{L}_\pm\}$, respectively. As done for dS_3, it is useful to briefly establish our conventions regarding the description of single-particle states living in AdS_3 as representations of $\mathfrak{sl}(2,\mathbb{R})_L \oplus \mathfrak{sl}(2,\mathbb{R})_R$. Lowest-weight representations R^{LW}_j are defined by a lowest-weight state, $|j,0\rangle_{\text{LW}}$, which is annihilated by L_- and labelled by its L_0 eigenvalue:

$$L_-|j,0\rangle_{\text{LW}} = 0 \ , \qquad\qquad L_0|j,0\rangle_{\text{LW}} = j|j,0\rangle_{\text{LW}} \ . \tag{204}$$

All other states in R^{LW}_j are generated by the action of L_+ acting on $|j,0\rangle_{\text{LW}}$. This representation has a character

$$\chi_{j,\text{LW}}(z) = \mathrm{Tr}_{\mathsf{R}^{\text{LW}}_j}\left(e^{i2\pi z L_0}\right) = \frac{e^{i\pi z(2j-1)}}{2\sinh(-i\pi z)} \ . \tag{205}$$

Highest-weight representations R_j^{HW} are defined by a highest-weight state, $|j,0\rangle_{\text{HW}}$, which is instead annihilated by L_+. We will take by convention its L_0 eigenvalue to be $-j$:

$$L_+|j,0\rangle_{\text{HW}} = 0 \ , \qquad\qquad L_0|j,0\rangle_{\text{HW}} = -j|j,0\rangle_{\text{HW}} \ . \tag{206}$$

All other states of R_j^{HW} are generated by the action of L_- on $|j,0\rangle_{\text{HW}}$. This representation has a character

$$\chi_{j,\text{HW}}(z) = \text{Tr}_{\mathsf{R}_j^{\text{HW}}}\left(e^{i2\pi z L_0}\right) = \frac{e^{-i\pi z(2j-1)}}{2\sinh(i\pi z)} \ . \tag{207}$$

Our conventions have been chosen so that the Casimir of both highest and lowest-weight representations is given by

$$c_2^{\mathfrak{sl}(2,\mathbb{R})}|j,p\rangle_{\text{LW/HW}} = j(j-1)|j,p\rangle_{\text{LW/HW}} \ , \qquad\qquad \forall \ \ |j,p\rangle_{\text{LW/HW}} \in \mathsf{R}_j^{\text{LW/HW}} \ . \tag{208}$$

Returning to the functional determinant (201), we now want to implement the steps in section 11.1: the "mass-shell condition" and **Conditions I & II**. To start, we write the Laplace-Beltrami operator in terms of the Casimir of the isometries of AdS_3 [66]:

$$2c_2^{\mathfrak{sl}(2,\mathbb{R})_L} + 2c_2^{\mathfrak{sl}(2,\mathbb{R})_R} = \nabla^2_{(\mathsf{s})} + \mathsf{s}(\mathsf{s}+1) \ . \tag{209}$$

In this language, a pole contributing to $Z^2_{\Delta,\mathsf{s}}$ corresponds to a state $|\psi\rangle$ in a representation of $\mathfrak{sl}(2,\mathbb{R})_L \oplus \mathfrak{sl}(2,\mathbb{R})_R$ satisfying

$$\left(2c_2^{\mathfrak{sl}(2,\mathbb{R})_L} + 2c_2^{\mathfrak{sl}(2,\mathbb{R})_R}\right)|\psi\rangle = \left(\Delta(\Delta-2) + \mathsf{s}^2\right)|\psi\rangle \ . \tag{210}$$

This is the "mass-shell condition" and it is satisfied for pairs of highest and lowest-weight representations labeled by

$$j_L = \frac{\Delta \pm \mathsf{s}}{2} \ , \qquad j_R = \frac{\Delta \mp \mathsf{s}}{2} \ . \tag{211}$$

Note that representations labeled by (211) with Δ replaced by $\bar{\Delta} = 2-\Delta$ share the same Casimir. However, unlike in dS_3, we are forced to choose either Δ or $\bar{\Delta}$ due to Dirichlet boundary conditions tacitly imposed on solutions contributing to (201).[31] For

[31] Dirichlet boundary conditions also exclude $\mathfrak{sl}(2,\mathbb{R})$ representations that are neither highest nor lowest-weight from contributing to the one-loop determinant.

what follows we will assume, without loss of generality, that Δ corresponds to a normalizable massive spin-s solution.[32] Fixing $j_{L/R}$ as in (211) with the upper sign, we can then have pole contributions from any representation appearing in

$$\mathcal{R}_{\Delta,\mathsf{s}} = \mathcal{R}^{\text{HW}}_{\Delta,\mathsf{s}} \cup \mathcal{R}^{\text{LW}}_{\Delta,\mathsf{s}} = \left\{\mathsf{R}^{\text{HW}}_{j_L}\otimes\mathsf{R}^{\text{HW}}_{j_R}, \mathsf{R}^{\text{HW}}_{j_R}\otimes\mathsf{R}^{\text{HW}}_{j_L}\right\}\cup\left\{\mathsf{R}^{\text{LW}}_{j_L}\otimes\mathsf{R}^{\text{LW}}_{j_R}, \mathsf{R}^{\text{LW}}_{j_R}\otimes\mathsf{R}^{\text{LW}}_{j_L}\right\} \ . \tag{212}$$

For scalars, $\mathsf{s}=0$, we have the reduced set

$$\mathcal{R}_{\Delta,\text{scalar}} = \mathcal{R}^{\text{HW}}_{\Delta,\text{scalar}} \cup \mathcal{R}^{\text{LW}}_{\Delta,\text{scalar}} = \left\{\mathsf{R}^{\text{HW}}_{j}\otimes\mathsf{R}^{\text{HW}}_{j}\right\}\cup\left\{\mathsf{R}^{\text{LW}}_{j}\otimes\mathsf{R}^{\text{LW}}_{j}\right\} \ . \tag{213}$$

We now apply the group theoretic conditions of section 11.1 to the representations contributing to (201). **Conditions I & II** as they are stated in section 11.1 can be readily applied to the one-loop determinant on BTZ, and they respectively imply:

Condition I. Single-valued solutions. In Euclidean signature, BTZ is a solid torus that has two cycles that characterize its global properties. Requiring solutions to be single-valued around the contractible thermal cycle, γ_{th}, of the BTZ geometry requires $\mathsf{s}\in\mathbb{Z}$. Requiring solutions to be single-valued around the non-contractible spatial cycle, γ_{sp}, of the BTZ geometry requires a weight $(\lambda_L,\lambda_R)\in\mathsf{R}_L\otimes\mathsf{R}_R$ to satisfy

$$\lambda_L \mathsf{h}_L - \lambda_R \mathsf{h}_R \in \mathbb{Z} \ , \tag{214}$$

where $\mathsf{h}_{L,R}$ are holonomies around the non-contractible cycle

$$\mathsf{h}_L = -\frac{1}{\tau} \ , \qquad \mathsf{h}_R = -\frac{1}{\bar{\tau}} \ , \tag{215}$$

and τ is the modular parameter defining the geometry.

Condition II. Globally regular solutions. A representation $\mathsf{R}_L\otimes\mathsf{R}_R\in\mathcal{R}_{\Delta,\mathsf{s}}$ is required to lift to a group representation of $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$. In contrast to section 11.1, in the present case **Condition II** is trivial since every representation of $\mathfrak{sl}(2,\mathbb{R})$ lifts to a (not necessarily unitary) representation of the universal cover of $SL(2,\mathbb{R})$ [**?**].

[32] In the special cases where Δ and $\bar{\Delta}$ both correspond to normalizable solutions, we simply choose Δ, again without loss of generality.

Thus we fix $\mathsf{s} \in \mathbb{Z}$ and the one-loop determinant (201) is then the product of poles in the complex Δ plane given by

$$Z_{\Delta,\mathsf{s}} = \prod_{\mathcal{R}_{\Delta,\mathsf{s}}} \prod_{(\lambda_L,\lambda_R)} \prod_{N\in\mathbb{Z}} \left(|N| - \lambda_L \mathsf{h}_L + \lambda_R \mathsf{h}_R\right)^{-1/4} \left(|N| + \lambda_L \mathsf{h}_L - \lambda_R \mathsf{h}_R\right)^{-1/4} . \tag{216}$$

Following (85), we next implement a Schwinger parameterization of the logarithm of this expression which reads

$$\log(Z_{\Delta,\mathsf{s}}) = \frac{1}{4}\int_{\times}^{\infty} \frac{\mathrm{d}\alpha}{\alpha}\frac{\cosh\left(\frac{\alpha}{2}\right)}{\sinh\left(\frac{\alpha}{2}\right)} \sum_{\mathcal{R}_{\Delta,\mathsf{s}}}\sum_{(\lambda_L,\lambda_R)} \left(e^{\alpha(\lambda_L\mathsf{h}_L-\lambda_R\mathsf{h}_R)} + e^{-\alpha(\lambda_L\mathsf{h}_L-\lambda_R\mathsf{h}_R)}\right) , \tag{217}$$

where we have performed the sum over N as in (90). We will regulate the $\alpha \to 0$ divergence of the above expression by combining the two terms in the bracket into a single contour integral regulated about the origin by an $i\varepsilon$ prescription. To ensure convergence of the representation traces, a separate $i\varepsilon$ prescription must be given to the highest and lowest-weight representations appearing in $\mathcal{R}_{\Delta,\mathsf{s}}$:

$$\log(Z_{\Delta,\mathsf{s}}) = \frac{1}{4}\sum_{\mathcal{R}_{\Delta,\mathsf{s}}}\int_{-\infty\pm i\varepsilon}^{\infty\pm i\varepsilon} \frac{\mathrm{d}\alpha}{\alpha}\frac{\cosh\left(\frac{\alpha}{2}\right)}{\sinh\left(\frac{\alpha}{2}\right)} \sum_{(\lambda_L,\lambda_R)} e^{\alpha(\lambda_L\mathsf{h}_L-\lambda_R\mathsf{h}_R)} . \tag{218}$$

with the (+/-) sign applying to representations in $\mathcal{R}^{\text{LW/HW}}_{\Delta,\mathsf{s}}$, respectively. Recognizing the sum over weights as a representation trace and redefining $\alpha \to -i\alpha$ we then can write this suggestively as

$$\begin{aligned}\log(Z_{\Delta,\mathsf{s}}) &= \frac{i}{4}\sum_{\mathcal{R}^{\text{LW}}_{\Delta,\mathsf{s}}}\int_{\mathcal{C}_+} \frac{\mathrm{d}\alpha}{\alpha}\frac{\cos\left(\frac{\alpha}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)} \mathrm{Tr}_{\mathsf{R}_L}\left(\mathcal{P}e^{\frac{\alpha}{2\pi}\oint_{\gamma_{\text{sp}}} a_L}\right) \mathrm{Tr}_{\mathsf{R}_R}\left(\mathcal{P}e^{-\frac{\alpha}{2\pi}\oint_{\gamma_{\text{sp}}} a_R}\right) \\ &+ \frac{i}{4}\sum_{\mathcal{R}^{\text{HW}}_{\Delta,\mathsf{s}}}\int_{\mathcal{C}_-} \frac{\mathrm{d}\alpha}{\alpha}\frac{\cos\left(\frac{\alpha}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)} \mathrm{Tr}_{\mathsf{R}_L}\left(\mathcal{P}e^{\frac{\alpha}{2\pi}\oint_{\gamma_{\text{sp}}} a_L}\right) \mathrm{Tr}_{\mathsf{R}_R}\left(\mathcal{P}e^{-\frac{\alpha}{2\pi}\oint_{\gamma_{\text{sp}}} a_R}\right) ,\end{aligned} \tag{219}$$

where the contours $\mathcal{C}_\pm$ are depicted in Fig. 4 and applied separately to the lowest and highest-weight representations appearing in $\mathcal{R}_{\Delta,\mathsf{s}}$. We have written $\log Z_{\Delta,\mathsf{s}}$ in this form to draw comparison to the dS_3

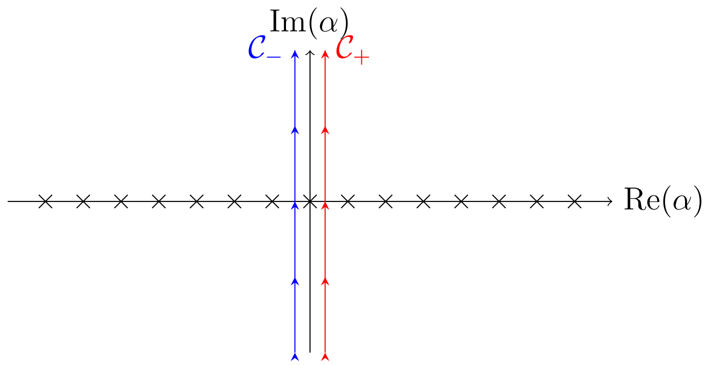

Figure 4: The integration contours relevant for the Wilson spool in AdS_3 gravity. Highest-weight representations are integrated against $\mathcal{C}_-$ (in blue) while lowest-weight representations are integrated against $\mathcal{C}_+$ (in red). The black crosses depict the poles of the integration measure, $\frac{1}{\alpha}\frac{\cos\alpha/2}{\sin\alpha/2}$. Not depicted are the poles of the representation traces which lie along the imaginary α axis.

Wilson spool in section 11.2. In the construction from that section, all representations are integrated along both $\mathcal{C}_\pm$. This is natural: as was argued in section 11.1, the poles of the one-loop determinant lie on states of finite-dimensional $\mathfrak{su}(2)$ representations which are simultaneously highest and lowest-weight. In the present case, $\mathcal{C}_\pm$ appear distinctly because highest and lowest-weight representations are distinct for $\mathfrak{sl}(2,\mathbb{R})$.

With this comparison noted, we can express $\log Z_{\Delta,\mathsf{s}}$ purely in terms of lowest-weight representations, which are more standard in the AdS/CFT dictionary, by returning to (218) and recalling that every weight of a highest-weight representation is the negative of a weight in a corresponding lowest-weight representation. This allows to write $\log Z_{\Delta,\mathsf{s}}$ in the form conjectured in the supplemental material of [42]

$$\log(Z_{\Delta,\mathsf{s}}) = \frac{i}{4}\sum_{\mathcal{R}^{\mathrm{LW}}_{\Delta,\mathsf{s}}}\int_{2\mathcal{C}_+}\frac{\mathrm{d}\alpha}{\alpha}\frac{\cos\left(\frac{\alpha}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)}\mathrm{Tr}_{\mathsf{R}_L}\left(\mathcal{P}e^{\frac{\alpha}{2\pi}\oint_{\gamma_{\mathrm{sp}}}a_L}\right)\mathrm{Tr}_{\mathsf{R}_R}\left(\mathcal{P}e^{-\frac{\alpha}{2\pi}\oint_{\gamma_{\mathrm{sp}}}a_R}\right). \tag{220}$$

providing a principled derivation of that result. It was shown there that utilizing the holonomies of the background connections corresponding to a spinning BTZ black hole, the right-hand side of

(220) evaluates to the correct one-loop determinant of massive spin-s fields on that background:

$$\log Z_{\Delta,\mathsf{s}} = -\sum_{\pm}\sum_{l,\bar{l}=0}^{\infty} \log\left(1 - q^{\frac{\Delta\pm\mathsf{s}}{2}+l}\bar{q}^{\frac{\Delta\mp\mathsf{s}}{2}+\bar{l}}\right) , \qquad q = e^{i\frac{2\pi}{\tau}} , \qquad \bar{q} = e^{-i\frac{2\pi}{\bar{\tau}}} . \tag{221}$$

This agrees with the one-loop determinant of BTZ for spinning fields as reported in, for example, [65, 67].

We finish this section by reconciling the spinning Wilson spool in AdS$_3$ gravity with our main result (28), namely the lack of the s^2 correction to the α integration measure. This lack stems from two places in our derivation: (i) the absence of normalizable zero-modes and (ii) the triviality of **Condition II** which leaves the product over the weight spaces unrestricted. However it is interesting to note that, on-shell, $\mathrm{Tr}_{\mathsf{R}_{L/R}}\mathcal{P}e^{\pm\frac{\alpha}{2\pi}\oint_{\gamma_{\rm sp}} a_{L/R}}$ only have poles along the imaginary α axis [42]. Thus this correction term is completely regular along the real α axis and integrates to zero. We may then include it into the definition of the spinning Wilson spool and state succinctly, for both signs of the cosmological constant,

$$\log Z_{\Delta,\mathsf{s}} = \frac{1}{4}\mathbb{W}_{j_L,j_R}[a_L, a_R] , \tag{222}$$

with $\mathbb{W}_{j_L,j_R}$ appearing in (28) and the representations $\mathsf{R}_{j_{L,R}}$, the integration contour, and the background-connections $a_{L,R}$ chosen appropriately.

16 Gravity Discussion

In this paper we extended the Wilson spool constructions of [40, 42] to incorporate one-loop determinants of massive spinning fields in both Euclidean dS$_3$ and AdS$_3$ backgrounds. This construction was based upon arranging the quasinormal mode spectra of the respective backgrounds into the representation theory of the isometry algebra. Along the way we codified two important principles for evaluating one-loop determinants in a representation-theoretic framework. These conditions lead naturally to a spinning Wilson spool expression for the local path integral, $\log Z_{\Delta,\mathrm{s}}$, of a massive spinning field, (28). This expression mimics the scalar Wilson spool: it is an integral over gauge invariant Wilson loop operators with the integral providing a mechanism for "wrapping" the loops around cycles of the base manifold. The spinning Wilson spool only departs from the scalar expression in its integration measure which cleanly captures the "edge contributions" of [37]. We posited an off-shell expression for the spinning Wilson spool which allows its insertion into the Chern-Simons path integral. In the context of a Euclidean dS_3 background, exact methods in Chern-Simons theory provide efficient methods for evaluating quantum gravity corrections to $\log Z_{\Delta,\mathrm{s}}$. We discussed two renormalization conditions in which these quantum corrections can be interpreted as renormalizing either the particle mass or Newton's constant. This provides concrete predictions for testing the correspondence between Chern-Simons theory and three-dimensional quantum gravity.

There are multiple comments in order about our results, as well as future directions to which the spinning Wilson spool may prove useful. We briefly discuss these below.

Group theoretic perspective on one-loop determinants

One important result of section 11.1 is the restating of the quasinormal mode method in a manner directly utilizing aspects of the representation theory and isometries of the background. At the core of that section are **Conditions I & II** which precisely isolate the representations and the states of a representation that contribute poles to massive one-loop determinants. We regard these conditions to be a significant new asset to the evaluation of one-loop determinants. For one, **Conditions I & II** will play an important role in establishing the Wilson spool for matter on Euclidean saddles beyond S^3, which is ongoing work [68]. For instance, our preliminary findings indicate that these two conditions correctly predict the placement and the degeneracies of the poles of scalar one-loop determinants on Lens spaces. This gives us strong credence that a

Wilson spool built from **Conditions I & II** will accurately describe the physics of matter on those backgrounds.

More broadly, we have taken care to state the content of these conditions in a manner not specific to spacetime dimension or sign of cosmological constant. Even outside the context of 3d quantum gravity, it is our expectation that **Conditions I & II** will provide a powerful asset to organizing quasinormal modes and evaluating one-loop determinants for any manifold which possesses a transitive group action, regardless of dimension.

On the role of finite dimensional $\mathfrak{su}(2)$ representations

In section 10 we took great care to build $\mathfrak{su}(2)$ representations corresponding to massive spinning particle states; these representations are infinite dimensional and non-standard. The reader might then find it surprising that finite dimensional representations played such important role in section 11.1 (particularly in the implementation of **Condition II**). It is worth disentangling the roles of these two separate series of representations.

We regard the mass-shell condition, (66), as always providing a link between representation theory and the particle mass. For physical values of the mass, these representations are generically non-standard. The implementation of **Condition II** is a separate statement about the analytic structure of Z^2_{STT}, namely it encounters poles on finite dimensional representations of $\mathfrak{su}(2)$. These however do not (necessarily) lie on physical values of the mass. In evaluating $\mathbb{W}_{j_L,j_R}$ our goal is not to land on a pole; our goal is to evaluate a one-loop determinant for a physical field. As such the representations appearing in (28) should be appropriate for a physical value of Δ and are, again, generically non-standard. Obtaining the right hand side of (95) is a non-trivial verification of this fact.

We can further illustrate this distinction by investigating what happens if we insert finite dimensional representations $j_{L/R} \in \frac{1}{2}\mathbb{Z}$ directly into the Wilson spool (say, for the classical S^3 background). For the sake of this illustration we will take a non-spinning case

$j_L = j_R = j$:

$$\mathbb{W}_j\big|_{\text{finite dim rep}} = \frac{i}{2}\int_{\mathcal{C}} \frac{\mathrm{d}\alpha}{\alpha} \frac{\cos\left(\frac{\alpha}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)} \frac{\sin^2\left((2j+1)\frac{\alpha}{2}\right)}{\sin^2\left(\frac{\alpha}{2}\right)} \ , \tag{223}$$

where we have inserted directly the finite dimensional $\mathfrak{su}(2)$ characters $\chi_j(z) = \frac{\sin(\pi(2j+1)z)}{\sin(\pi z)}$ and utilized the holonomies around $\gamma_{\text{hor.}}$, (72). However it is clear what will happen if we wrap $\mathcal{C}$ as in Fig. 2: the poles at $\alpha \in 2\pi\mathbb{Z}_{\neq 0}$ are now only first order and now without any exponential damping. We thus find

$$\mathbb{W}_j\big|_{\text{finite dim rep}} = \sum_{n=1}^{\infty} \frac{(2j+1)^2}{n} = (2j+1)^2\zeta(1) = \infty \ , \tag{224}$$

where ζ is the Riemann zeta function. This illustrates the importance of utilizing the non-standard representations (corresponding to physical masses) in $\mathbb{W}$.

Massive versus massless spinning fields

It is worth commenting on the important distinction between describing massive spinning fields (the focus of this work) versus massless (higher) spin fields in the Chern-Simons formalism. Our comments are very much motivated by viewing Chern-Simons gravity as an effective field theory. This effective field theory provides a description of the physics below some mass gap; indeed we can expect that massive degrees of freedom can be "integrated" out, leaving behind an effective response on the remaining low-energy degrees of freedom. This is precisely what the spinning Wilson spool encapsulates: the response of massive degrees of freedom directly in variables natural to the Chern-Simons path integral.

We contrast this with massless degrees of freedom which cannot be integrated out. Instead their presence must alter the low-energy effective field theory. This is an indication that the one-loop determinants of massless higher-spin fields do not have a Wilson spool description. Instead, we already know they are described by modifying the effective field theory to a theory of higher-spin gravity. For example, one simple way of describing massless higher-spin fields up to spin-N is by replacing $SL(2,\mathbb{R}) \to SL(N,\mathbb{R})$ in asymptotically AdS_3 spacetimes [58–60] and $SU(2) \to SU(N)$ in asymptotically Euclidean dS_3 spacetimes [37, 61].

These above comments lead naturally to a line of inquiry: "How does one couple massive matter to a theory of higher-spin gravity?" The relations between Wilson lines and particle worldlines in higher-spin gravity have been explored. However we believe the Wilson spool (generalized to, e.g., $SU(N)$ or $SL(N,\mathbb{R})$) will be an invaluable tool for a more complete answer to this important open question.

Generalized symmetries in 3D gravity

Let us offer a final, speculative, remark on the insights that the Wilson spool can possibly lend to our understanding of gravity as an effective field theory. A modern perspective on organizing low-energy physics is the extension of the Landau paradigm of broken symmetries to higher-form and non-invertible (what we will collectively call, generalized) symmetries [69, 70]. Attempts to categorize gravity as a low-energy phase through this lens include [71–73]. An important conjecture along these lines is the wide-held belief that all global symmetries are either gauged or explicitly broken in a UV theory of quantum gravity [74] and these statements extend to generalized symmetries.

Chern-Simons theory provides a natural area for exploring these discussions applied to three-dimensional gravity: the Wilson line operators of Chern-Simons theory both generate and are mutually charged under a generalized symmetry. It naturally follows that the Wilson spool is also charged under this generalized symmetry and its non-zero expectation value is a direct sign of the explicit breaking of this symmetry through the inclusion of matter. While the spectrum of operators in Chern-Simons theory might be too large, we note the special role non-standard representation theory plays in this statement: as noted above, the Wilson spool associated to a finite-dimensional $\mathfrak{su}(2)$ representation diverges even classically, (224). It would be very interesting to make more precise the relation between the Wilson spool and the generalized symmetries of Chern-Simons theory as well as what these relations lend to the question, "What is 3D gravity?"

17 Conclusion

Lattice methods have been developed for CS theory canonical quantization [75, 76]. These make possible developing numerical calculations for composite states.

The supersymmetric Chern-Simons actions, (5) and (154) provide a new unified view of matter and quantum gravity, i.e. spacetime. While naturally important in itself, it may have further ramifications.

References

[1] Risto Raitio, A Model of Lepton and Quark Structure. Physica Scripta, 22, 197 (1980). PS 22,197. viXra:1903.0224
The core of this model was conceived in November 1974 at SLAC. I proposed that the c-quark would be an excitation of the u-quark, both composites of three 'subquarks'. The idea was opposed by the community and was therefore not written down until five years later.

[2] Risto Raitio, Supersymmetric preons and the standard model, Nuclear Physics B931, 283–290 (2018). DOI: 10.1016/j.nuclphysb.2018.04.021 arXiv:1805.03013

[3] J. Wess and B. Zumino, Supergauge transformations in four dimensions, Nucl. Phys. B 70, 39 (1974). DOI: 10.1016/0550-3213(74)90355-1

[4] Roberto D. Peccei and Helen R. Quinn, CP Conservation in the Presence of Pseudoparticles, Phys. Rev. Lett. 38 (25) 1440–1443 (1977).

[5] Risto Raitio, The fate of supersymmetry in quantum field theories, Journal of High Energy Physics, Gravitation and Cosmology, Vol.10, No.2 (2024). DOI: 10.4236/jhepgc.2024.102038. arXiv:2307.13017

[6] S. Deser, R. Jackiw and S. Templeton, Three-dimensional massive gauge theories, Phys. Rev. Lett. 48, 975 (1982); Ann. Phys. (NY) 140, 372 (1982).

[7] R. Jackiw, Two lectures on Two-Dimensional Gravity. arXiv:gr-qc/9511048

[8] Juan Maldacena, The illusion of Gravity, Scientific American, April 1 (2007).

[9] Wenning Liang and Chengbo Zhai, Existence of bound state solutions for the generalized Chern–Simons–Schrödinger system in $H^1(R^2)$, Applied Mathematics Letters 100, 106028 (2020).

[10] S.-S. Chern and J. Simons, Characteristic forms and geometric invariants, Annals of Mathematics. 99 (1): 48–69 (1974).

[11] Witten, Edward, Quantum Field Theory and the Jones Polynomial, Commun. Math. Phys. 121, 351-399 (1989). DOI: 10.1007/BF01217730.

[12] H. Belich, O. M. Del Cima, M. M. Ferreira Jr. and J. A. Helayël-Neto, Electron-Electron Bound States in Maxwell-Chern-Simons-Proca QED_3, Eur. Phys. J. B 32, 145–155 (2003). arXiv:hep-th/0212285

[13] M. A. De Andrade, O. M. Del Cima and J. A. Helayel-Neto, Il Nuovo Cimento 111, 1145 (1998).

[14] O. M. Del Cima, D.H.T. Franco, J. A. Helayel-Neto and O. Piguet, Phys. Lett. B 410, 250 (1997) and Phys. Lett. B 416, 402 (1998).

[15] B. Binegar, J. Math. Phys. 23, 1511 (1982); S. Deser and R. Jackiw, Phys. Lett. B263, 431 (1991); R. Jackiw and V. P. Nair, Phys. Rev. D43, 1933 (1991); J. Fröhlich and P. A. Marchetti, Lett. in Math. Phys. 16, 347 (1988).

[16] Ya. I. Kogan, JETP Lett. 49, 225 (1989).

[17] M. I. Dobroliubov, D. Eliezer, I. I. Kogan, G.W. Semenoff and R.J. Szabo, Mod. Phys. Lett. A, 8, 2177 (1993).

[18] N. Aghanim et al., Planck 2018 results. VI. Cosmological parameters. arXiv:1807.06209

[19] A. Starobinsky, A New Type of Isotropic Cosmological Models without Singularity, Phys. Lett. B, Vol. 91, 1 (1980).

[20] John Ellis, Marcos A. G. García, Natsumi Nagata, Dimitri V. Nanopoulos, Keith A. Olive and Sarunas Verner, Building Models of Inflation in No-Scale Supergravity. IJMPD Vol. 29, No. 16, 2030011 (2020). https://doi.org/10.1142/S0218271820300116. arXiv:2009.01709

[21] Evan McDonough, Alan H. Guth, and David I. Kaiser, Nonminimal Couplings and the Forgotten Field of Axion Inflation. arXiv:2010.04179

[22] T. D. Brennan, F. Carta and C. Vafa, The String Landscape, the Swampland, and the Missing Corner. arXiv:1711.00864

[23] Lavinia Heisenberg, Matthias Bartelmann, Robert Brandenberger, and Alexandre Refregier, Model Independent Analysis of Supernova Data, Dark Energy, Trans-Planckian Censorship and the Swampland. arXiv:2003.13283

[24] Weijie Jin, Robert Brandenberger, and Lavinia Heisenberg, Axion Monodromy Inflation, Trapping Mechanisms and the Swampland. arXiv:2010.04999

[25] John Ellis, Dimitri V. Nanopoulos, Keith A. Olive, No-Scale Supergravity Realization of the Starobinsky Model of Inflation, Phys. Rev. Lett. 111, 111301 (2013). arXiv:1305.1247

[26] K. S. Stelle, Gen. Rel. Grav. 9 (1978) 353; B. Whitt, Phys. Lett. 145B, 176 (1984).

[27] A. D. Sakharov Violation of CP invariance, C asymmetry, and baryon asymmetry of the universe, Journal of Experimental and Theoretical Physics Letters. 5: 24–27 (1967).

[28] P.A. Zyla et al. (Particle Data Group), Prog. Theor. Exp. Phys. 2020, 083C01 (2020) and 2021 update.

[29] N. Cabibbo, Unitary Symmetry and Leptonic Decays, Phys. Rev. Lett. 10, 531 (1963).

[30] M. Kobayashi and T. Maskawa, CP Violation in the Renormalizable Theory of Weak Interaction, Prog. Theor. Phys. 49, 652 (1973).

[31] Quentin Bonnefoy, Emanuele Gendy, Christophe Grojean and Joshua T. Ruderman, Beyond Jarlskog: 699 invariants for CP violation in SMEFT, JHEP 08, 032 (2022). arXiv:2112.03889

[32] Quentin Bonnefoy, Emanuele Gendy, Christophe Grojean and Joshua T. Ruderman, Opportunistic CP Violation. arXiv:2302.07288

[33] Robert Bourne, Alejandra Castro, and Jackson R. Fliss, Spinning up the spool: Massive spinning fields in 3d quantum gravity. arXiv:2407.09608

[34] E. Witten, (2+1)-Dimensional Gravity as an Exactly Soluble System, Nucl. Phys. B 311 (1988) 46.

[35] Collier, Scott and Eberhardt, Lorenz and Zhang, Mengyang, Solving 3d gravity with Virasoro TQFT, SciPost Phys., 15, 4 (2023). DOI: 10.21468/SciPostPhys.15.4.151. arXiv:2304.13650

[36] Castro, Alejandra and Lashkari, Nima and Maloney, Alexander, A de Sitter Farey Tail, Phys. Rev. D 83, 124027 (2011). DOI: 10.1103/PhysRevD.83.124027. arXiv:1103.4620

[37] Anninos, Dionysios and Denef, Frederik and Law, Y. T. Albert and Sun, Zimo, Quantum de Sitter horizon entropy from quasicanonical bulk, edge, sphere and topological string partition functions, JHEP 01, 088 (2022). DOI: 10.1007/JHEP01(2022)088. arXiv:2009.12464

[38] Denef, Frederik and Hartnoll, Sean A. and Sachdev, Subir, Black hole determinants and quasinormal modes, Class. Quant. Grav. 27,125001 (2010). DOI: 10.1088/0264-9381/27/12/125001. arXiv:0908.2657 j

[39] Ooguri, Hirosi and Vafa, Cumrun, Knot invariants and topological strings, Nucl. Phys. B 577, 419-438 (2000). DOI: 10.1016/S0550-3213(00)00118-8. arXiv:hep-th/9912123

[40] A. Castro, I. Coman, J. R. Fliss, and C. Zukowski, Keeping matter in the loop in dS3 quantum gravity, JHEP 07 (2023) 120. arXiv:2302.12281

[41] Zinoviev, Yu. M., On massive high spin particles in (A)dS. arXiv:hep-th/0108192

[42] Alejandra Castro, Ioana Coman, Jackson R. Fliss, and Claire Zukowski, Coupling Fields to 3D Quantum Gravity via Chern-Simons Theory, Phys. Rev. Lett. 131, 171602 (2023). DOI: 10.1103/PhysRevLett.131.171602. arXiv:2304.02668

[43] He, Miao, One-loop partition functions in $T\overline{T}$-deformed AdS_3, JHEP 05, 067 (2024). DOI: 10.1007/JHEP05(2024)067. arXiv:2401.09879

[44] Blau, Matthias and Thompson, George, Chern-Simons Theory on Seifert 3-Manifolds, JHEP 09, 033 (2013). DOI: 10.1007/JHEP09(2013)033. arXiv:1306.3381

[45] Marcos Mariño, Lectures on localization and matrix models in supersymmetric Chern–Simons–matter theories, J. Phys. A: Math. Theor. 44 463001 (2011). DOI: 10.1088/1751-8113/44/46/463001. arXiv:1104.0783

[46] Kapustin, Anton and Willett, Brian and Yaakov, Itamar, Exact Results for Wilson Loops in Superconformal Chern-

Simons Theories with Matter, JHEP 03, 089 (2010). DOI: 10.1007/JHEP03(2010)089. arXiv:0909.4559

[47] Blau, Matthias and Thompson, George, Chern-Simons theory on S1-bundles: Abelianisation and q-deformed Yang-Mills theory, JHEP 05, 003 (2006). DOI: 10.1088/1126-6708/2006/05/003. arXiv:hep-th/0601068

[48] Witten, Edward, Quantization of Chern-Simons Gauge Theory with Complex Gauge Group, Commun. Math. Phys., 137, 29-66 (1991). DOI: 10.1007/BF02099116

[49] Polchinski, Joseph, The Phase of the Sum Over Spheres, Phys. Lett. B 219, 251-257 (1989). DOI: 10.1016/0370-2693(89)90387-0.

[50] Joung, E. and Mourad, J. and Parentani, R., Group theoretical approach to quantum fields in de Sitter space. I. The Principle series, JHEP 08, 082 (2006). DOI: 10.1088/1126-6708/2006/08/082. arXiv:0606119

[51] Joung, Euihun and Mourad, Jihad and Parentani, Renaud, Group theoretical approach to quantum fields in de Sitter space. II. The complementary and discrete series, JHEP 09, 030 (2007). DOI: 10.1088/1126-6708/2007/09/030. arXiv:0707.2907

[52] Basile, Thomas and Bekaert, Xavier and Boulanger, Nicolas, Mixed-symmetry fields in de Sitter space: a group theoretical glance, JHEP 05, 081 (2017). DOI: 10.1007/JHEP05(2017)081. arXiv:1612.08166

[53] Sun, Zimo, A note on the representations of SO$(1, d+1)$ (2021). arXiv:2111.04591

[54] Penedones, Joao and Salehi Vaziri, Kamran and Sun, Zimo, Hilbert space of Quantum Field Theory in de Sitter spacetime (2023). arXiv:2301.04146

[55] Castro, Alejandra and Sabella-Garnier, Philippe and Zukowski, Claire, Gravitational Wilson Lines in 3D de Sitter,

JHEP 07, 202 (2020). DOI: 10.1007/JHEP07(2020)202. arXiv:2001.09998

[56] Jafferis, Daniel L. and Lupsasca, Alexandru and Lysov, Vyacheslav and Ng, Gim Seng and Strominger, Andrew, Quasinormal quantization in de Sitter spacetime, JHEP 01, 004 (2015). DOI: 10.1007/JHEP01(2015)004. arXiv:1305.5523

[57] Law, Y. T. Albert and Parmentier, Klaas, Black hole scattering and partition functions, JHEP 10, 039 (2022). DOI: 10.1007/JHEP10(2022)039. arXiv:2207.07024

[58] Blencowe, M. P., A Consistent Interacting Massless Higher Spin Field Theory in $D = (2{+}1)$, Class. Quant. Grav. 6,443 (1989). DOI: 10.1088/0264-9381/6/4/005.

[59] Bergshoeff, E. and Blencowe, M. P. and Stelle, K. S., Area Preserving Diffeomorphisms and Higher Spin Algebra, Commun. Math. Phys. 128, 213 (1990). DOI: 10.1007/BF02108779.

[60] Campoleoni, Andrea and Fredenhagen, Stefan and Pfenninger, Stefan and Theisen, Stefan, Asymptotic symmetries of three-dimensional gravity coupled to higher-spin fields, JHEP 11, 007 (2010). DOI: 10.1007/JHEP11(2010)007. arXiv:1008.4744

[61] Joung, Euihun and Mkrtchyan, Karapet, Notes on higher-spin algebras: minimal representations and structure constants, JHEP 05, 103 (2014). DOI: 10.1007/JHEP05(2014)103. arXiv:1401.7977

[62] Beasley, Chris, Localization for Wilson Loops in Chern-Simons Theory, Adv. Theor. Math. Phys. 17, 1-240 (2013). DOI: 10.4310/ATMP.2013.v17.n1.a1. arXiv:0911.2687

[63] Hikida, Yasuaki and Nishioka, Tatsuma and Takayanagi, Tadashi and Taki, Yusuke, Holography in de Sitter Space via Chern-Simons Gauge Theory, Phys. Rev. Lett. 129, 4, 041601 (2022). DOI: 10.1103/PhysRevLett.129.041601. arXiv:2110.03197

[64] Leston, Mauricio and Goya, Andrés and Pérez-Nadal, Guillem and Passaglia, Mario and Giribet, Gaston, 3D Quantum Grav-

ity Partition Function at Three Loops, Phys. Rev. Lett. 131, 18, 181601 (2023). DOI: 10.1103/PhysRevLett.131.181601. arXiv:2307.03830

[65] Datta, Shouvik and David, Justin R., Higher Spin Quasinormal Modes and One-Loop Determinants in the BTZ black Hole, JHEP 03, 079 (2012). DOI: 10.1007/JHEP03(2012)079. arXiv:1112.4619

[66] Sleight, Charlotte, Metric-like Methods in Higher Spin Holography, PoS Modave2016, 003 (2017). DOI: 10.22323/1.296.0003. arXiv:1701.08360

[67] Castro, Alejandra and Keeler, Cynthia and Szepietowski, Phillip, Tweaking one-loop determinants in AdS_3, JHEP 10, 070 (2017). DOI: 10.1007/JHEP10(2017)070. arXiv:1707.06245

[68] Bourne, Robert and Castro, Alejandra and Fliss, Jackson R. and Law, Albert, Work in progress.

[69] Gaiotto, Davide and Kapustin, Anton and Seiberg, Nathan and Willett, Brian, Generalized Global Symmetries, JHEP 02, 172 (2015). DOI: 10.1007/JHEP02(2015)172. arXiv:1412.5148

[70] McGreevy, John, Generalized Symmetries in Condensed Matter, Ann. Rev. Condensed Matter Phys. 14, 57-82 (2023). DOI: 10.1146/annurev-conmatphys-040721-021029 arXiv:2204.03045

[71] Benedetti, Valentin and Casini, Horacio and Magan, Javier M., Generalized symmetries of the graviton, JHEP 05, 045 (2022). DOI: 10.1007/JHEP05(2022)045. arXiv:2111.12089

[72] Hinterbichler, Kurt and Hofman, Diego M. and Joyce, Austin and Mathys, Grégoire, Gravity as a gapless phase and biform symmetries, JHEP 02, 151 (2023). DOI: 10.1007/JHEP02(2023)151. arXiv:2205.12272

[73] Benini, Francesco and Copetti, Christian and Di Pietro, Lorenzo, Factorization and global symmetries in hologra-

phy, SciPost Phys. 14, 2, 019 (2023). DOI: 10.21468/SciPost-Phys.14.2.019. arXiv:2203.09537

[74] Harlow, Daniel and Ooguri, Hirosi, Symmetries in quantum field theory and quantum gravity, Commun. Math. Phys. 383, 3, 1669-1804 (2021). DOI: 10.1007/s00220-021-04040-y. arXiv:1810.05338

[75] Theodore Jacobson and Tin Sulejmanpasic, Canonical quantization of lattice Chern-Simons theory. arXiv:2401.09597

[76] Bingnan Zhang, Abelian Chern-Simons gauge theory on the lattice, Phys. Rev. D 105, 014507 (2022). arXiv:2109.13411