Seismic Anomaly. Analytical Solutions

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Content:

Abstract

In this article we provide some analytical solutions of seismic equations with the different sources for a media, consisting of Uniform Half-Space (Air or Water or…) & solid Uniform Half-Space (Earth), containing a localized Anomaly. Such solutions allow building the very fast computer-based programs to decipher near-surface caves, karsts, tunnels, engineering applications, etc. This way particularly allows to go along a curved line, discovering already built tunnels without noise detection.

We consider sources and model, which are practical for onshore and offshore (in deep water) seismic explorations. One may apply some forms of seismic solutions for a **deep** exploration of the slightly inclined multi-layer underground structures to find oil-gas-minerals-water-bearing lenses (see for example [9]). Here we apply the found solutions for **shallow** sounding to describe an effect from Anomaly.

Also we showed a math similarity of uniform fluid (within Navier-Stokes equation) with seismic isotropic linear media (excluding the boundary layers). The details of boundary conditions are discussed, as well as the first orders of decomposition theory for the Fourier-Bessel representations of the seismic displacements. It is noted that within reasonable survey parameters an azimuth component can be ignored. Besides, it is observed, that we cannot stitch non-viscous fluid with solids directly, instead of this we must consider a limit transfer of solid-solid interaction. The radial and vertical displacements inside solid half-space are obtained as well as effects from localized Anomalies. For the cases of distributed Anomalies (karsts, tunnels, etc.) a convergence of analytical solutions was shown.

Basic Assumptions and Boundary Conditions

We assume: source $\vec{F}^{\,\text{Source}}\left[\frac{N}{\text{m}^3}\right]$ (Hammer, Accelerated Weight-Drop, Explosive, etc.) is applied near center of the coordinates $\left(x=y=0,\ 0_- = z^{\text{S}}_{\text{min}} < z < z^{\text{S}}_{\text{max}} = 0^+\right)$, see Fig.2. Environment contains **U**pper-**H**alf-**S**pace (UHS, $z<0$) and **L**ower **M**edium (LM, $z>0$).

The UHS is a viscous fluid (air or water); $\left(\tilde{P}_F, \tilde{\rho}_F, \tilde{V}_F, \tilde{\zeta}_F, \tilde{\eta}_F\right)$ are general properties. UHS is uniform at $z<-\delta z_\rho$ with properties $\left(\tilde{P}, \tilde{\rho}, \tilde{V}, \tilde{\zeta}, \tilde{\eta}\right)_F = \left(P, \rho, V, \zeta, \eta\right)_0 = \text{Const}$ (not depending on $\vec{R}, t$); $\delta z_\rho > 0$ is a thin boundary layer's thickness, P_0 - pressure; ρ_0 - density, ζ_0 - volume viscosity, η_0 - share viscosity. Sound velocity $\tilde{V}_F(z)$ can be expressed by $\tilde{V}_F^2 = \left(\partial\tilde{P}_F / \partial\tilde{\rho}_F\right)$ in adiabatic process; it is mainly responsible for the propagation of the pressure-waves in fluid, similar to $\tilde{V}_{\text{P}}^{\,2} = \left(\tilde{\lambda} + 2\tilde{\mu}\right)/\tilde{\rho}$ in solids. In fluids, using Navier-Stokes and continuity equations [3], we can obtain the eq. (1.F), describing a behavior of the displacement (strain)'s component $l_n\left(\vec{R}, t\right)$ by projecting them on unit coordinate vector $\vec{i}_n$:

$$\begin{aligned}\tilde{\rho}_F \cdot \frac{\partial^2 l_n}{\partial t^2} &= \frac{\partial}{\partial r_n}\left(\tilde{V}_F^{\,2} \cdot \text{div}\left(\tilde{\rho}_F \cdot \vec{l}\right) + \left(\tilde{\zeta}_F - \frac{2}{3}\tilde{\eta}_F\right) \cdot \text{div}\frac{\partial \vec{l}}{\partial t}\right) + \\ &+ \sum_{\text{m=x,y,z}} \frac{\partial}{\partial r_m}\left(\tilde{\eta}_F \cdot \frac{\partial}{\partial t}\left(\frac{\partial l_n}{\partial r_m} + \frac{\partial l_m}{\partial r_n}\right)\right) - \frac{\partial \tilde{P}_F}{\partial r_n} + F^{\text{Source}}_{\text{n=x,y,z}}\end{aligned} \qquad (1.\text{F})$$

Note 1: a known term $\partial\tilde{P}_F / \partial r_n$ (if non-zero) can be included into $F^{\text{Source}}_{\text{n}}$.

Note 2: Fig.1 shows natural assumptions: $\tilde{\rho}_F^{\,z<-\delta z_\rho} = \rho_0,\ \tilde{\rho}_F^{\,-\delta z_\rho<z<-0_-} = \text{var}(z),\ \tilde{\rho}_F^{\,z=-0_-} = 0$, $\tilde{V}_F^{\,z<-\delta z_V} = \text{const} = V_0,\ \tilde{V}_F^{\,-\delta z_V<z<-0_-} = \text{var}(z),\ \tilde{V}_F^{\,z=-0_-} = 0$. Indeed: at the pure solid boundary $z=+0^+$ we obviously have $\tilde{\rho}^{\text{Solid}}_{z=+0^+} = 0,\ \tilde{\lambda}^{\text{Solid}}_{z=+0^+} = \tilde{\mu}^{\text{Solid}}_{z=+0^+} = 0$ (especially for a very rigid LM, like granite); and $\tilde{\rho}$ - continuity requires $\tilde{\rho}^{\text{Fluid}}_{z=-0_-} = \tilde{\rho}^{\text{Solid}}_{z=+0_+}$.

Note 3: we also use $\delta z_V / \delta z_\rho \ll 1$, allowing $\tilde{V}_F^{\,z<-\delta z_V} = V_0$ almost everywhere in UHS, it can be justified by $\tilde{V}_F^{\,2} = \left(\partial\tilde{P}_F / \partial\tilde{\rho}_F\right)_{\text{ad.}}$ and $\delta z_P \sim \delta z_\rho$ (no δz_P on Fig.1); but $\tilde{V}_F^{\,z=-0_-} = 0$.

Note 4: at typical surveys we can assume *low* f_{Hz}, where UHS acts as **N**on-**V**iscous **F**luid (NVF). Indeed, one can show: $(\zeta_0,\eta_0)_{\mathrm{air}} \neq 0$ bring error < 0.5% to the solution for onshore surveys, if $f_{\mathrm{Hz}} < 20\mathrm{KHz}$; and in offshore survey (UHS = deep water) the error < 0.5% will be reached, if $f_{\mathrm{Hz}} < 5\mathrm{MHz}$. In the NVF a normal strain $l_z \neq 0$ and its stress $\tilde{\sigma}_{zz} \neq 0$; however tangential strains $l_x = l_y = 0$ and their stresses $\tilde{\sigma}_{xz} = \tilde{\sigma}_{yz} = 0$ [2-p.30](but we rarely can use it in stitching with solids due to inconsistency of their equations, see below; instead of this we make a limit transfer $\mu_0 \to 0$ from initial full-size system). If for such UHS = Air = NVF we also ignore effects, caused by a very *low* air-density ρ_0, we actually disregard small air-waves, arising from normal strain. Thus a full ρ_0 -ignorance **and** *low* f_{Hz} make such UHS = Air $(\rho_0 = 0,\ \zeta_0 = \eta_0 = 0)$ the same as UHS = Vacuum (in which *all* the strains with their stresses are zero).

LM contains: **S**olid **H**alf-**S**pace (SHS), *uniform* at $z > \delta z^+$ (non-uniform by z in a thin boundary layer δz^+) and narrow Anomaly, localized in LM around its central point (x_A, y_A, z_A); we assume that Anomaly does not touch or cross the boundary $z = 0$. The SHS's uniform seismic properties (not depending on $\vec{R}, t$) are: (ρ, λ, μ), where (λ, μ) - Lame coefficients; we suppose that Anomaly (see Fig.1) is not δ -Function: $(\rho,\lambda,\mu)^{\mathrm{inside}}_{\mathrm{Anom.}} < \infty$, so for $(\Delta\rho, \Delta\lambda, \Delta\mu)_{\mathrm{A}} = (\rho,\lambda,\mu)_{\mathrm{A}} - (\rho,\lambda,\mu)$ we apply: $(\Delta\rho, \Delta\lambda, \Delta\mu)^{\mathrm{inside}}_{\mathrm{Anom.}} < \infty$. Also we assume that Anomaly is bounded: $(\rho,\lambda,\mu)^{\mathrm{outside}}_{\mathrm{Anom.}} = (\rho,\lambda,\mu)$, so $(\Delta\rho, \Delta\lambda, \Delta\mu)^{\mathrm{outside}}_{\mathrm{Anom.}} = 0$. Also we apply $(\tilde{\rho},\tilde{\lambda},\tilde{\mu})^{\mathrm{any}\ z} = \mathrm{Var}(z)$ as general seismic properties; but for the uniform parts of UHS and SHS we apply:

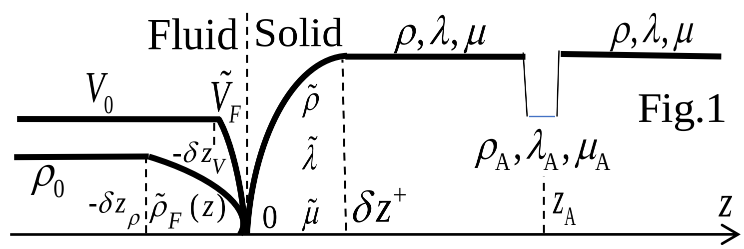
$$(\hat{\rho},\hat{\lambda},\hat{\mu}) = \begin{pmatrix} (\rho_0, \lambda_0, \mu_0) \text{ for } z \leq -\delta z_- \\ (\rho,\ \lambda,\ \mu\ ) \text{ for } z \geq +\delta z^+ \end{pmatrix} \quad (1.0)$$

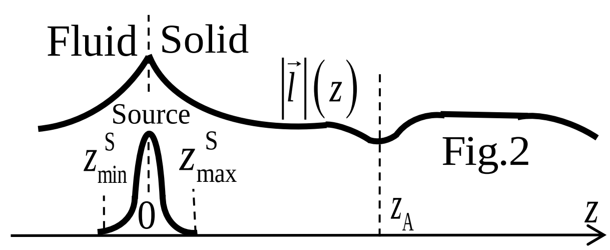
Note: there is a *thin* layer between Anomaly and solid surrounding. Expecting the Anomaly's impact on the solution to be small, we ignore the effects from such *thin* layer, considering just an influence from the Anomaly's volume $\Delta\Omega_{\mathrm{A}}$.

For solid isotropic linear media (in general case $(\tilde{\rho},\tilde{\lambda},\tilde{\mu}) = \mathrm{Var}$) we use a well-known system of differential equations for the displacements $l_{\mathrm{n}}(\vec{R},t)$ by projecting them on a unit coordinate vector $\vec{i}_n$ [4],[1] and using $(\tilde{\lambda},\tilde{\mu})$:

$$\tilde{\rho}\cdot\frac{\partial^2 l_n}{\partial t^2}=\frac{\partial}{\partial r_n}\left(\tilde{\lambda}\cdot \mathrm{div}\,\vec{l}\right)+\sum_{\mathrm{m=x,y,z}}\frac{\partial}{\partial r_m}\left(\tilde{\mu}\cdot\left(\frac{\partial l_n}{\partial r_m}+\frac{\partial l_m}{\partial r_n}\right)\right)+F^{\mathrm{Source}}_{\mathrm{n=x,y,z}}\left(\vec{R},t\right)\quad (1.S)$$

Note: if Anomaly is solid, we can use (1.S) everywhere in LM, but if it is fluid - we should apply (1.F) to Anomaly with $\left(\tilde{P},\tilde{\rho},\tilde{V},\tilde{\zeta},\tilde{\eta}\right)_{\mathrm{A}}=\left(\tilde{P},\tilde{\rho},\tilde{V},\tilde{\zeta},\tilde{\eta}\right)_{\mathrm{F}}$ and $l_{\mathrm{n}}=l_{\mathrm{n}}^{\mathrm{Anom.}}$.





We apply Fourier spectrum $\vec{l}^{\,\omega}\left(\vec{R}\right)$ by representation $\left(\omega=2\pi\cdot f_{\mathrm{Hz}}\right)$

$$\vec{l}\left(t,\vec{R}\right)=\int_{-\infty}^{\infty}d\omega\cdot e^{-i\omega t}\cdot\vec{l}^{\,\omega}\left(\vec{R}\right)=\int_{0}^{\infty}d\omega\cdot e^{-i\omega t}\cdot\vec{l}^{\,\omega}\left(\vec{R}\right)+C.C.\quad (2)$$

where $\vec{l}^{\,\omega}\left(\vec{R},t\right)=\int_{-\infty}^{\infty}dt\cdot e^{+i\omega\cdot t}\cdot\vec{l}\left(\vec{R},t\right)/\left(2\pi\right)$. Note: here, as usual: $\vec{l}^{\,-\omega}=\left(\vec{l}^{\,+\omega}\right)^{*}$.

Note: transfer to the spectrum for $\vec{F}^{\mathrm{Source}}=\left(F_x,F_y,F_z\right)\rightarrow\vec{f}^{\,\omega}=\left(f^{\omega}_x,f^{\omega}_y,f^{\omega}_z\right)$ is similar to (2).

Note: applying (2) to (1), we can replace: $\vec{l}\rightarrow\vec{l}^{\,\omega},\ \frac{\partial\vec{l}}{\partial t}\rightarrow -i\cdot\omega\cdot\vec{l}^{\,\omega},\ \frac{\partial^2\vec{l}}{\partial t^2}\rightarrow-\omega^2\cdot\vec{l}^{\,\omega}$.

The fluid properties $\left(..,\tilde{V}_F,..\right)$ can be transformed to the seismic set $\left(\tilde{\lambda}_F,\tilde{\mu}_F\right)$, when $\tilde{\rho}_F=\mathrm{const}$, assigning: $\tilde{\lambda}_F\left(z\right)=\tilde{\rho}_F\cdot\tilde{V}_F^2-i\cdot\omega\cdot\left(\tilde{\zeta}_F-2\cdot\tilde{\eta}_F/3\right),\ \ \tilde{\mu}_F\left(z\right)=-i\cdot\omega\cdot\tilde{\eta}_F$ (3)

Note: for the fluid Anomaly we can use (3), expressing $\left(\lambda_A,\mu_A\right)$ via $\left(...,V_A,...\right)$.

Note: for NVF, when $\left(\tilde{\zeta}_F,\tilde{\eta}_F\right)\rightarrow 0$, (3) gives: $\tilde{\lambda}_F=\tilde{\rho}_F\cdot\tilde{V}_F^2\pm i\cdot 0,\ \ \tilde{\mu}_F=-i\cdot 0$ (3.a)

Note: the sign of $\tilde{\lambda}_F^{\mathrm{Im}}$ in (3) is defined by ratio between $\tilde{\zeta}_F^{\mathrm{Re}}>0$ and $\tilde{\eta}_F^{\mathrm{Re}}>0$.

Note: for uniform fluid part the eq.(3) gives (pressure & share)'s waves velocities:

$$\tilde{V}_{\mathrm{P}}^2=\frac{\left(\tilde{\lambda}_F+2\tilde{\mu}_F\right)}{\tilde{\rho}_F}=\tilde{V}_F^2-i\cdot\omega\cdot\frac{\left(\tilde{\zeta}_F+4\cdot\tilde{\eta}_F/3\right)}{\tilde{\rho}_F}\qquad \tilde{V}_{\mathrm{S}}^2=\frac{\tilde{\mu}_F}{\tilde{\rho}_F}=\frac{-i\cdot\omega\cdot\tilde{\eta}_F}{\tilde{\rho}_F}\quad (3.b)$$

Note: signs of $\left(\tilde{\lambda}_F,\tilde{\mu}_F\right)_{\mathrm{Re}}^{\mathrm{Im}}$ in (3) must satisfy **R**adiation **C**onditions (RC) for $V_{\mathrm{P,S}}^2$, using $e^{-i\cdot\omega\cdot t}$ factor in (2): $\left(V_{\mathrm{P,S}}^2\right)_{\mathrm{Im}}<0$ and as usual $\left(V_{\mathrm{P,S}}^2\right)_{\mathrm{Re}}>0$; it is obvious due to $\tilde{\zeta}_F^{\mathrm{Re}}>0,\ \tilde{\eta}_F^{\mathrm{Re}}>0$.

Note: assuming $\tilde{\rho}_F^{-\delta z_\rho<z<-0_-}=\mathrm{var}(z)$, $\tilde{V}_F^{-\delta z_V<z<-0_-}=\mathrm{var}(z)$ (see Note 2), and due to identity $\mathrm{div}\left(\tilde{\rho}_F\cdot\vec{l}\right)=\tilde{\rho}_F\cdot\mathrm{div}\vec{l}+\vec{l}\cdot\mathrm{grad}\,\tilde{\rho}_F$, a term $\sim\vec{l}\cdot\mathrm{grad}\,\tilde{\rho}_F$ appears in (1.F) for fluids, but not in (1.S) for solids, giving *structural difference* between (1.F) & (1.S), which should be taken into account, analyzing the UHS z-ranges (see below). However, for uniform fluid part the term $\sim\vec{l}\cdot\mathrm{grad}\,\tilde{\rho}_F$ is absent in (1.F) (for example: UHS portion with $\tilde{\rho}_F=\rho_0=\mathrm{const}$, see Fig.1); here we can match the *structure* of (1.F) (converted to the spectrums l_n^ω) with the *structure* of eq.(1.S) (for l_n^ω also), applying (2) to (1.F,S); thus using (3), we get:

$$\lambda_0=\rho_0\cdot V_0^2-i\cdot\omega\cdot\left(\zeta_0-2\cdot\eta_0/3\right),\quad \mu_0=-i\cdot\omega\cdot\eta_0 \tag{3.c}$$

Note: $\left(\tilde{\lambda},\tilde{\mu}\right)_F$ in (3) depend on ω (unlike ω-independent $\left(\tilde{\lambda},\tilde{\mu}\right)$ in LM), but $\left(\tilde{\lambda},\tilde{\mu}\right)_F$ are not Fourier spectrums from $\left(\tilde{\lambda},\tilde{\mu}\right)_F^{\mathrm{Init}}$, similar to (2); such view will be incorrect due to independence of $\left(\tilde{\lambda},\tilde{\mu}\right)_F^{\mathrm{Init}}$ on t; $\left(\tilde{\lambda},\tilde{\mu}\right)_F$ in (3) are just artificial coefficients (particularly allowing (in uniform z-range) to match structure (1.F) with (1.S) (converted to l_n^ω)).

The **B**oundary **C**onditions (BC) near the surface $z=0$ (where $\vec{f}^{\,\omega}$ is applied, and the properties can jump) could be obtained, integrating (1.F,S) around $z=0$. We assign: $\{A\}_{-\delta z_-}^{+\delta z^+}=\int_{-\delta z_-}^{+\delta z^+}dz\cdot A(z)=\{A\}_{-\delta z_-}^{-0_-}+\{A\}_{-0_-}^{+0^+}+\{A\}_{+0^+}^{+\delta z^+}$ is a boundary integral, and $\left[A\right]_{z_1}^{z_2}=A(z_2)-A(z_1)$ is a discontinuity jump. Here $\delta z_-=\max\left(\delta z_\rho,\delta z_V\right)=\delta z_\rho$ (taking into account Note 3).

Calculating $\{\mathrm{LM}\}_{+0^+}^{+\delta z^+}$ at $\delta z^+\to 0$, we should integrate (1.S) (converted to spectrum (2) for $\vec{l}^{\,\omega}$). Here the LHS of (1.S) gives zero due to absence of $\partial/\partial z$. The first term of RHS $\left\{\mathrm{grad}\left(\tilde{\lambda}\cdot\mathrm{div}\vec{l}^{\,\omega}\right)\right\}_{+0^+}^{+\delta z^+}$ comes to $\vec{i}_z\cdot\left[\tilde{\lambda}\cdot\mathrm{div}\,\vec{l}^{\,\omega}\right]_{+0^+}^{+\delta z^+}$ due to $\vec{i}_\perp\left\{\partial\left(\tilde{\lambda}\cdot\mathrm{div}\,\vec{l}^{\,\omega}\right)/\partial r_\perp\right\}_{+0^+}^{+\delta z^+}=0$ $\left(r_\perp=x\text{ or }y\right)$. Indeed, assigning $\mathrm{div}_\perp\vec{l}_\perp^{\,\omega}=\frac{\partial l_x^\omega}{\partial x}+\frac{\partial l_y^\omega}{\partial y}$, we have: $\mathrm{div}\vec{l}^{\,\omega}=\mathrm{div}_\perp\vec{l}_\perp^{\,\omega}+\frac{\partial l_z^\omega}{\partial z}$.

Also $\left\{\frac{\partial}{\partial r_\perp}\left(\tilde{\lambda}\cdot\mathrm{div}_\perp\vec{l}_\perp^{\,\omega}\right)\right\}_{+0^+}^{+\delta z^+}=0$ due to absence of $\frac{\partial}{\partial z}$. The last of $\frac{\partial}{\partial r_\perp}\left\{\tilde{\lambda}\cdot\mathrm{div}\,\vec{l}^{\,\omega}\right\}_{+0^+}^{+\delta z^+}$ terms is $\partial\left\{\tilde{\lambda}\cdot\partial l_z^\omega/\partial z\right\}_{+0^+}^{+\delta z^+}/\partial r_\perp$, which is zero due to (4.a) (we will use (4.b) later)

$\left\{\tilde{\vartheta}\cdot\partial l_n^\omega/\partial z\right\}_{+0^+}^{+\delta z^+}=0$ (4.a) $\left\{\tilde{\vartheta}\cdot\partial^2 l_n^\omega/\partial z^2\right\}_{+0^+}^{+\delta z^+}=\left[\tilde{\vartheta}\cdot\partial l_n^\omega/\partial z\right]_{+0^+}^{+\delta z^+}$ (4.b) at $\delta z^+\to 0$.

Note: we assign $\tilde{\vartheta}$ as any of $\left(\tilde{\rho},\tilde{\lambda},\tilde{\mu}\right)$ ((4) can be proven, integrating $\{\ \}_{+0^+}^{+\delta z^+}$ by parts).

Next terms in (1.S), contributing to BC, are $\vec{i}_n\cdot\left\{\sum_{\mathrm{m=x,y,z}}\frac{\partial}{\partial r_m}\left(\tilde{\mu}\cdot\left(\frac{\partial l_n^\omega}{\partial r_m}+\frac{\partial l_m^\omega}{\partial r_n}\right)\right)\right\}_{+0^+}^{+\delta z^+}$.

For $\left(r_m=r_n=r_\perp\right)$ these terms are zero (NO $\partial/\partial z$). For $\left(r_m=r_\perp,\ r_n=z\right)$ such terms also are zero due to (4.a). If $\left(r_m=z,\ r_n=r_\perp\right)$, we come to $\vec{i}_\perp\cdot\left[\tilde{\mu}\left(\frac{\partial l_\perp^\omega}{\partial z}+\frac{\partial l_z^\omega}{\partial r_\perp}\right)\right]_{+0^+}^{+\delta z^+}$ for $l_{x,y}^\omega$.

For $r_m=r_n=z$, we come to $\vec{i}_z\cdot\left[2\cdot\tilde{\mu}\cdot\partial l_z^\omega/\partial z\right]_{+0^+}^{+\delta z^+}$. Combining all this, for $\left\{1.\mathrm{S}\right\}_{+0^+}^{+\delta z^+}$ we get:

$$\vec{i}_x\cdot\left[\tilde{\mu}\left(\frac{\partial l_x^\omega}{\partial z}+\frac{\partial l_z^\omega}{\partial x}\right)\right]_{+0^+}^{+\delta z^+}+\vec{i}_y\cdot\left[\tilde{\mu}\left(\frac{\partial l_y^\omega}{\partial z}+\frac{\partial l_z^\omega}{\partial y}\right)\right]_{+0^+}^{+\delta z^+}+\vec{i}_z\cdot\left[\tilde{\lambda}\cdot\mathrm{div}\,\vec{l}^{\,\omega}+2\cdot\tilde{\mu}\cdot\frac{\partial l_z^\omega}{\partial z}\right]_{+0^+}^{+\delta z^+}=-\left\{\vec{f}^{\,\omega}\right\}_{+0^+}^{+\delta z^+}.$$

Calculating $\left\{\mathrm{UHS}\right\}_{-\delta z_-}^{-0_-}$, we should integrate (1.F)(converted to spectrum (2) for $\vec{l}^{\,\omega}$). Here, applying $\partial\tilde{\rho}_F\left(z<0\right)/\partial r_\perp=0$, we have $\mathrm{div}\left(\tilde{\rho}_F\cdot\vec{l}\right)=\tilde{\rho}_F\cdot\mathrm{div}\vec{l}+l_z\cdot\partial\tilde{\rho}_F/\partial z$. Also we use (3) for $\tilde{\lambda}_F=\tilde{\lambda}_{z<0}$ (in spite of $\partial\tilde{\rho}_F/\partial z\neq 0$) and $\vec{l}^{\,\omega}$ - continuity $\vec{l}^{\,\omega}\big|_{-\delta z_-<z<-0_-}\approx\vec{l}^{\,\omega}\big|_{z=0}$ (this becomes exact equality, if $\delta z_-\to 0$), so at $\delta z_-\approx 0$ the first term of RHS in (1.F) gives:

$$\left\{\mathrm{grad}\left(l_z^\omega\cdot\tilde{V}_F^{\,2}\frac{\partial\tilde{\rho}_F}{\partial z}+\left(\tilde{\rho}_F\tilde{V}_F^{\,2}-i\omega\cdot\left(\tilde{\zeta}_F-\frac{2}{3}\tilde{\eta}_F\right)\right)\cdot\mathrm{div}\vec{l}^{\,\omega}\right)\right\}_{-\delta z_-}^{-0_-}\approx\left(\vec{i}_x\frac{\partial l_z^\omega}{\partial x}+\vec{i}_y\frac{\partial l_z^\omega}{\partial y}\right)\bigg|_{z=0}\left\{\tilde{V}_F^{\,2}\frac{\partial\tilde{\rho}_F}{\partial z}\right\}_{-\delta z_-}^{-0_-}+$$

$$+\vec{i}_z\,l_z^\omega\big|_{z=0}\left[\tilde{V}_F^{\,2}\frac{\partial\tilde{\rho}_F}{\partial z}\right]_{-\delta z_-}^{-0_-}+\left\{\mathrm{grad}\left(\tilde{\lambda}_F\left(z\right)\cdot\mathrm{div}\vec{l}^{\,\omega}\right)\right\}_{-\delta z_-}^{-0_-}.$$ Here the integrations of the last term

and $\sum_{\mathrm{m=x,y,z}} \frac{\partial}{\partial r_m}\left(\tilde{\eta}_F \cdot \frac{\partial}{\partial t}\left(\frac{\partial l_n}{\partial r_m} + \frac{\partial l_m}{\partial r_n} \right)\right)$ with $F^{\mathrm{Source}}_{\mathrm{n=x,y,z}}$ in (1.F) are the same as for the related terms in (1.S) (see above). Also $\left[\tilde{V}_F^{\,2} \cdot \frac{\partial \tilde{\rho}_F}{\partial z} \right]_{-\delta z_-}^{-0_-} = 0$ due to $\tilde{V}_F^{\,z=-0_-} = 0$ and $\frac{\partial \tilde{\rho}_F^{\,z=-\delta z_-}}{\partial z} = 0$.

The integration $\left\{ \tilde{V}_F^{\,2} \cdot \partial \tilde{\rho}_F / \partial z \right\}_{-\delta z_-}^{-0_-}$ at $\delta z_- \approx 0$ is questionable, because the derivative $\partial \tilde{\rho}_F / \partial z$ near $z = 0$ comes to infinity. In spite of this $\left\{ \tilde{V}_F^{\,2} \cdot \partial \tilde{\rho}_F / \partial z \right\}_{-\delta z_-}^{-0_-} \to -\rho_0 \cdot V_0^{\,2}$. Indeed, $\delta z_V \ll \delta z_\rho$ (see Note 3), so in the most part of UHS we have $\tilde{V}_F^{\,z<-0_-} = V_0 = \mathrm{const}$, allowing trivial integration $\{\ \}_{-\delta z_-}^{-0_-}$. Simple example:

$\tilde{V}_F\left(z < -\delta z_V \right) = V_0,\ \ \tilde{V}_F^{\,2}\left(-\delta z_V < z < -0_- \right) = -z \cdot V_0^{\,2} / \delta z_V;\ \ \tilde{\rho}_F\left(z < -\delta z_\rho \right) = \rho_0$,

$\tilde{\rho}_F\left(-\delta z_\rho < z < -0_- \right) = -z \cdot \rho_0 / \delta z_\rho$; here $\left\{ \tilde{V}_F^{\,2} \cdot \partial \tilde{\rho}_F / \partial z \right\}_{-\delta z_-}^{-0_-} = -\rho_0 \cdot V_0^{\,2} \cdot \left(1 - 1.5 * \delta z_V / \delta z_\rho \right)$.

Now, integrating $\{\mathrm{eq.1}\}_{-\delta z_-}^{+\delta z^+} = \{1.\mathrm{F}\}_{-\delta z_-}^{-0_-} + \{\ \}_{-0_-}^{+0^+} + \{1.\mathrm{S}\}_{+0^+}^{+\delta z^+}$, using $\tilde{\vartheta}_{z=0} = 0$ and finite $\partial l_m^\omega / \partial r_n$ (see Fig.1,2 with assumed $\delta z_-^+ \to 0$), we get:

$$\vec{i}_x \cdot \left[\tilde{\mu} \cdot \left(\frac{\partial l_x^\omega}{\partial z} + \frac{\partial l_z^\omega}{\partial x} \right) \right]_{-\delta z_-}^{+\delta z^+} + \vec{i}_y \cdot \left[\tilde{\mu} \cdot \left(\frac{\partial l_y^\omega}{\partial z} + \frac{\partial l_z^\omega}{\partial y} \right) \right]_{-\delta z_-}^{+\delta z^+} + \vec{i}_z \cdot \left[\tilde{\lambda} \cdot \mathrm{div}\, \vec{l}^{\,\omega} + 2\tilde{\mu} \cdot \frac{\partial l_z^\omega}{\partial z} \right]_{-\delta z_-}^{+\delta z^+} -$$

$$-\rho_0 \cdot V_0^{\,2} \cdot \left(\vec{i}_x \cdot \partial l_z^\omega / \partial x + \vec{i}_y \cdot \partial l_z^\omega / \partial y \right)\Big|_{z=0} = -\left\{ \vec{f}^{\,\omega} \right\}_{-\delta z_-}^{+\delta z^+}$$

Note: if $z_{\mathrm{min,max}}^{\mathrm{source}}$ does not embrace $z = 0$: RHS here and in following BC are zeroes.

Note: we do not use typical for UHS = NVF values $l_{x,y}^\omega\big|_{z<0} = 0$ in BC and in (1.F).

Note: generally the tangential and normal stresses are:

$$\tilde{\sigma}_{xz}^\omega \equiv \tilde{\mu} \cdot \left(\frac{\partial l_x^\omega}{\partial z} + \frac{\partial l_z^\omega}{\partial x} \right),\ \ \tilde{\sigma}_{yz}^\omega \equiv \tilde{\mu} \cdot \left(\frac{\partial l_y^\omega}{\partial z} + \frac{\partial l_z^\omega}{\partial y} \right),\ \ \tilde{\sigma}_{zz}^\omega \equiv \tilde{\lambda} \cdot \mathrm{div}\, \vec{l}^{\,\omega} + 2\tilde{\mu} \cdot \frac{\partial l_z^\omega}{\partial z}.$$

Note: if UHS = NVF & $\tilde{\rho}_{\mathrm{F}} = \mathrm{const}$: (3.a) gives $\tilde{\lambda}_F = \tilde{\rho}_F \tilde{V}_F^2,\ \tilde{\mu}_F = 0$, so $\tilde{\sigma}_{zz}^\omega\big|_{z<0} = \tilde{\rho}_F \tilde{V}_F^2 \left(\partial l_z^\omega / \partial z \right)$.

Now with $\vec{l}^{\,\omega}$- continuity around $z=0$, we have final BC (5) (at $\delta z_{-}^{+} \to 0$):

$$\left[l_x^{\omega} \right]_{-\delta z_-}^{+\delta z^+} = \left[l_y^{\omega} \right]_{-\delta z_-}^{+\delta z^+} = \left[l_z^{\omega} \right]_{-\delta z_-}^{+\delta z^+} = \mathbf{0} \qquad (5.0)$$

$$\left[\tilde{\mu} \cdot \frac{\partial l_x^{\omega}}{\partial z} \right]_{-\delta z_-}^{+\delta z^+} + \left(\mu - \mu_0 - \rho_0 \cdot V_0^{\,2} \right) \cdot \left. \frac{\partial l_z^{\omega}}{\partial x} \right|_{z=0} = -\left\{ \vec{f}_x^{\,\omega} \right\}_{-\delta z_-}^{+\delta z^+} \qquad (5.x)$$

$$\left[\tilde{\mu} \cdot \frac{\partial l_y^{\omega}}{\partial z} \right]_{-\delta z_-}^{+\delta z^+} + \left(\mu - \mu_0 - \rho_0 \cdot V_0^{\,2} \right) \cdot \left. \frac{\partial l_z^{\omega}}{\partial y} \right|_{z=0} = -\left\{ \vec{f}_y^{\,\omega} \right\}_{-\delta z_-}^{+\delta z^+} \qquad (5.y)$$

$$\left[\left(\tilde{\lambda} + 2\tilde{\mu} \right) \cdot \frac{\partial l_z^{\omega}}{\partial z} \right]_{-\delta z_-}^{+\delta z^+} + \left(\lambda - \lambda_0 \right) \cdot \left. \left(\frac{\partial l_x^{\omega}}{\partial x} + \frac{\partial l_y^{\omega}}{\partial y} \right) \right|_{z=0} = -\left\{ \vec{f}_z^{\,\omega} \right\}_{-\delta z_-}^{+\delta z^+} \qquad (5.z)$$

Note: (5.0) means smooth $\left. \vec{l}^{\,\omega} \right|_{z\sim 0}$: no infinite derivatives, but $\left. \partial \vec{l}^{\,\omega} / \partial z \right|_{z=0_-} \neq \left. \partial \vec{l}^{\,\omega} / \partial z \right|_{z=0^+}$.

Using the cylindrical coordinates, (5) comes to (6):

$$\left[l_R^{\omega} \right]_{-\delta z_-}^{+\delta z^+} = \left[l_{\varphi}^{\omega} \right]_{-\delta z_-}^{+\delta z^+} = \left[l_z^{\omega} \right]_{-\delta z_-}^{+\delta z^+} = \mathbf{0} \qquad (6.0)$$

$$\left[\tilde{\mu} \cdot \frac{\partial l_R^{\omega}}{\partial z} \right]_{-\delta z_-}^{+\delta z^+} + \left(\mu - \mu_0 - \rho_0 \cdot V_0^{\,2} \right) \cdot \frac{\partial \left. l_z^{\omega} \right|_{z=0}}{\partial R} = -\left\{ f_R^{\omega} \right\}_{-\delta z_-}^{+\delta z^+} \qquad (6.R)$$

$$\left[\tilde{\mu} \cdot \frac{\partial l_{\varphi}^{\omega}}{\partial z} \right]_{-\delta z_-}^{+\delta z^+} + \left(\mu - \mu_0 - \rho_0 \cdot V_0^{\,2} \right) \cdot \frac{\partial \left. l_z^{\omega} \right|_{z=0}}{R \cdot \partial \varphi} = -\left\{ f_{\varphi}^{\omega} \right\}_{-\delta z_-}^{+\delta z^+} \qquad (6.\varphi)$$

$$\left[\left(\tilde{\lambda} + 2\tilde{\mu} \right) \cdot \frac{\partial l_z^{\omega}}{\partial z} \right]_{-\delta z_-}^{+\delta z^+} + \left(\lambda - \lambda_0 \right) \cdot \left(\frac{\partial l_R^{\omega}}{\partial R} + \frac{l_R^{\omega}}{R} + \frac{1}{R} \cdot \frac{\partial l_{\varphi}^{\omega}}{\partial \varphi} \right)_{z=0} = -\left\{ f_z^{\omega} \right\}_{-\delta z_-}^{+\delta z^+} \qquad (6.z)$$

Note: for vacuum-solid boundary, taking into account (3.a) & Note 4, we can make a limit transfer $\rho_0 \to 0,\ \lambda_0 = \mu_0 \to 0$ due to $\zeta_0 = \eta_0 = 0$ and get: $\left. \vec{l}^{\,\omega} \right|_{z=0^+} = 0$ or $\left(l_R^{\omega} = l_{\varphi}^{\omega} = l_z^{\omega} \right)_{z=0^+} = 0$ (6.0*)

$$\left[\tilde{\sigma}_{Rz}^{\omega} \right]_{-\delta z_-}^{+\delta z^+} = -\left\{ f_R^{\omega} \right\}_{-\delta z_-}^{+\delta z^+} \text{ or } \mu \cdot \left(\frac{\partial l_R^{\omega}}{\partial z} + \frac{\partial l_z^{\omega}}{\partial R} \right)_{z=0^+} = -\left\{ f_R^{\omega} \right\}_{-\delta z_-}^{+\delta z^+} \qquad (6.R^*)$$

$$\left[\tilde{\sigma}_{\varphi z}^{\omega} \right]_{-\delta z_-}^{+\delta z^+} = -\left\{ f_{\varphi}^{\omega} \right\}_{-\delta z_-}^{+\delta z^+} \text{ or } \mu \cdot \left(\frac{\partial l_{\varphi}^{\omega}}{\partial z} + \frac{1}{R} \cdot \frac{\partial l_z^{\omega}}{\partial \varphi} \right)_{z=0^+} = -\left\{ f_{\varphi}^{\omega} \right\}_{-\delta z_-}^{+\delta z^+} \qquad (6.\varphi\ ^*)$$

$$\left[\tilde{\sigma}_{zz}^{\omega} \right]_{-\delta z_-}^{+\delta z^+} = -\left\{ f_z^{\omega} \right\}_{-\delta z_-}^{+\delta z^+} \text{ or } \lambda \cdot \left(\frac{\partial l_R^{\omega}}{\partial R} + \frac{l_R^{\omega}}{R} + \frac{1}{R} \cdot \frac{\partial l_{\varphi}^{\omega}}{\partial \varphi} + \frac{\partial l_z^{\omega}}{\partial z} \right)_{z=0^+} + 2\mu \cdot \left. \frac{\partial l_z^{\omega}}{\partial z} \right|_{z=0^+} = -\left\{ f_z^{\omega} \right\}_{-\delta z_-}^{+\delta z^+} \qquad (6.z^*)$$

Below we denote $\hat{\vartheta} = \left(\vartheta_0^{z<0}, \vartheta^{z>0}\right)$ as any of $\left(\hat{\rho}, \hat{\lambda}, \hat{\mu}\right)$ to satisfy (1.0)(in (4) we assigned $\tilde{\vartheta}$ as any of $\left(\tilde{\rho}, \tilde{\lambda}, \tilde{\mu}\right)$) and $\Delta\vartheta_{\mathrm{A}}$ as any of the Anomaly properties' deviations $\left(\Delta\rho_{\mathrm{A}}, \Delta\lambda_{\mathrm{A}}, \Delta\mu_{\mathrm{A}}\right)$ ($\Delta\vartheta_{\mathrm{A}}$ can be as big as $\hat{\vartheta}$). Also we denote ξ as any of the cylindrical coordinates $\left(R, \left(R\cdot\varphi\right), z\right)$, and $\Delta\xi_{\mathrm{A}}$ as Anomaly's sizes. We suppose small $\Delta\xi_{\mathrm{A}}$, assuming that the Anomaly's influence on the strains $\vec{l}$ should come to zero, if $\Delta\xi_{\mathrm{A}} \to 0$ (see above the Anomaly's type: it is not a δ-Function, which can affect $\vec{l}$, even when $\Delta\xi_{\mathrm{A}} \to 0$). Particularly we have: $l_{\varphi}^{\Delta\xi_{\mathrm{A}}\to 0} \to 0,\ \ \partial l_{R,z}^{\Delta\xi_{\mathrm{A}}\to 0} / \partial\varphi \to 0$, because for No-Anomaly case we have cylindrical symmetry, so $l_{\varphi}^{\Delta\xi_{\mathrm{A}}=0} = 0;\ \ \partial l_{R,z}^{\Delta\xi_{\mathrm{A}}=0} / \partial\varphi = 0$.

Note: we assumed (see above) that the Anomaly does not touch or cross the edge $z = 0$, so we can replace $\tilde{\vartheta}\left(z \sim 0^{+}\right) \equiv \hat{\vartheta}$ (but not in general equations).

Note: below it is shown that the Anomaly's effect on $\vec{l}$ can be represented as triple integrals like $\int_{\xi_{\min}^{\mathrm{Anom}}}^{\xi_{\max}^{\mathrm{Anom}}} d\xi \cdot a \cdot \frac{\partial\left(b\cdot\Delta\tilde{\vartheta}\right)}{\partial\xi} = -\int_{\xi_{\min}^{\mathrm{Anom}}}^{\xi_{\max}^{\mathrm{Anom}}} d\xi \cdot \frac{\partial a}{\partial\xi} \cdot b \cdot \Delta\tilde{\vartheta} = -\Delta\xi_{\mathrm{A}} \cdot \frac{\partial a}{\partial\xi_{\mathrm{A}}} \cdot b_{\mathrm{A}} \cdot \Delta\vartheta_{\mathrm{A}}$ (here $\Delta\vartheta\left(\xi_{\mathrm{min,max}}^{\mathrm{Anomaly}}\right) = 0$ were taken into account, while integrating by parts) or $\int_{\xi_{\min}^{\mathrm{Anom.}}}^{\xi_{\max}^{\mathrm{Anom.}}} d\xi \cdot a \cdot \Delta\tilde{\vartheta} = \Delta\xi_{\mathrm{A}} \cdot a_{\mathrm{A}} \cdot \Delta\vartheta_{\mathrm{A}}$, so this effect finally is proportional to Anomaly's small Volume $\Delta\Omega_{\mathrm{A}} = R_{\mathrm{A}} \cdot \Delta R_{\mathrm{A}} \cdot \Delta\varphi_{\mathrm{A}} \cdot \Delta z_{\mathrm{A}}$, so influence of $\Delta\vartheta_{\mathrm{A}}$ on $\vec{l}$ will always go with small factor $\Delta\Omega_{\mathrm{A}}$. Thus, in decompositions of the integral kernels like $\tilde{\vartheta} = \hat{\vartheta} + \Delta\vartheta_{\mathrm{A}}$, we can consider $\Delta\vartheta_{\mathrm{A}}$ as small in spite of their really big values (we assume $\Delta\vartheta_{\mathrm{A}} \sim \hat{\vartheta}$). Note: the derivatives $\partial\vartheta_{\mathrm{A}} / \partial\xi$ can be big also, due to finite $\Delta\vartheta_{\mathrm{A}}$ and small thicknesses $\Delta\xi_{\mathrm{A}}^{\mathrm{thickness}} \to 0$. However self-cancelling happens for $\partial\vartheta_{\mathrm{A}} / \partial\xi$, if $\Delta\xi_{\mathrm{A}}^{\mathrm{thickness}} \to 0$, because here $\partial\vartheta_{\mathrm{A}} / \partial\xi$ at opposite edges of narrow Anomaly come with opposite signs.

Below we use cylindrical source $\vec{f}_{\mathrm{source}}^{\,\omega} = \left(f_R^{\omega}, f_{\varphi}^{\omega}, f_z^{\omega}\right)$, particularly allowing analytical solution of axis-symmetrical seismic task for LM = SHS: $f_{\varphi}^{\omega} = 0 \left(l_{\varphi}^{\omega} = \partial l_{R,z}^{\omega} / \partial\varphi = 0\right)$. Also, when $z_{\mathrm{min,max}}^{\mathrm{source}}$ embraces $z = 0$, we should leave $\vec{f}^{\,\omega}$ in RHS of BC, but remove $\vec{f}^{\,\omega}$ from (1). However, for general dynamic equations we allow $f_{\varphi}^{\omega} \neq 0$. Thus we re-write general (1.S) (converted for $\vec{l}^{\,\omega}$) in the cylindrical coordinates at $f_{\varphi}^{\omega} \neq 0$ (the same for (1.F) at $z < \delta z_{-}$), and keep $\left(\tilde{\rho}, \tilde{\lambda}, \tilde{\mu}\right)$ under derivatives.

Basic Equations

$$\frac{\partial}{\partial R}\left(\tilde{\lambda}\cdot\left(\frac{\partial l_R^\omega}{\partial R}+\frac{l_R^\omega}{R}+\frac{1}{R}\cdot\frac{\partial l_\varphi^\omega}{\partial\varphi}+\frac{\partial l_z^\omega}{\partial z}\right)\right)+2\cdot\frac{\partial}{\partial R}\left(\frac{\tilde{\mu}}{R}\cdot\left(l_R^\omega+\frac{\partial l_\varphi^\omega}{\partial\varphi}\right)\right)+$$

$$+\frac{2}{R}\cdot\frac{\partial}{\partial R}\left(\tilde{\mu}\cdot\left(R\cdot\frac{\partial l_R^\omega}{\partial R}-l_R^\omega-\frac{\partial l_\varphi^\omega}{\partial\varphi}\right)\right)+\frac{1}{R^2}\cdot\frac{\partial}{\partial\varphi}\left(\tilde{\mu}\cdot\left(\frac{\partial l_R^\omega}{\partial\varphi}-l_\varphi^\omega+R\cdot\frac{\partial l_\varphi^\omega}{\partial R}\right)\right)+ \qquad (7.R)$$

$$+\frac{\partial}{\partial z}\left(\tilde{\mu}\cdot\left(\frac{\partial l_R^\omega}{\partial z}+\frac{\partial l_z^\omega}{\partial R}\right)\right)+\tilde{\rho}\cdot\omega^2\cdot l_R^\omega+f_R^\omega=0$$

$$\frac{1}{R}\cdot\frac{\partial}{\partial\varphi}\left(\tilde{\lambda}\cdot\left(\frac{\partial l_R^\omega}{\partial R}+\frac{l_R^\omega}{R}+\frac{1}{R}\cdot\frac{\partial l_\varphi^\omega}{\partial\varphi}+\frac{\partial l_z^\omega}{\partial z}\right)\right)+\frac{1}{R^2}\cdot\frac{\partial}{\partial R}\left(\tilde{\mu}\cdot R\cdot\left(\frac{\partial l_R^\omega}{\partial\varphi}-l_\varphi^\omega\right)\right)+$$

$$+\frac{2}{R}\cdot\frac{\partial}{\partial R}\left(\tilde{\mu}\cdot R\cdot\frac{\partial l_\varphi^\omega}{\partial R}\right)-\frac{\partial}{\partial R}\left(\tilde{\mu}\cdot\frac{\partial l_\varphi^\omega}{\partial R}\right)+\frac{2}{R^2}\cdot\frac{\partial}{\partial\varphi}\left(\tilde{\mu}\cdot\left(l_R^\omega+\frac{\partial l_\varphi^\omega}{\partial\varphi}\right)\right)+ \qquad (7.\varphi)$$

$$+\frac{\partial}{\partial z}\left[\tilde{\mu}\cdot\left(\frac{\partial l_\varphi^\omega}{\partial z}+\frac{1}{R}\cdot\frac{\partial l_z^\omega}{\partial\varphi}\right)\right]+\tilde{\rho}\cdot\omega^2\cdot l_\varphi^\omega+f_\varphi^\omega=0$$

$$\frac{\partial}{\partial z}\left(\tilde{\lambda}\cdot\left(\frac{\partial l_R^\omega}{\partial R}+\frac{l_R^\omega}{R}+\frac{1}{R}\cdot\frac{\partial l_\varphi^\omega}{\partial\varphi}+\frac{\partial l_z^\omega}{\partial z}\right)\right)+\frac{1}{R}\cdot\frac{\partial}{\partial R}\left(\tilde{\mu}\cdot R\cdot\left(\frac{\partial l_R^\omega}{\partial z}+\frac{\partial l_z^\omega}{\partial R}\right)\right)+$$

$$+\frac{1}{R}\cdot\frac{\partial}{\partial\varphi}\left(\tilde{\mu}\cdot\left(\frac{\partial l_\varphi^\omega}{\partial z}+\frac{1}{R}\cdot\frac{\partial l_z^\omega}{\partial\varphi}\right)\right)+2\frac{\partial}{\partial z}\left(\tilde{\mu}\cdot\frac{\partial l_z^\omega}{\partial z}\right)+\tilde{\rho}\cdot\omega^2\cdot l_z^\omega+f_z^\omega=0 \qquad (7.z)$$

Let $\vec{\ell}^\omega(R,z)$ represent a seismic solution of the simplest axis-symmetrical task: $f_\varphi^\omega=0$; LM = SHS, NO Anomaly ($\Delta\xi_A=0$). Here $\ell_\varphi^\omega=0,\ \partial\ell_{R,z}^\omega/\partial\varphi=0$, and in (7) we can neglect changes of general properties $\tilde{\vartheta}$, caused by Anomaly-related $\Delta\vartheta_A$. This allows to take out $\tilde{\vartheta}$ from under the derivatives (replacing $\tilde{\vartheta}\to\hat{\vartheta}$), thus for the zero-order decomposition terms $\vec{l}^\omega(R,\varphi,z)\to\vec{\ell}^\omega(R,z)$, the eq.(7.R,z) in SHS give:

$$\hat{\mu}\cdot\frac{\partial^2 \boldsymbol{\ell}_R^{\omega}}{\partial z^2}+\left(\hat{\lambda}+2\hat{\mu}\right)\cdot\left(\frac{\partial^2 \boldsymbol{\ell}_R^{\omega}}{\partial R^2}+\frac{1}{R}\frac{\partial \boldsymbol{\ell}_R^{\omega}}{\partial R}-\frac{\boldsymbol{\ell}_R^{\omega}}{R^2}\right)+\left(\hat{\lambda}+\hat{\mu}\right)\cdot\frac{\partial^2 \boldsymbol{\ell}_z^{\omega}}{\partial R\cdot\partial z}+\omega^2\cdot\hat{\rho}\cdot\boldsymbol{\ell}_R^{\omega}=-f_R^{\omega} \quad \text{(8.R)}$$

$$\left(\hat{\lambda}+2\hat{\mu}\right)\cdot\frac{\partial^2 \boldsymbol{\ell}_z^{\omega}}{\partial z^2}+\hat{\mu}\cdot\left(\frac{\partial^2 \boldsymbol{\ell}_z^{\omega}}{\partial R^2}+\frac{1}{R}\cdot\frac{\partial \boldsymbol{\ell}_z^{\omega}}{\partial R}\right)+\left(\hat{\lambda}+\hat{\mu}\right)\cdot\frac{\partial}{\partial z}\left(\frac{\partial \boldsymbol{\ell}_R^{\omega}}{\partial R}+\frac{\boldsymbol{\ell}_R^{\omega}}{R}\right)+\omega^2\cdot\hat{\rho}\cdot\boldsymbol{\ell}_z^{\omega}=-f_z^{\omega} \quad \text{(8.z)}$$

Here we can use [5] for general case (NO φ-symmetry), ignoring ω-notations:

$$\text{grad}\ a=\left(\frac{\partial a}{\partial R},\ \frac{1}{R}\frac{\partial a}{\partial \varphi},\ \frac{\partial a}{\partial z}\right), \qquad \text{di}\upsilon\vec{A}=\frac{\partial A_R}{\partial R}+\frac{A_R}{R}+\frac{1}{R}\frac{\partial A_\varphi}{\partial \varphi}+\frac{\partial A_z}{\partial z}$$

$$\Delta a=\frac{\partial^2 a}{\partial R^2}+\frac{1}{R}\cdot\frac{\partial a}{\partial R}+\frac{1}{R^2}\frac{\partial^2 a}{\partial \varphi^2}+\frac{\partial^2 a}{\partial z^2}, \qquad \left(\Delta\vec{A}\right)_R=\Delta\left(A_R\right)-\frac{A_R}{R^2}-\frac{2}{R^2}\frac{\partial A_\varphi}{\partial \varphi}$$

$$\left(\Delta\vec{A}\right)_\varphi=\Delta\left(A_\varphi\right)-\frac{A_\varphi}{R^2}+\frac{2}{R^2}\frac{\partial A_R}{\partial \varphi}, \qquad \left(\Delta\vec{A}\right)_z=\Delta\left(A_z\right) \quad \text{(9)}$$

Applying the related parts of Laplacians (9) and omitting terms $\boldsymbol{\ell}_\varphi^{\omega}=0,\ \partial\boldsymbol{\ell}_{R,z}^{\omega}/\partial\varphi=0$, we can re-write (8) for $\vec{\boldsymbol{\ell}}^{\omega}$, taking into account its φ-symmetry:

$$\left(\hat{\lambda}+\hat{\mu}\right)\cdot\text{grad}_{\xi=R,z}\text{di}\upsilon\,\vec{\boldsymbol{\ell}}^{\omega}+\hat{\mu}\cdot\left(\Delta\vec{\boldsymbol{\ell}}^{\omega}\right)_\xi+\omega^2\cdot\hat{\rho}\cdot\boldsymbol{\ell}_\xi^{\omega}=-f_\xi^{\omega} \quad \text{(10)}$$

Let $\mathbb{u}_{R,z}^{\omega}\ll\boldsymbol{\ell}_{R,z}^{\omega}$ represent the Anomaly's small influence on $l_{R,z}^{\omega}$ (due to small $\Delta\xi_A$), so

$$\vec{l}^{\,\omega}\left(R,\varphi,z\right)=\vec{\boldsymbol{\ell}}^{\omega}\left(R,z\right)+\vec{\mathbb{u}}^{\omega}\left(R,\varphi,z\right) \quad \text{(11)}$$

Note: even when LM contains narrow Anomaly (where $\Delta\xi_A\sim 0$): $\mathbb{u}_\varphi^{\omega}$ and $\partial\mathbb{u}_{R,z}^{\omega}/\partial\varphi$ are small. Indeed, $\mathbb{u}_{\varphi,\omega}^{\Delta\xi_A\to 0}\to 0$ because $\mathbb{u}_{\varphi,\omega}^{\Delta\xi_A=0}=l_{\varphi,\omega}^{\Delta\xi_A=0}=0$ due to $\boldsymbol{\ell}_\varphi^{\omega}=0$. Also $\partial\mathbb{u}_{R,z,\omega}^{\Delta\xi_A\to 0}/\partial\varphi\to 0$ because $\partial\mathbb{u}_{R,z,\omega}^{\Delta\xi_A=0}/\partial\varphi=0$ due to $\partial l_{R,z,\omega}^{\Delta\xi_A=0}/\partial\varphi=\partial\boldsymbol{\ell}_{R,z}^{\omega}/\partial\varphi=0$.

In (7) the products similar to $\tilde{\mu}\cdot\vec{l}^{\,\omega}=\left(\hat{\mu}+\Delta\mu_A\right)\cdot\left(\vec{\boldsymbol{\ell}}^{\omega}+\vec{\mathbb{u}}^{\omega}\right)$ can be decomposed, ignoring the second-order by $\Delta\xi_A\sim 0$ terms: $\tilde{\mu}\cdot\vec{l}^{\,\omega}=\left(\hat{\mu}\cdot\vec{\boldsymbol{\ell}}^{\omega}\right)+\left(\Delta\mu_A\cdot\vec{\boldsymbol{\ell}}^{\omega}+\hat{\mu}\cdot\vec{\mathbb{u}}^{\omega}\right)+\ldots$.

Taking into account the zero-order decomposition terms for $\vec{\boldsymbol{\ell}}^{\omega}$ (8) or (10), the first-order terms from (7,11) give:

$$\hat{\mu}\cdot\left(\frac{\partial^2 \mathrm{u}_R^\omega}{\partial z^2}+\frac{1}{R^2}\cdot\frac{\partial^2 \mathrm{u}_R^\omega}{\partial \varphi^2}-\frac{2}{R^2}\cdot\frac{\partial l_\varphi^\omega}{\partial \varphi}\right)+\left(\hat{\lambda}+\hat{\mu}\right)\cdot\left(\frac{1}{R}\cdot\frac{\partial^2 l_\varphi^\omega}{\partial R\cdot\partial \varphi}-\frac{1}{R^2}\cdot\frac{\partial l_\varphi^\omega}{\partial \varphi}+\frac{\partial^2 \mathrm{u}_z^\omega}{\partial R\cdot\partial z}\right)+$$

$$+\left(\hat{\lambda}+2\hat{\mu}\right)\cdot\left(\frac{\partial^2 \mathrm{u}_R^\omega}{\partial R^2}+\frac{1}{R}\cdot\frac{\partial \mathrm{u}_R^\omega}{\partial R}-\frac{\mathrm{u}_R^\omega}{R^2}\right)+\omega^2\cdot\hat{\rho}\cdot\mathrm{u}_R^\omega=-\hat{\rho}\cdot\theta_R^\omega \qquad \text{(12.R,φ,z)}$$

$$\hat{\mu}\cdot\left(\frac{\partial^2 l_\varphi^\omega}{\partial z^2}+\frac{\partial^2 l_\varphi^\omega}{\partial R^2}+\frac{1}{R}\cdot\frac{\partial l_\varphi^\omega}{\partial R}-\frac{l_\varphi^\omega}{R^2}+\frac{1}{R^2}\cdot\frac{\partial^2 l_\varphi^\omega}{\partial \varphi^2}+\frac{2}{R^2}\cdot\frac{\partial \mathrm{u}_R^\omega}{\partial \varphi}\right)+$$

$$+\frac{\left(\hat{\lambda}+\hat{\mu}\right)}{R}\cdot\frac{\partial}{\partial \varphi}\left(\frac{\partial \mathrm{u}_R^\omega}{\partial R}+\frac{\mathrm{u}_R^\omega}{R}+\frac{1}{R}\cdot\frac{\partial l_\varphi^\omega}{\partial \varphi}+\frac{\partial \mathrm{u}_z^\omega}{\partial z}\right)+\omega^2\cdot\hat{\rho}\cdot l_\varphi^\omega=-\hat{\rho}\cdot\theta_\varphi^\omega$$

$$\hat{\mu}\cdot\left(\frac{\partial^2 \mathrm{u}_z^\omega}{\partial z^2}+\frac{\partial^2 \mathrm{u}_z^\omega}{\partial R^2}+\frac{1}{R}\cdot\frac{\partial \mathrm{u}_z^\omega}{\partial R}+\frac{1}{R^2}\cdot\frac{\partial^2 \mathrm{u}_z^\omega}{\partial \varphi^2}\right)+$$

$$+\left(\hat{\lambda}+\hat{\mu}\right)\cdot\frac{\partial}{\partial z}\left(\frac{\partial \mathrm{u}_z^\omega}{\partial z}+\frac{\partial \mathrm{u}_R^\omega}{\partial R}+\frac{\mathrm{u}_R^\omega}{R}+\frac{1}{R}\cdot\frac{\partial l_\varphi^\omega}{\partial \varphi}\right)+\omega^2\cdot\hat{\rho}\cdot\mathrm{u}_z^\omega=-\hat{\rho}\cdot\theta_z^\omega$$

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where the components $\theta_{\xi=(R,\varphi,z)}^\omega\left(R,\varphi,z\right)$ are:

$$\hat{\rho}\cdot\theta_R^\omega=\frac{\partial}{\partial R}\left(\Delta\lambda_A\cdot\left(\frac{\partial \boldsymbol{\ell}_R^\omega}{\partial R}+\frac{\boldsymbol{\ell}_R^\omega}{R}+\frac{\partial \boldsymbol{\ell}_z^\omega}{\partial z}\right)\right)+\frac{2}{R}\cdot\frac{\partial}{\partial R}\left(\Delta\mu_A\cdot\left(R\cdot\frac{\partial \boldsymbol{\ell}_R^\omega}{\partial R}-\boldsymbol{\ell}_R^\omega\right)\right)+$$

$$+2\cdot\frac{\partial}{\partial R}\left(\Delta\mu_A\cdot\frac{\boldsymbol{\ell}_R^\omega}{R}\right)+\frac{\partial}{\partial z}\left(\Delta\mu_A\cdot\left(\frac{\partial \boldsymbol{\ell}_R^\omega}{\partial z}+\frac{\partial \boldsymbol{\ell}_z^\omega}{\partial R}\right)\right)+\omega^2\cdot\Delta\rho_A\cdot\boldsymbol{\ell}_R^\omega$$

$$\hat{\rho}\cdot\theta_\varphi^\omega=\frac{\partial\Delta\lambda_A}{\partial \varphi}\cdot\frac{1}{R}\cdot\left(\frac{\partial \boldsymbol{\ell}_R^\omega}{\partial R}+\frac{\boldsymbol{\ell}_R^\omega}{R}+\frac{\partial \boldsymbol{\ell}_z^\omega}{\partial z}\right)+2\cdot\frac{\partial\Delta\mu_A}{\partial \varphi}\cdot\frac{\boldsymbol{\ell}_R^\omega}{R^2} \qquad \text{(13.R,φ,z)}$$

$$\hat{\rho}\cdot\theta_z^\omega=\frac{\partial}{\partial z}\left(\Delta\lambda_A\cdot\left(\frac{\partial \boldsymbol{\ell}_R^\omega}{\partial R}+\frac{\boldsymbol{\ell}_R^\omega}{R}+\frac{\partial \boldsymbol{\ell}_z^\omega}{\partial z}\right)\right)+2\cdot\frac{\partial}{\partial z}\left(\Delta\mu_A\cdot\frac{\partial \boldsymbol{\ell}_z^\omega}{\partial z}\right)+$$

$$+\frac{1}{R}\cdot\frac{\partial}{\partial R}\left(\Delta\mu_A\cdot R\cdot\left(\frac{\partial \boldsymbol{\ell}_R^\omega}{\partial z}+\frac{\partial \boldsymbol{\ell}_z^\omega}{\partial R}\right)\right)+\omega^2\cdot\Delta\rho_A\cdot\boldsymbol{\ell}_z^\omega$$

Note: both $\theta^{\omega}_{R,z}$ components (13.R,z) do not contain $\partial\Delta\vartheta_A/\partial\varphi$. Also, using (10), we re-write the first decomposition terms (12) for $\vec{\mathrm{u}}^{\omega}$ in similar form:

$$\left(\hat{\lambda}+\hat{\mu}\right)\cdot \mathrm{grad}_{\xi=R,\varphi,z}\mathrm{di\upsilon}\,\vec{\mathrm{u}}^{\omega}+\hat{\mu}\cdot\left(\Delta\vec{\mathrm{u}}^{\omega}\right)_{\xi}+\omega^2\cdot\hat{\rho}\cdot\mathrm{u}^{\omega}_{\xi}=-\hat{\rho}\cdot\theta^{\omega}_{\xi} \qquad (14)$$

Now we apply Fourier-Bessel representations for the φ-symmetrical $\ell^{\omega}_{R,z}$ (15.R,z)

$$\ell^{\omega}_R\left(R,z\right)=\int_0^{\infty}d\text{æ}\cdot\text{æ}\cdot J_1\left(\text{æ}\cdot R\right)\cdot\Lambda^{\omega}_R\left(\text{æ},z\right),$$

$$\Lambda^{\omega}_R\left(\text{æ},z\right)=\int_0^{\infty}dr\cdot r\cdot J_1\left(\text{æ}\cdot r\right)\cdot\ell^{\omega}_R\left(r,z\right)\ \ell^{\omega}_z\left(R,z\right)=\int_0^{\infty}d\text{æ}\cdot\text{æ}\cdot J_0\left(\text{æ}\cdot R\right)\cdot\Lambda^{\omega}_z\left(\text{æ},z\right),$$

$$\Lambda^{\omega}_z\left(\text{æ},z\right)=\int_0^{\infty}dr\cdot r\cdot J_0\left(\text{æ}\cdot r\right)\cdot\ell^{\omega}_z\left(r,z\right)$$

For φ-asymmetrical components $\mathrm{u}^{\omega}_{R,z}$ and $\mathrm{u}^{\omega}_{\varphi}\equiv l^{\omega}_{\varphi}$ we apply (16.a,b)

$$\mathrm{u}^{\omega}_{R,\varphi,z}\left(R,\varphi,z\right)=\sum_{\mathrm{k}=-\infty}^{\infty}e^{i\cdot\mathrm{k}\cdot\varphi}\cdot\int_0^{\infty}d\text{æ}\cdot\text{æ}\cdot J_{\mathrm{m_k},\mathrm{L_k},\mathrm{n_k}}\left(\text{æ}\cdot R\right)\cdot\mathbb{U}^{\omega}_{R,\varphi,z}\left(\text{æ},\mathrm{k},z\right)$$

$$\mathbb{U}^{\omega}_{R,\varphi,z}\left(\text{æ},\mathrm{k},z\right)=\int_0^{2\pi}\frac{d\psi}{2\pi}\cdot e^{-i\cdot\mathrm{k}\cdot\psi}\cdot\int_0^{\infty}dr\cdot r\cdot J_{\mathrm{m_k},\mathrm{L_k},\mathrm{n_k}}\left(\text{æ}\cdot r\right)\cdot\mathrm{u}^{\omega}_{R,\varphi,z}\left(r,\psi,z\right)$$

with Bessel orders $\alpha_{\mathrm{k}}=\left(\mathrm{m_k},\mathrm{L_k},\mathrm{n_k}\right)$ for $\xi=\left(R,\varphi,z\right)$ components accordingly. Note: if any of α_{k} in (16) does not depend on $\left(\mathrm{k}\right)$, and $\mathrm{u}^{\omega}_{\xi}$ does not depend on $\left(\varphi\right)$- we can use a well-known relation:

$\int_0^{2\pi}\frac{d\psi}{2\pi}\cdot\sum_{\mathrm{k}=-\infty}^{\infty}e^{i\cdot\mathrm{k}\cdot(\varphi-\psi)}=1$ due to $\sum_{\mathrm{k}=-\infty}^{\infty}e^{i\cdot\mathrm{k}\cdot x}=2\pi\cdot\delta\left(x\right)$, which brings (16) to analog of (15): $\mathrm{u}^{\omega}_{\xi}\left(R,z\right)=\int_0^{\infty}d\text{æ}\cdot\text{æ}\cdot J_{\alpha}\left(\text{æ}\cdot R\right)\cdot\int_0^{\infty}dr\cdot r\cdot J_{\alpha}\left(\text{æ}\cdot r\right)\cdot\mathrm{u}^{\omega}_{\xi}\left(r,z\right)$.

Similar representations we can write for $\theta^{\omega}_{\xi=\left(R,\varphi,z\right)}$ (13):

$$\theta^{\omega}_{R,\varphi,z}\left(R,\varphi,z\right)=\sum_{\mathrm{k}=-\infty}^{\infty}e^{i\cdot\mathrm{k}\cdot\varphi}\cdot\int_0^{\infty}d\text{æ}\cdot\text{æ}\cdot J_{\mathrm{m_k},\mathrm{L_k},\mathrm{n_k}}\left(\text{æ}\cdot R\right)\cdot\Phi^{\omega}_{R,\varphi,z}\left(\text{æ},\mathrm{k},z\right) \qquad (17.a)$$

$$\Phi^{\omega}_{R,\varphi,z}\left(\text{æ},\mathrm{k},z\right)=\int_0^{2\pi}\frac{d\psi}{2\pi}\cdot e^{-i\cdot\mathrm{k}\cdot\psi}\cdot\int_0^{\infty}dr\cdot r\cdot J_{\mathrm{m_k},\mathrm{L_k},\mathrm{n_k}}\left(\text{æ}\cdot r\right)\cdot\theta^{\omega}_{R,\varphi,z}\left(r,\psi,z\right) \qquad (17.b)$$

Now we can obtain (18.R,φ,z) below from (12.R,φ,z) and (16,17). We apply $x = æ \cdot R$; omit common operator $I = \sum_{k=-\infty}^{\infty} e^{i \cdot k \cdot \varphi} \int_0^\infty dæ \cdot æ$; take into account $\frac{\partial \mathbb{u}^{\omega}_{R,\varphi,z}}{\partial R} \to æ \frac{dJ_{m,L,n}}{dx} \mathbb{U}^{\omega}_{R,\varphi,z}$, $\partial \mathbb{u}^{\omega}_{R,\varphi,z} / \partial \varphi \to i \cdot k \cdot J_{m,L,n} \cdot \mathbb{U}^{\omega}_{R,\varphi,z}$. The result is: (18.R,φ,z)

$$\left(\hat{\lambda}+\hat{\mu}\right)\cdot\left(i\cdot k\cdot\frac{æ^2}{x}\cdot\left(\frac{dJ_L}{dx}-\frac{J_L}{x}\right)\cdot\mathbb{U}^{\omega}_{\varphi}+æ\cdot\frac{dJ_n}{dx}\cdot\frac{\partial\mathbb{U}^{\omega}_{z}}{\partial z}\right)+$$

$$+\hat{\mu}\cdot\left(J_m\cdot\frac{\partial^2\mathbb{U}^{\omega}_{R}}{\partial z^2}-k^2\cdot\frac{æ^2}{x^2}\cdot J_m\cdot\mathbb{U}^{\omega}_{R}-2i\cdot k\cdot\frac{æ^2}{x^2}\cdot J_L\cdot\mathbb{U}^{\omega}_{\varphi}\right)+$$

$$+\left(\hat{\lambda}+2\hat{\mu}\right)\cdot æ^2\cdot\left(\frac{d^2J_m}{dx^2}+\frac{1}{x}\frac{dJ_m}{dx}-\frac{J_m}{x^2}\right)\cdot\mathbb{U}^{\omega}_{R}+\omega^2\cdot\hat{\rho}\cdot J_m\cdot\mathbb{U}^{\omega}_{R}=-\hat{\rho}\cdot J_m\cdot\Phi^{\omega}_{R}$$

$$\left(\hat{\lambda}+\hat{\mu}\right)\cdot\left(i\cdot k\cdot\frac{æ^2}{x}\cdot\left(\frac{dJ_m}{dx}+\frac{J_m}{x}\right)\cdot\mathbb{U}^{\omega}_{R}-k^2\cdot\frac{æ^2}{x^2}\cdot J_L\cdot\mathbb{U}^{\omega}_{\varphi}+i\cdot k\cdot\frac{æ}{x}\cdot J_n\cdot\frac{\partial\mathbb{U}^{\omega}_{z}}{\partial z}\right)+$$

$$+\hat{\mu}\cdot\left(\begin{array}{l} J_L\,\frac{\partial^2\mathbb{U}^{\omega}_{\varphi}}{\partial z^2}+æ^2\left(\frac{d^2J_L}{dx^2}+\frac{1}{x}\frac{dJ_L}{dx}-\frac{J_L}{x^2}\right)\mathbb{U}^{\omega}_{\varphi}- \\ -æ^2\cdot\frac{k^2}{x^2}\cdot J_L\cdot\mathbb{U}^{\omega}_{\varphi}+2i\cdot k\cdot\frac{æ^2}{x^2}\cdot J_m\cdot\mathbb{U}^{\omega}_{R}\end{array}\right)+\omega^2\hat{\rho}\cdot J_L\cdot\mathbb{U}^{\omega}_{\varphi}=-\hat{\rho}\cdot J_L\cdot\Phi^{\omega}_{\varphi}$$

$$\left(\hat{\lambda}+\hat{\mu}\right)\cdot\left(J_n\cdot\frac{\partial^2\mathbb{U}^{\omega}_{z}}{\partial z^2}+æ\cdot\left(\frac{dJ_m}{dx}+\frac{J_m}{x}\right)\cdot\frac{\partial\mathbb{U}^{\omega}_{R}}{\partial z}+i\cdot k\cdot\frac{æ}{x}\cdot J_L\cdot\frac{\partial\mathbb{U}^{\omega}_{\varphi}}{\partial z}\right)+$$

$$\hat{\mu}\cdot\left(J_n\cdot\frac{\partial^2\mathbb{U}^{\omega}_{z}}{\partial z^2}+æ^2\cdot\left(\frac{d^2J_n}{dx^2}+\frac{1}{x}\cdot\frac{dJ_n}{dx}-k^2\cdot\frac{J_n}{x^2}\right)\cdot\mathbb{U}^{\omega}_{z}\right)+\omega^2\hat{\rho}\cdot J_n\cdot\mathbb{U}^{\omega}_{z}=-\hat{\rho}\cdot J_n\cdot\Phi^{\omega}_{z}$$

Applying the decompositions (16,17), we assume that the terms in (18), containing $x = æ \cdot R$, $J_M(x)$, J'_M, J''_M, should vanish. Some terms in (18) can be excluded, using one of Bessel Functions definitions and properties [7-8.472.1,2], [8]:

$$\frac{d^2J_M}{dx^2}+\frac{1}{x}\frac{dJ_M}{dx}=\left(\frac{M^2}{x^2}-1\right)J_M(x),\quad \frac{dJ_M}{dx}=\frac{M}{x}J_M-J_{M+1}=J_{M-1}-\frac{M}{x}J_M,\quad \frac{dJ_0}{dx}=-J_1,\quad \frac{dJ_1}{dx}+\frac{J_1}{x}=J_0 \quad (19)$$

This allows obtaining Bessel orders, using (18):

$$\mathrm{m}_{\mathrm{k}}^2 = 1 + \mathrm{k}^2 \cdot \hat{\upsilon}, \quad \mathrm{L}_{\mathrm{k}}^2 = 1 + \mathrm{k}^2 / \hat{\upsilon}, \quad \mathrm{n}_{\mathrm{k}} = \mathrm{k}, \text{ where } \hat{\upsilon} = \hat{\mu} / \left(\hat{\lambda} + 2\hat{\mu}\right) \qquad (20)$$

and (21.R,φ,z)

$$\hat{\mu} \cdot J_{\mathrm{m_k}} \cdot \frac{\partial^2 \mathbb{U}_R^\omega}{\partial z^2} - \left(æ^2 \cdot \left(\hat{\lambda} + 2\hat{\mu}\right) - \omega^2 \cdot \rho\right) \cdot J_{\mathrm{m_k}} \cdot \mathbb{U}_R^\omega + \left(\hat{\lambda} + \hat{\mu}\right) \cdot æ \cdot \frac{dJ_{\mathrm{n_k}}}{dx} \cdot \frac{\partial \mathbb{U}_z^\omega}{\partial z} +$$

$$+ \hat{\rho} \cdot J_{\mathrm{m_k}} \cdot \Phi_R^\omega = i \cdot \mathrm{k} \cdot \frac{æ^2}{x} \cdot \mathbb{U}_\varphi^\omega \cdot \left(2\hat{\mu} \cdot \frac{J_{\mathrm{L_k}}}{x} - \left(\hat{\lambda} + \hat{\mu}\right) \cdot \left(\frac{dJ_{\mathrm{L_k}}}{dx} - \frac{J_{\mathrm{L_k}}}{x}\right)\right)$$

$$\hat{\mu} \cdot J_{\mathrm{L_k}} \cdot \frac{\partial^2 \mathbb{U}_\varphi^\omega}{\partial z^2} - \left(æ^2 \cdot \hat{\mu} - \omega^2 \cdot \hat{\rho}\right) \cdot J_{\mathrm{L_k}} \cdot \mathbb{U}_\varphi^\omega + \hat{\rho} \cdot J_{\mathrm{L_k}} \cdot \Phi_\varphi^\omega =$$

$$= -\left(\hat{\lambda} + \hat{\mu}\right) \cdot i \cdot \mathrm{k} \cdot \frac{æ}{x} \cdot \left(æ \cdot \left(\frac{dJ_{\mathrm{m_k}}}{dx} + \frac{J_{\mathrm{m_k}}}{x}\right) \cdot \mathbb{U}_R^\omega + J_{\mathrm{n_k}} \cdot \frac{\partial \mathbb{U}_z^\omega}{\partial z}\right) - \hat{\mu} \cdot 2i \cdot \mathrm{k} \cdot \frac{æ^2}{x^2} \cdot J_{\mathrm{m_k}} \cdot \mathbb{U}_R^\omega$$

$$\left(\hat{\lambda} + 2\hat{\mu}\right) \cdot J_{\mathrm{n_k}} \cdot \frac{\partial^2 \mathbb{U}_z^\omega}{\partial z^2} - \left(æ^2 \cdot \hat{\mu} - \omega^2 \cdot \hat{\rho}\right) \cdot J_{\mathrm{n_k}} \cdot \mathbb{U}_z^\omega + \left(\hat{\lambda} + \hat{\mu}\right) \cdot \left(\frac{dJ_{\mathrm{m_k}}}{dx} + \frac{J_{\mathrm{m_k}}}{x}\right) \cdot æ \cdot \frac{\partial \mathbb{U}_R^\omega}{\partial z} +$$

$$+ \hat{\rho} \cdot J_{\mathrm{n_k}} \cdot \Phi_z^\omega = -\left(\hat{\lambda} + \hat{\mu}\right) \cdot i \cdot \mathrm{k} \cdot \frac{æ}{x} \cdot J_{\mathrm{L_k}} \cdot \frac{\partial \mathbb{U}_\varphi^\omega}{\partial z}$$

Note: for UHS = NVF we can apply (3.a), using $\mu_0 = 0$ - even for onshore survey, where ρ_0 is small, but not zero, so $\lambda_0 \neq 0$; moreover, for offshore survey, where $\rho_0 \simeq 1000\,\mathrm{kg/m^3}$. Thus for UHS = NVF we have $\hat{\upsilon}_{z<0} = 0$ from (20), so $\left(\mathrm{m}_{\mathrm{k}\neq0}^{z<0} = 1,\ \mathrm{L}_{\mathrm{k}\neq0}^{z<0} = \infty\right)_{z<0}$; Actually for high f_{Hz}, when UHS-viscosity is taken into account (so UHS ≠ NVF), the small terms, deviated from $\left(\mathrm{m}_{\mathrm{k}\neq0}^{z<0} = 1,\ \mathrm{L}_{\mathrm{k}\neq0}^{z<0} = \infty\right)_{z<0}$ can be moved to the RHS of (21) (which is also small, even vanishing at $\mathrm{k} = 0$). One can use such terms in iterative process, or just apply the first approximations of these terms.

For $z>0$ we estimate Bessel orders at a typical level of $V_S^{\text{shallow}} \sim V_P/3$. Given that $\hat{\upsilon}_{z>0} = V_S^2/V_P^2$, for the main terms $\mathrm{k}=(0;\ 1;\ 2)$ we get: $\mathrm{m}_{\mathrm{k}}^{z>0} \approx (1;\ 1.05;\ 1.20;...)$, $\mathrm{L}_{\mathrm{k}}^{z>0} \approx (1;\ 3.2;\ 6.1;...)$. According to (16) and RHS of (21.R,z), the effects on $\mathbb{U}_{R,z}^{\omega}$ from Anomaly come from $\int_0^\infty dæ \cdot J_{\mathrm{L}}(æ \cdot R)...$, and the main contributions bring values $J_{\mathrm{L}}(æ \cdot R \sim 1)$. Thus, taking into account $J_{\mathrm{L}=3}(1) \approx 2.0*10^{-2}$, $J_{\mathrm{L}=6}(1) \approx 2.1*10^{-5}$ (see [7-p.209]), we can ignore the terms $\sim J_{\mathrm{L_k}}$ in (21.R,z), which bring to $\mathbb{U}_{R,z}^{\omega}$ an acceptable relative error ~2% of its value. Such ignorance also allows avoiding R -dependence $(x = æ \cdot R)$ in RHS of (21.R,z).

Note: for $\mathrm{m}_{\mathrm{k}}^{z>0} \approx 1$ above we can replace $J'_{\mathrm{m_k}} + J_{\mathrm{m_k}}/x \approx J_0$ in (21.z). Indeed, from (19) we have: $J'_{\mathrm{m_k}} + J_{\mathrm{m_k}}/x \equiv J_{\mathrm{m_k}-1} - (\mathrm{m_k}-1) J_{\mathrm{m_k}}/x$, and $(\mathrm{m_k}-1) J_{\mathrm{m_k}}/x \ll J_{\mathrm{m_k}-1}$.

Also the eq. (21.φ) for $\mathbb{U}_{\varphi}^{\omega}$ can be ignored, amplifying the reasons to ignore $\mathbb{U}_{\varphi}^{\omega}$ in the eq. (21.R,z) for $\mathbb{U}_{R,z}^{\omega}$ at UHS = NVF case $\zeta_0 = \eta_0 = 0$. Indeed: at $\mathrm{k} \neq 0$ the LHS of (21.φ)(which mostly affects $\mathbb{U}_{\varphi}^{\omega}$) is small due to $J_{\mathrm{L_{k\neq0}}}(x = æ \cdot R \sim 1) \sim 0$ for the typical parameters (see above). For $\mathrm{k}=0$ (main terms) the RHS of (21.φ) vanishes in general case UHS ≠ NVF due to proportionality of *all* the RHS of (21.R,φ,z) to (k).

The LHS of (21.φ) vanishes also (when UHS = NVF) in spite of $J_{\mathrm{L_{k=0}}}(1) \sim 1$. The proof: even for general case UHS ≠ NVF the eq.(21.φ) for $\mathbb{U}_{\varphi,\omega}^{\mathrm{k}=0}(æ, z)$ comes to

$$\hat{\mu}\frac{\partial^2 \mathbb{U}_{\varphi,\omega}^{\mathrm{k}=0}}{\partial z^2} - (æ^2\hat{\mu} - \omega^2\hat{\rho})\mathbb{U}_{\varphi,\omega}^{\mathrm{k}=0} = -\hat{\rho} \cdot \Phi_{\varphi,\omega}^{\mathrm{k}=0} \qquad (21.\varphi^*)$$

and here the RHS vanishes due to $\Phi_{\varphi,\omega}^{\mathrm{k}=0}(æ, z) \sim \int_0^{2\pi} d\psi \cdot \theta_{\varphi}^{\omega}(r,\psi,z) = 0$ in (17). Indeed, (13.φ) for $\theta_{\varphi}^{\omega}(R,\varphi,z)$ contains $\theta_{\varphi}^{\omega}(R,\varphi,z) \sim c(R,z) \cdot \partial\Delta\vartheta_A/\partial\varphi$, and the factors $c(R,z)$ do not depend on φ; this makes the above integral = 0 (there are no terms ~ $\Delta\vartheta_A(R,\varphi,z)$ in $\theta_{\varphi}^{\omega}(R,\varphi,z)$, which provide integral ≠ 0). Thus instead of (21.φ*) we get (21.φ**):

$$\frac{\partial^2 \mathbb{U}_{\varphi,\omega}^{\mathrm{k}=0}}{\partial z^2} - (æ^2 - \hat{k}_S^2)\mathbb{U}_{\varphi,\omega}^{\mathrm{k}=0} = 0 \text{ with } \hat{k}_S^2 = \frac{\omega^2\hat{\rho}}{\hat{\mu}}, \text{ giving } \mathbb{U}_{\varphi,\omega}^{\mathrm{k}=0} = \hat{A}_S \cdot e^{+\sqrt{æ^2-\hat{k}_S^2}\cdot z} + \hat{B}_S \cdot e^{-\sqrt{æ^2-\hat{k}_S^2}\cdot z},$$

with 4 unknown factors $(\hat{A}_S = A_S^{\pm},\ \hat{B}_S = B_S^{\pm})$, where (+) means: $z>0$, and (-) means: $z<0$.

Note: eq.(21.φ**) for $\mathbb{U}_{\varphi,\omega}^{\mathrm{k}=0}$ is a second order uniform ODE (no mix terms $\mathbb{U}_{R,z,\omega}^{\mathrm{k}=0}$, and no P -waves terms, unlike fourth order ODE-system for $\mathbb{U}_{R,z,\omega}^{\mathrm{k}=0}$ below).

Assuming decay at $z=\pm\infty$, we get: $A_S^{+}=0$, and $\mathbb{U}_{\varphi,\omega}^{k=0}(æ,z)_{z<0}=A_S^{-}\cdot e^{+\sqrt{æ^2-k_{0S}^2}\cdot z}$ with $k_{0S}^2=\omega^2\rho_0/\mu_0$; $B_S^{-}=0$, and $\mathbb{U}_{\varphi,\omega}^{k=0}(æ,z)_{z>0}=B_S^{+}\cdot e^{-\sqrt{æ^2-k_S^2}\cdot z}$ (reminder: here we assume general UHS $\neq$ NVF case: $l_{\varphi}^{\omega}\big|_{z<0}\neq 0$). For UHS = NVF $(\rho_0\neq 0,\ \zeta_0=\eta_0=0)$ (when from (3.a): $\mu_0=0$, so $k_{0S}^2=+i\cdot\infty$), assuming as usual $\mathrm{Re}\left(z\cdot\sqrt{æ^2-k_{0S}^2}\right)_{z<0}<0$, we have $\mathrm{Re}\left(z\sqrt{æ^2-k_{0S}^2}\right)_{z<0}=-\infty$. Thus $\mathbb{U}_{\varphi,\zeta_0=0,\eta_0=0}^{\omega,k=0,\rho_0\neq 0}(æ,z)_{z<0}=0$. The continuity of l_{φ}^{ω} (6.0) (transferred by (11,16.a) to $\mathbb{U}_{\varphi,\omega}^{k=0}$) requires $\mathbb{U}_{\varphi,k=0}^{\omega,z=0^+}(æ)=\mathbb{U}_{\varphi,k=0}^{\omega,z=0_-}(æ)$, so $B_S^{+}=0$, thus $\mathbb{U}_{\varphi,\zeta_0=0,\eta_0=0}^{\omega,k=0,\rho_0\neq 0}(æ,z)\equiv 0$ for all (z), making needless (6.φ) usage for $\partial\mathbb{U}_{\varphi,\omega}^{k=0}/\partial z$ jump (this logic does not apply to the system for $\mathbb{U}_{R,z,\omega}^{k=0}$, see below).

Note: both $\theta_{R,z}^{\omega}(R,\varphi,z)$ in (13.R,z) depend on φ proportionally to $\Delta\vartheta_A(R,\varphi,z)$ or $\partial\Delta\vartheta_A/\partial R,\ \partial\Delta\vartheta_A/\partial z$ instead of $\partial\Delta\vartheta_A/\partial\varphi$, so this logic does not generally applies to (17.b), and $\Phi_{\omega,R,z}^{k=0}(æ,z)\sim\int_0^{2\pi}d\psi\cdot\theta_{R,z}^{\omega}(r,\psi,z)\neq 0$.

Now, returning to general $l_{R,\varphi,z}^{\omega}\big|_{z<0}\neq 0$ case, we focus on $\mathbb{U}_{R,z}$ - eq.(21.R,z). Ignoring RHS $\left(\sim k\cdot J_{L_k}\right)$, for both $k=0$ and $k\neq 0$ (due to $J_{L_{k\neq 0}}(x\sim 1)\ll 1$), we get: (22.R,z)

$$J_{m_k}\cdot\left(\hat{\mu}\cdot\frac{\partial^2\mathbb{U}_R^{\omega}}{\partial z^2}-\left(æ^2\cdot\left(\hat{\lambda}+2\hat{\mu}\right)-\omega^2\cdot\hat{\rho}\right)\cdot\mathbb{U}_R^{\omega}+\hat{\rho}\cdot\Phi_R^{\omega}\right)+\frac{dJ_k}{dx}\cdot\left(\hat{\lambda}+\hat{\mu}\right)\cdot æ\cdot\frac{\partial\mathbb{U}_z^{\omega}}{\partial z}=0$$

$$J_k\cdot\left(\left(\hat{\lambda}+2\hat{\mu}\right)\cdot\frac{\partial^2\mathbb{U}_z^{\omega}}{\partial z^2}-\left(æ^2\cdot\hat{\mu}-\omega^2\cdot\hat{\rho}\right)\cdot\mathbb{U}_z^{\omega}+\hat{\rho}\cdot\Phi_z^{\omega}\right)+\left(\frac{dJ_{m_k}}{dx}+\frac{J_{m_k}}{x}\right)\cdot\left(\hat{\lambda}+\hat{\mu}\right)\cdot æ\cdot\frac{\partial\mathbb{U}_R^{\omega}}{\partial z}=0$$

Note: the term $\sim\partial\mathbb{U}_z/\partial z$ in (22.R) prevents from cutting down the factor $J_{m_{k\neq 0}}(x)$. Similar term $\sim\partial\mathbb{U}_R/\partial z$ in (22.z) prevents from cutting down the factor $J_{k\neq 0}(x)$. However for $k=0$ from (20): $m=1,\ n=0$, so we can cut down Bessel-Functions in (22), thus for $\mathbb{U}_{R,z,\omega}^{k=0}(æ,z)$ we get:

$$\hat{\mu}\cdot\frac{\partial^2\mathbb{U}_{R,\omega}^{k=0}}{\partial z^2}-\left(æ^2\cdot\left(\hat{\lambda}+2\hat{\mu}\right)-\omega^2\cdot\hat{\rho}\right)\cdot\mathbb{U}_{R,\omega}^{k=0}-\left(\hat{\lambda}+\hat{\mu}\right)\cdot æ\cdot\frac{\partial\mathbb{U}_{z,\omega}^{k=0}}{\partial z}=-\hat{\rho}\cdot\Phi_{R,\omega}^{k=0}\quad\text{(23.R)}$$

$$\left(\hat{\lambda}+2\hat{\mu}\right)\cdot\frac{\partial^2\mathbb{U}_{z,\omega}^{k=0}}{\partial z^2}-\left(æ^2\cdot\hat{\mu}-\omega^2\cdot\hat{\rho}\right)\cdot\mathbb{U}_{z,\omega}^{k=0}+\left(\hat{\lambda}+\hat{\mu}\right)\cdot æ\cdot\frac{\partial\mathbb{U}_{R,\omega}^{k=0}}{\partial z}=-\hat{\rho}\cdot\Phi_{z,\omega}^{k=0}\quad\text{(23.z)}$$

This allows analytical solution (see below). For $\mathrm{k}\neq 0$ the situation is more complex, even assuming $\mathrm{m_k}\approx 1$ for a typical case $V_{\mathrm{S}}^{\mathrm{shallow}}\sim V_{\mathrm{P}}/3$ (see above): the factors $J_{\mathrm{m}_{\mathrm{k}\neq 0}}\left(x=æR\right)$ vs. $J_{\mathrm{k}}'\left(x\right)$ in (22.R) have different æ-zeroes, same for the factors $J_{\mathrm{k}\neq 0}\left(x\right)$ vs. $\left(J_{\mathrm{m_k}}'+J_{\mathrm{m_k}}/x\right)$ in (22.z); the resulting R-dependence can't be canceled for $\mathrm{k}\neq 0$. Thus (22) (and more generally (21.R,z)) can be solved only approximately, decompositions (16) do not work for exact solutions, even assuming $\mathrm{m}_{\mathrm{k}\neq 0}\approx 1$ and excluding small terms ~ $J_{\mathrm{L}_{\mathrm{k}\neq 0}}\left(1\right)$. However, ignoring terms ~ $J_{\mathrm{L}_{\mathrm{k}\neq 0}}$ will be enough to simplify general chain (12.R,z)→(18.R,z)→(21.R,z)→(22.R,z), coming to (24) below. Indeed, we can return from $\mathbb{U}_{R,z}^{\omega}$ to $\mathrm{u}_{R,z}^{\omega}$, applying to (22.R,z) the operator (see above) $I=\sum_{\mathrm{k}=-\infty}^{\infty}e^{i\cdot\mathrm{k}\cdot\varphi}\int_0^{\infty}dæ\cdot æ$ (to be consistent with (16,17)), and using (19,20,22) with following relation: $J_{\mathrm{m_k}}\left(x\right)=\frac{\mathrm{m_k^2}}{x^2}\cdot J_{\mathrm{m_k}}-\frac{d^2J_{\mathrm{m_k}}}{dx^2}-\frac{1}{x}\frac{dJ_{\mathrm{m_k}}}{dx}$. Thus $æ^2J_{\mathrm{m_k}}\mathbb{U}_R^{\omega}\rightarrow\frac{\mathrm{u}_R^{\omega}}{R^2}-\frac{\partial^2\mathrm{u}_R^{\omega}}{\partial R^2}-\frac{1}{R}\frac{\partial\mathrm{u}_R^{\omega}}{\partial R}-\frac{\hat{\upsilon}}{R^2}\frac{\partial^2\mathrm{u}_R^{\omega}}{\partial\varphi^2}$, $æ^2J_{\mathrm{k}}\mathbb{U}_z^{\omega}\rightarrow\frac{-1}{R^2}\frac{\partial^2\mathrm{u}_z^{\omega}}{\partial\varphi^2}-\frac{\partial^2\mathrm{u}_z^{\omega}}{\partial R^2}-\frac{1}{R}\frac{\partial\mathrm{u}_z^{\omega}}{\partial R}$, and $\frac{dJ_{\mathrm{k}}}{dx}\cdot æ\cdot\mathbb{U}_z^{\omega}\rightarrow\frac{\partial\mathrm{u}_z^{\omega}}{\partial R}$, $\left(\frac{dJ_{\mathrm{m_k}}}{dx}+\frac{J_{\mathrm{m_k}}}{x}\right)\cdot æ\cdot\mathbb{U}_R^{\omega}\rightarrow\frac{\partial\mathrm{u}_R^{\omega}}{\partial R}+\frac{\mathrm{u}_R^{\omega}}{R}$ bring to:

$$\hat{\mu}\cdot\left(\frac{\partial^2\mathrm{u}_R^{\omega}}{\partial z^2}+\frac{1}{R^2}\cdot\frac{\partial^2\mathrm{u}_R^{\omega}}{\partial\varphi^2}\right)+\left(\hat{\lambda}+2\hat{\mu}\right)\cdot\left(\frac{\partial^2\mathrm{u}_R^{\omega}}{\partial R^2}+\frac{1}{R}\cdot\frac{\partial\mathrm{u}_R^{\omega}}{\partial R}-\frac{\mathrm{u}_R^{\omega}}{R^2}\right)+$$
$$+\omega^2\cdot\hat{\rho}\cdot\mathrm{u}_R^{\omega}+\left(\hat{\lambda}+\hat{\mu}\right)\cdot\frac{\partial^2\mathrm{u}_z^{\omega}}{\partial R\cdot\partial z}=-\hat{\rho}\cdot\theta_R^{\omega} \qquad \text{(24.R)}$$

$$\left(\hat{\lambda}+2\hat{\mu}\right)\cdot\frac{\partial^2\mathrm{u}_z^{\omega}}{\partial z^2}+\hat{\mu}\cdot\left(\frac{\partial^2\mathrm{u}_z^{\omega}}{\partial R^2}+\frac{1}{R}\cdot\frac{\partial\mathrm{u}_z^{\omega}}{\partial R}+\frac{1}{R^2}\cdot\frac{\partial^2\mathrm{u}_z^{\omega}}{\partial\varphi^2}\right)+$$
$$+\omega^2\cdot\hat{\rho}\cdot\mathrm{u}_z^{\omega}+\left(\hat{\lambda}+\hat{\mu}\right)\cdot\frac{\partial}{\partial z}\left(\frac{\partial\mathrm{u}_R^{\omega}}{\partial R}+\frac{\mathrm{u}_R^{\omega}}{R}\right)=-\hat{\rho}\cdot\theta_z^{\omega} \qquad \text{(24.z)}$$

Note: comparing (24) with general (12), we see: the terms in (12.R,z), containing $\partial l_\varphi^\omega / \partial\varphi$, are ignored due to neglecting ~ $J_{\mathrm{L_{k\neq 0}}}$ in (21.R,z)→(22.R,z):

$\left(\hat{\lambda}+\hat{\mu}\right)\cdot\frac{\partial}{\partial R}\left(\frac{1}{R}\cdot\frac{\partial l_\varphi^\omega}{\partial\varphi}\right)-\frac{2\hat{\mu}}{R^2}\cdot\frac{\partial l_\varphi^\omega}{\partial\varphi}$ in (12.R) and $\left(\hat{\lambda}+\hat{\mu}\right)\cdot\frac{\partial}{\partial z}\left(\frac{1}{R}\cdot\frac{\partial l_\varphi^\omega}{\partial\varphi}\right)$ in (12.z).

Note: $\mathbb{u}_{R,z}^\omega$ must depend on φ (to take into account φ_A), this is guaranteed by (16.a) even without dependency of Bessel orders $\alpha_\mathrm{k}=\left(\mathrm{m_k},\mathrm{L_k},\mathrm{n_k}\right)$ on (k) due to $\mathbb{U}_{R,z}^\omega\left(æ,\mathrm{k},z\right)$ dependency on (k). For example, (22) contains explicit k-dependency on $\Phi_{R,z}^\omega\left(æ,\mathrm{k},z\right)$, which are functions (17.b) of $\theta_{R,z}^\omega\left(R,\varphi,z\right)$($\vec{\theta}^{\,\omega}$ depends explicitly on φ in (13.R,z) via $\left(\Delta\rho_A,\Delta\lambda_A,\Delta\mu_A\right)$), and $\mathbb{u}_{R,z}^\omega$ in (24) depends on φ via $\vec{\theta}^{\,\omega}$ in RHS. The dependency $\mathbb{u}_z^\omega$ on φ can be achieved by using **B**essel **A**ddition **T**heorem BAT (see below) after solving (22) for $\mathbb{U}_{R,z}^\omega\left(æ,\mathrm{k},z\right)$ with (16.a), or directly (24) for $\mathbb{u}_{R,z}^\omega$. The BAT implementation requires Bessel order to be equal to exponential factor k (at $e^{i\cdot\mathrm{k}\cdot\varphi}$ in (16.a), where Bessel is $J_\mathrm{k}(x)$). The only chance for such scenario in (22) has (22.z) for $\mathbb{U}_z^\omega$, but not (22.R) for $\mathbb{U}_R^\omega$ ($J_{\mathrm{m_k}}$ in (22.R) exclude this option).

Now we introduce the new representations, taking into account (11):

$\mathbb{u}_R^\omega\left(R,\varphi,z\right)=\mathbf{u}_R^\omega\left(R,z\right)+u_R^\omega\left(R,\varphi,z\right)$ or $l_R^\omega\left(R,\varphi,z\right)=\boldsymbol{\ell}_R^\omega\left(R,z\right)+\mathbf{u}_R^\omega\left(R,z\right)+u_R^\omega\left(R,\varphi,z\right)$ (25.R)

$\mathbb{u}_z^\omega\left(R,\varphi,z\right)=\mathbf{u}_z^\omega\left(R,z\right)+u_z^\omega\left(R,\varphi,z\right)$ or $l_z^\omega\left(R,\varphi,z\right)=\boldsymbol{\ell}_z^\omega\left(R,z\right)+\mathbf{u}_z^\omega\left(R,z\right)+u_z^\omega\left(R,\varphi,z\right)$ (25.z)

Note: the main terms $\mathbf{u}_{R,z}^\omega\left(R,z\right)$ do not have φ-dependency, they can represent a special axis-symmetrical Anomaly (impractical thin ring, bounded by $\left(\Delta R_\mathrm{A},\Delta z_\mathrm{A}\right)$; or disk, bounded by Δz_A). However, if we need to integrate by Anomaly's volume $d\Omega_\mathrm{A}$, we must allow $R_\mathrm{A}\left(\varphi_\mathrm{A}\right), z_\mathrm{A}\left(R_\mathrm{A},\varphi_\mathrm{A}\right)$. The second terms $u_{R,z}^\omega\left(R,\varphi,z\right)$ relate to a real Anomaly, bounded not only by $\left(\Delta R_\mathrm{A},\Delta z_\mathrm{A}\right)$, but also by $\Delta\varphi_A$, so $\left|u_{R,z}^\omega\right|\sim\Delta\Omega_\mathrm{A}\approx\Delta R_\mathrm{A}\cdot R_\mathrm{A}\cdot\Delta\varphi_\mathrm{A}\cdot\Delta z_\mathrm{A}$.

Note: eq.(24.z) is the main eq. for the main component $\mathbb{u}_z^\omega$; (24.R) is the supporting eq. for the minor component $\mathbb{u}_R^\omega$. Thus in (24.z) we use full (25.z) for $\mathbb{u}_z^\omega\left(R,\varphi,z\right)$, but only the main term $\mathbf{u}_R^\omega\left(R,z\right)$ (i.e. $u_R^\omega=0$) instead of full $\mathbb{u}_R^\omega$ (such approach also allows to use BAT for u_z^ω, see below); in (24.R) we use both $\mathbf{u}_{R,z}^\omega$ instead of full $\mathbb{u}_{R,z}^\omega$. Here:

$$\mathbf{u}_R^{\omega}(R,z)=\int_0^{\infty}dæ\cdot æ\cdot J_1(æ\cdot R)\cdot\mathbf{U}_R^{\omega}(æ,z),\ \mathbf{U}_R^{\omega}(æ,z)=\int_0^{\infty}dr\cdot r\cdot J_1(æ\cdot r)\cdot\mathbf{u}_R^{\omega}(r,z)\quad(26.R)$$

$$\mathbf{u}_z^{\omega}(R,z)=\int_0^{\infty}dæ\cdot æ\cdot J_0(æ\cdot R)\cdot\mathbf{U}_z^{\omega}(æ,z),\ \mathbf{U}_z^{\omega}(æ,z)=\int_0^{\infty}dr\cdot r\cdot J_0(æ\cdot r)\cdot\mathbf{u}_z^{\omega}(r,z)\quad(26.z)$$

$$u_z^{\omega}(R,\varphi,z)=\sum_{\mathrm{k}=-\infty}^{\infty}e^{i\cdot\mathrm{k}\cdot\varphi}\cdot\int_0^{\infty}dæ\cdot æ\cdot J_{\mathrm{k}}(æ\cdot R)\cdot U_z^{\omega}(æ,\mathrm{k},z)\quad(26.z.a)$$

$$U_z^{\omega}(æ,\mathrm{k},z)=\int_0^{2\pi}\frac{d\psi}{2\pi}\cdot e^{-i\cdot\mathrm{k}\cdot\psi}\cdot\int_0^{\infty}dr\cdot r\cdot J_{\mathrm{k}}(æ\cdot r)\cdot u_z^{\omega}(r,\psi,z)\quad(26.z.b)$$

Replacing $\mathbb{u}_{R,z}^{\omega}\to\mathbf{u}_{R,z}^{\omega}$ in (24.R,z), both RHS should be φ-averaged; thus instead of $\theta_{R,z}^{\omega}(R,\varphi,z)$, $\Phi_{R,z}^{\omega}(æ,\mathrm{k},z)$ in (17.a,b), we use: $\Theta_{R,z}^{\omega}(R,z)=\int_0^{2\pi}\frac{d\varphi}{2\pi}\cdot\theta_{R,z}^{\omega}(R,\varphi,z)$, and $\mathbf{\Phi}_{R,z}^{\omega}(æ,z)$, connected by decompositions

$$\Theta_R^{\omega}(R,z)=\int_0^{\infty}dæ\cdot æ\cdot J_1(æ\cdot R)\cdot\mathbf{\Phi}_R^{\omega}(æ,z),\ \mathbf{\Phi}_R^{\omega}(æ,z)=\int_0^{\infty}dr\cdot r\cdot J_1(æ\cdot r)\cdot\Theta_R^{\omega}(r,z)\quad(27.R)$$

$$\Theta_z^{\omega}(R,z)=\int_0^{\infty}dæ\cdot æ\cdot J_0(æ\cdot R)\cdot\mathbf{\Phi}_z^{\omega}(æ,z),\ \mathbf{\Phi}_z^{\omega}(æ,z)=\int_0^{\infty}dr\cdot r\cdot J_0(æ\cdot r)\cdot\Theta_z^{\omega}(r,z)\quad(27.z)$$

Note: such φ-averaging brings to $\Phi_{R,z,\omega}^{\mathrm{k}=0}(æ,z)=\mathbf{\Phi}_{R,z}^{\omega}(æ,z)$ (using (17.a,b)), and in (13.R,z) instead of $\Delta\vartheta_A(R,\varphi,z)$ (where $\Delta\vartheta_A=$ any of $(\Delta\rho_A,\Delta\lambda_A,\Delta\mu_A)$) we should use $\Delta\bar{\vartheta}_A(R,z)=\int_0^{2\pi}d\varphi\cdot\Delta\vartheta_A(R,\varphi,z)/(2\pi)$ for $\Theta_{R,z}^{\omega}(R,z)$ (instead of $\theta_{R,z}^{\omega}(R,\varphi,z)$). For localized bounded Anomaly it gives: $\Delta\bar{\vartheta}_A(R,z)\approx\Delta\vartheta_A(R,\varphi_A,z)\cdot\Delta\varphi_A/(2\pi)$.

Note: we cannot use such approach for $\theta_{\varphi}^{\omega}$ due to $\Phi_{\varphi,\omega}^{\mathrm{k}=0}(æ,z)\sim\int_0^{2\pi}d\psi\cdot\theta_{\varphi}^{\omega}(r,\psi,z)=0$ (see explanation between (21.φ*) and (21.φ**)).

Note: obviously, functions $\mathbf{u}_{R,z}^{\omega}(R,z)$ and following $\mathbf{U}_{R,z}^{\omega}(æ,z)$ (26) can be chosen quite arbitrary, if $\mathbf{u}_{R,z}^{\omega}$ do not depend on φ. We choose a specific system (28) below for $\mathbf{U}_{R,z}^{\omega}$, applying the mentioned above φ-averaging. It allows an analytical solution, and brings to (29.z) (if $\mathbf{\Phi}_{R,z}^{\omega}(æ,z)$ is replaced with $\Phi_{R,z,\omega}^{\mathrm{k}=0}(æ,z)$):

$$\hat{\mu}\cdot\frac{\partial^2\mathbf{U}_R^\omega}{\partial z^2}-\left(æ^2\cdot\left(\hat{\lambda}+2\hat{\mu}\right)-\omega^2\cdot\hat{\rho}\right)\cdot\mathbf{U}_R^\omega-\left(\hat{\lambda}+\hat{\mu}\right)\cdot æ\cdot\frac{\partial\mathbf{U}_z^\omega}{\partial z}=-\hat{\rho}\cdot\mathbf{\Phi}_R^\omega \quad (28.R)$$

$$\left(\hat{\lambda}+2\hat{\mu}\right)\cdot\frac{\partial^2\mathbf{U}_z^\omega}{\partial z^2}-\left(æ^2\cdot\hat{\mu}-\omega^2\cdot\hat{\rho}\right)\cdot\mathbf{U}_z^\omega+\left(\hat{\lambda}+\hat{\mu}\right)\cdot æ\cdot\frac{\partial\mathbf{U}_R^\omega}{\partial z}=0 \quad (28.z)$$

Note: (28) correlates with (23), except: ($\mathbf{U}_{R,z}^\omega$ in LHS of (28) vs. $\mathbb{U}_{R,z,\omega}^{k=0}$ in LHS of (23)) and (($\mathbf{\Phi}_R^\omega$, 0) in RHS of (28) vs. $\Phi_{R,z,\omega}^{k=0}$ in RHS of (23)).

For $U_z^\omega(æ,k,z)$ the equations (24.z→28.z; 17.a) with $u_R^\omega(R,\varphi,z)\equiv 0$ provide (29) below at both $k=0$ and $k\neq 0$ (for BAT implementation, see below).
Same result (29) could be obtained, applying (25-28,16) directly to (22.z) and using $u_R^\omega(R,\varphi,z)\equiv 0$ in (25.R) or $\left(\mathbb{U}_{R,\omega}^{k=0}=\mathbf{U}_R^\omega,\ \mathbb{U}_{R,\omega}^{k\neq 0}\equiv 0\right)$, see (16.b) with $\mathbb{u}_R^\omega\to\mathbf{u}_R^\omega(R,z)$ and (26.a); also from (25.z, 26.z, 27.z, 16.b) we have: $\left(\mathbb{U}_{z,\omega}^{k=0}=\mathbf{U}_z^\omega+U_{z,\omega}^{k=0},\ \mathbb{U}_{z,\omega}^{k\neq 0}=U_{z,\omega}^{k\neq 0}\right)$.
Finally we get for any k:

$$\left(\hat{\lambda}+2\hat{\mu}\right)\cdot\frac{\partial^2 U_z^\omega}{\partial z^2}-\left(æ^2\cdot\hat{\mu}-\omega^2\cdot\hat{\rho}\right)\cdot U_z^\omega\left(æ,\ \text{any}\,k,\ z\right)=-\hat{\rho}\cdot\Phi_z^\omega\left(æ,\ \text{any}\,k,\ z\right) \quad (29.z)$$

Note: in (29.z) the main part of $l_{\omega,R}^{\text{main}}=\boldsymbol{\ell}_R^\omega+\mathbf{u}_R^\omega$ was taken into account automatically.
Note: formally (29.z) reminds (21.φ*), but unlike resultant $\mathbb{U}_{\varphi,\zeta_0=0,\eta_0=0}^{\omega,\,k=0,\,\rho_0\neq 0}(æ,z)\equiv 0$, here $U_z^\omega(æ,k,z)\neq 0$, even if $\Phi_{z,\omega}^{k=0}(æ,z)=0$ and UHS = Air = NVF $\left(\rho_0\neq 0,\ \zeta_0=\eta_0=0\right)$ due to $\left(\mu_0=0,\ \lambda_0\neq 0\right)$, contributing to the coefficient $\left(\hat{\lambda}+2\hat{\mu}\right)_{z<0}$ at the first term.

Now we apply axis-symmetrical source $\vec{f}^\omega=\left(f_R^\omega,0,f_z^\omega\right)$, expressed via $\delta(z-H_0)$, where H_0 is the source's mid-depth, $\delta(z-H_0)$ - Dirac's Delta-Function:

$$\begin{pmatrix} f_R^\omega \\ f_z^\omega \end{pmatrix}=\delta(z-H_0)\cdot\int_0^\infty dæ\cdot æ\cdot\begin{pmatrix} J_1(æ\cdot R)\cdot\mathbb{Q}_R^\omega \\ J_0(æ\cdot R)\cdot\mathbb{Q}_z^\omega \end{pmatrix};\quad \begin{matrix}\mathbb{Q}_R^\omega=J_1(æ\cdot R_0)\cdot P_\omega\cdot R_0\cdot h_0 \\ \mathbb{Q}_z^\omega=J_1(æ\cdot R_0)\cdot P_\omega\cdot R_0/æ\end{matrix} \quad (30)$$

Here: $P_\omega\left[\mathrm{N\cdot s/m^2}\right]$, $(R_0,h_0)[\mathrm{m}]$ are pressure, (radius, height) of such source.

Below we often consider a limiting case $H_0 = 0$ (when $z^{\text{source}}_{\min,\max}$ embraces the center of coordinates $x = y = z = 0$) and UHS = NVF$(\zeta_0 = \eta_0 = 0)$. If $H_0 = 0$: in the dynamic equations for the displacements $\ell^{\omega}_{R,z}\big|_{|z|\neq 0}$ (8) or (10) the RHS should be reset to zeroes, because here the source will be absent in each UHS & SHS. However, for $H_0 = 0$ the transition area at $z \sim 0$ contains the source, so the RHS of BC (6.a,c) contain such source's terms. For $H_0 \neq 0$ we should apply an opposite approach: RHS in (8) or (10) should remain, but the RHS of BC (6.all) should be zeroes (this particularly allows analytics for **deep** exploration of a multi-layer underground structure).

Here we come from (8) or (10) for $\ell^{\omega}_{R,z}$ to a system for their Kernels $\Lambda^{\omega}_{R,z}$ (see (15)). Applying (15,19,30) to (8) or (10), for $H_0 = 0$ in (30) we get:

$$\hat{\mu} \cdot \frac{\partial^2 \Lambda^{\omega}_R}{\partial z^2} - \left(æ^2 \cdot \left(\hat{\lambda} + 2\hat{\mu} \right) - \omega^2 \cdot \hat{\rho} \right) \cdot \Lambda^{\omega}_R - \left(\hat{\lambda} + \hat{\mu} \right) \cdot æ \cdot \frac{\partial \Lambda^{\omega}_z}{\partial z} = 0 \qquad \text{(31.R)}$$

$$\left(\hat{\lambda} + 2\hat{\mu} \right) \cdot \frac{\partial^2 \Lambda^{\omega}_z}{\partial z^2} - \left(æ^2 \cdot \hat{\mu} - \omega^2 \cdot \hat{\rho} \right) \cdot \Lambda^{\omega}_z + \left(\hat{\lambda} + \hat{\mu} \right) \cdot æ \cdot \frac{\partial \Lambda^{\omega}_R}{\partial z} = 0 \qquad \text{(31.z)}$$

Note for $H_0 \neq 0$: RHS = 0 in (31.R) should be replaced with $\left(-\mathbb{Q}^{\omega}_R\right) \cdot \delta\left(z - H_0\right)$; and RHS = 0 in (31.z) should be replaced with $\left(-\mathbb{Q}^{\omega}_z\right) \cdot \delta\left(z - H_0\right)$.

Each of the following allows an analytical solution: system (31) for $\Lambda^{\omega}_{R,z}$; system (28) for $\mathbf{U}^{\omega}_{R,z}$; eq.(29) for U^{ω}_z; these 3 solutions with decompositions (15,26,27) solve the Anomaly problem. However, building analytics for (31,28,29), we should add the BC around $z = 0$, applying (6) for each of $\Lambda^{\omega}_{R,z}, \mathbf{U}^{\omega}_{R,z}, U^{\omega}_z$. For such BC we can't integrate (31,28,29) directly: they were designed, taking out $\tilde{\vartheta}$ from under derivatives in (7) at (8,10). For $\Lambda^{\omega}_{R,z}$ (31) from (6), using $\int_{-\delta z_-}^{+\delta z^+} dz \cdot \delta(z) = 1$ at $H_0 = 0$ in (30), we have:

$$\left[\Lambda_R^\omega\right]_{-\delta z_-}^{+\delta z^+} = \left[\Lambda_z^\omega\right]_{-\delta z_-}^{+\delta z^+} = 0 \qquad (32.0)$$

$$\left[\hat{\mu}\cdot\frac{\partial\Lambda_R^\omega}{\partial z}\right]_{-\delta z_-}^{+\delta z^+} - \left(\mu-\mu_0-\rho_0\cdot V_0^{\,2}\right)\cdot æ\cdot\Lambda_z^\omega\Big|_{z=0} = -\mathbb{Q}_R^\omega \qquad (32.R)$$

$$\left[\left(\hat{\lambda}+2\hat{\mu}\right)\cdot\frac{\partial\Lambda_z^\omega}{\partial z}\right]_{-\delta z_-}^{+\delta z^+} + \left(\lambda-\lambda_0\right)\cdot æ\cdot\Lambda_R^\omega\Big|_{z=0} = -\mathbb{Q}_z^\omega \qquad (32.z)$$

BC for $\mathbf{U}_{R,z}^\omega$ (28) & for U_z^ω (29) can be derived from BC (6.0,R,z), replacing $\tilde{\vartheta}_{z\sim0}\to\hat{\vartheta}_{z\sim0}$ (see Note after (6)) and using $H_0=0$ in (30). Particularly $\left[\mathbf{U}_R^\omega\right]_{-\delta z_-}^{+\delta z_+} = \left[\mathbf{U}_z^\omega\right]_{-\delta z_-}^{+\delta z_+} = 0$ (33.0)

For jumps $\left[\hat{\mu}\cdot\partial\mathbf{U}_R^\omega/\partial z\right]_{-\delta z_-}^{+\delta z^+}$ we apply $u_{R,z}^\omega=0$ to (6.R)(see Note before (26)), subtracting (32.a). Here, employing (15,26,27.z) and (19,25,30), we get:

$$\left[\hat{\mu}\cdot\frac{\partial\mathbf{U}_R^\omega}{\partial z}\right]_{-\delta z_-}^{+\delta z^+} - \left(\mu-\mu_0-\rho_0\cdot V_0^{\,2}\right)\cdot æ\cdot\mathbf{U}_z^\omega\Big|_{z=0} = 0 \qquad (33.R)$$

For jumps $\left[\left(\hat{\lambda}+2\hat{\mu}\right)\cdot\partial\mathbf{U}_z^\omega/\partial z\right]_{-\delta z_-}^{+\delta z^+}$ we apply only $u_R^\omega=0$ to (6.z), use $u_z^\omega\neq0$ (see Note before (26)) and employ $\partial l_\varphi/\partial\varphi=0$ due to $J_{\mathrm{L}_{k\neq0}}\sim0$ (see Note after (24)). Here, using (19,20,25,30) and (15,26,27.z), we get: $\left[\left(\hat{\lambda}+2\hat{\mu}\right)\cdot J_0\cdot\left(\frac{\partial\Lambda_z^\omega}{\partial z}+\frac{\partial\mathbf{U}_z^\omega}{\partial z}\right)\right]_{-\delta z_-}^{+\delta z^+} +$

$$+\left[\left(\hat{\lambda}+2\hat{\mu}\right)\cdot\sum_{\mathrm{k}=-\infty}^{\infty}e^{i\cdot\mathrm{k}\cdot\varphi}\cdot J_{\mathrm{k}}\cdot\frac{\partial U_z^\omega}{\partial z}\right]_{-\delta z_-}^{+\delta z^+} + \left(\lambda-\lambda_0\right)\cdot J_0\cdot æ\cdot\left(\Lambda_R^\omega+\mathbf{U}_R^\omega\right)_{z=0} = -J_0\cdot\mathbb{Q}_z^\omega \quad (*)$$

Here, subtracting (32.z) & in view of Note after (25)(i.e. $\left|U_z^\omega\right|/\left|\mathbf{U}_z^\omega\right|\sim\Delta\varphi/\left(2\pi\right)\ll1$):

$$\left[\left(\hat{\lambda}+2\hat{\mu}\right)\cdot\frac{\partial\mathbf{U}_z^\omega}{\partial z}\right]_{-\delta z_-}^{+\delta z^+} + \left(\lambda-\lambda_0\right)\cdot æ\cdot\mathbf{U}_R^\omega\Big|_{z=0} = 0 \qquad (33.z)$$

Using (6.0), we have $\left[U_z^{\omega}\right]_{-\delta z_-}^{+\delta z_+}=0$ (34.0); also subtracting (32.z),(33.z) from $(*)$ and taking into account independence of $e^{i\cdot \mathrm{k}\cdot\varphi}\cdot J_{\mathrm{k}}$ from each other (for various (k)), we get: $\left[\left(\hat{\lambda}+2\hat{\mu}\right)\cdot\partial U_z^{\omega}/\partial z\right]_{-\delta z_-}^{+\delta z^+}=0$ (34.z)

Note: in (34.z) the main part of $l_{\omega,R}^{\mathrm{main}}=\boldsymbol{\ell}_R^{\omega}+\mathbf{u}_R^{\omega}$ was taken into account automatically.

Now, combining the related dynamic equations with such BC: (31+32); (28+33); (29+34), we can get wanted analytical solutions. One can try to simplify them, allowing UHS = NVF by low freqs usage (Note 4); here $\mu_0=0$ & $l_R\big|_{z<0}=0$ (however, $\lambda_0\neq 0$ due to $\rho_0\neq 0$ in (3.a), and $l_z\big|_{z<0}\neq 0$). Nevertheless we will apply UHS ≠ NVF (so $l_{R,\varphi}^{\omega}\big|_{z<0}\neq 0$ and $\hat{\mu}\big|_{z<0}=\mu_0\neq 0$), assuming it as necessary generalization; otherwise the main terms $\mu_0\cdot\left(\partial^2\Lambda_R^{\omega}/\partial z^2\right)_{z<0}$ in (31.R) and $\mu_0\cdot\left(\partial^2\mathbf{U}_R^{\omega}/\partial z^2\right)_{z<0}$ in (28.R) are zeroes. Ignoring them brings inconsistency in each of the fourth-order differential systems: (31) for $\Lambda_{R,z}^{\omega}$ and (28) for $\mathbf{U}_{R,z}^{\omega}$ at $z<0$. The UHS = NVF case can be considered as $\mu_0\to 0$ limit transfer from the general case UHS ≠ NVF. Actually a difference between UHS ≠ NVF vs. UHS = NVF cases mainly appears in two thin δz_-^+ skin-layers around $z=0$, so we ignore it. For viscous UHS the situation becomes more complex, we do not focus on it here. Below for simplicity we assume $R_0<R$.

Expressing $\Lambda_{R,z}^{\omega}$ via unknown set $\hat{a}_{\mathrm{P,S}}(æ,z),\ \hat{b}_{\mathrm{P,S}}(æ,z)$, and using (19), from (31) we get:

$$\Lambda_R^{\omega}(æ,z)=+\hat{a}_{\mathrm{P}}\cdot e^{+æ\cdot\hat{q}_{\mathrm{P}}\cdot z}+\hat{b}_{\mathrm{P}}\cdot e^{-æ\cdot\hat{q}_{\mathrm{P}}\cdot z}-\hat{a}_{\mathrm{S}}\cdot\hat{q}_{\mathrm{S}}\cdot e^{+æ\cdot\hat{q}_{\mathrm{S}}\cdot z}+\hat{b}_{\mathrm{S}}\cdot\hat{q}_{\mathrm{S}}\cdot e^{-æ\cdot\hat{q}_{\mathrm{S}}\cdot z}\quad(35.\mathrm{R})$$

$$\Lambda_z^{\omega}(æ,z)=-\hat{a}_{\mathrm{P}}\cdot\hat{q}_{\mathrm{P}}\cdot e^{+æ\cdot\hat{q}_{\mathrm{P}}\cdot z}+\hat{b}_{\mathrm{P}}\cdot\hat{q}_{\mathrm{P}}\cdot e^{-æ\cdot\hat{q}_{\mathrm{P}}\cdot z}+\hat{a}_{\mathrm{S}}\cdot e^{+æ\cdot\hat{q}_{\mathrm{S}}\cdot z}+\hat{b}_{\mathrm{S}}\cdot e^{-æ\cdot\hat{q}_{\mathrm{S}}\cdot z}\quad(35.\mathrm{z})$$

where $\hat{q}_{\mathrm{P,S}}=\left(q_{\mathrm{P,S}}^{z>0},\ q_{\mathrm{0P,0S}}^{z<0}\right)$, $\hat{q}_{\mathrm{Re}}>0$, $\hat{q}_{\mathrm{Im}}<0$, so we apply (remembering (3.c)):

$$q_{\mathrm{P}}^2=1-\frac{\omega^2\rho/æ^2}{(\lambda+2\mu)},\quad q_{\mathrm{S}}^2=1-\frac{\omega^2\rho}{æ^2\mu},\quad q_{\mathrm{0P}}^2=1-\frac{\omega^2\rho_0/æ^2}{(\lambda_0+2\mu_0)},\quad q_{\mathrm{0S}}^2=1-\frac{\omega^2\rho_0}{æ^2\mu_0}\quad(35.0)$$

Also there are 8 (not 4) $\left(\hat{a}_{\mathrm{P,S}},\hat{b}_{\mathrm{P,S}}\right)$-factors in (35): $\left(\hat{a},\hat{b}\right)^{z<0}_{\mathrm{P,S}}=\left(a,\ b\right)_{\mathrm{0P,0S}}$, $\left(\hat{a},\hat{b}\right)^{z>0}_{\mathrm{P,S}}=\left(a,b\right)_{\mathrm{P,S}}$. 4 of them provide decay $\Lambda^{\omega}_{R,z}\Big|_{|z|=\infty}=0$ by $\left(a_{\mathrm{P,S}}=b_{\mathrm{0P,0S}}=0\right)$. 4 other factors $\left(a_{\mathrm{0P,0S}},b_{\mathrm{P,S}}\right)$ should satisfy four BC (32) with **non-uniform** RHS (32.R,z). Here we omit bulky calculations and assume: Anomaly is located in LM (see Fig.1, so $z_{\mathrm{A}}>0$); UHS = NVF$\left(\mu_0\to 0,\ \text{so}\ \ q^2_{\mathrm{0P}}\to 1-\omega^2\rho_0/\left(æ^2\cdot\lambda_0\right),\ \ q^2_{\mathrm{0S}}\to 1-i\cdot\infty\right)$. Thus, using $\mathbb{Q}^{\omega}_R/\mathbb{Q}^{\omega}_z=æ\cdot h_0$ (30); for the required below $\ell^{\omega}_{R,z}\left(R,z>0\right)$, $b_{\mathrm{P,S}}\left(\omega,æ\right)$ we get:

$$b_{\mathrm{P}}=\frac{P_\omega\cdot R_0\cdot J_1\left(æR_0\right)}{\omega^2\cdot\rho\cdot\blacktriangle}\cdot\left(æ\cdot h_0\cdot\left(\beta\cdot q_{\mathrm{0P}}\cdot q_{\mathrm{S}}-\frac{\rho_0}{\rho}\right)+q_{\mathrm{0P}}\cdot\left(1-\bar{\beta}\right)\right)\qquad(35.\mathrm{P})$$

$$b_{\mathrm{S}}=\frac{P_\omega\cdot R_0\cdot J_1\left(æR_0\right)}{\omega^2\cdot\rho\cdot\blacktriangle}\cdot\left(æ\cdot h_0\cdot\left(\frac{\rho_0}{\rho}\cdot q_{\mathrm{P}}+\left(1-\beta\right)\cdot q_{\mathrm{0P}}\right)+\bar{\beta}\cdot q_{\mathrm{0P}}\cdot q_{\mathrm{P}}\right)\qquad(35.\mathrm{S})$$

$$\beta=\frac{2æ^2\mu}{\omega^2\rho},\ \ \bar{\beta}=\beta\left(1-\frac{\rho_0 V_0^2}{2\mu}\right),\ \ \alpha=\frac{\rho_0}{\rho},\ \ \blacktriangle=\beta\cdot\bar{\beta}\cdot q_{\mathrm{0P}}q_{\mathrm{P}}q_{\mathrm{S}}-\left(1-\beta\right)\left(1-\bar{\beta}\right)q_{\mathrm{0P}}-\alpha\cdot q_{\mathrm{P}}\qquad(35.1)$$

Note: $\blacktriangle=\lim\left(\blacksquare/q_{\mathrm{0S}}\right)_{q_{\mathrm{0S}}\to\infty}$ $\quad\blacksquare=\left(\alpha+\beta-1\right)\cdot\left(\alpha+\bar{\beta}-1\right)-\left(1-\beta\right)\cdot\left(1-\bar{\beta}\right)\cdot q_{\mathrm{0P}}\cdot q_{\mathrm{0S}}+$

$$+\left(\alpha+\beta\right)\cdot\left(\alpha+\bar{\beta}\right)\cdot q_{\mathrm{P}}\cdot q_{\mathrm{S}}-\alpha\cdot\left(q_{\mathrm{0P}}\cdot q_{\mathrm{S}}+q_{\mathrm{P}}\cdot q_{\mathrm{0S}}\right)+\beta\cdot\bar{\beta}\cdot q_{\mathrm{0P}}\cdot q_{\mathrm{0S}}\cdot q_{\mathrm{P}}\cdot q_{\mathrm{S}}\qquad(35.2)$$

Note: $\blacksquare$ is the main determinant, describing UHS ≠ NVF $\left(\mu_0\neq 0,\ \text{so}\ q_{\mathrm{0S}}\ \text{is finite}\right)$.

Note: here the terms ~ h_0 actually ~ f^{ω}_R (30); these terms disappear if $f^{\omega}_R=0$, which means usage of Hammers, Weight-Drops, etc. for sources (but not Explosives).

Now, taking into account (15.R,z): $\ell^{\omega}_{R,z}=\int_0^\infty dæ\cdot æ\cdot J_{1,0}\left(æ\cdot R\right)\cdot\Lambda^{\omega}_{R,z}\left(æ,z\right)$ and from (35.R,z): $\Lambda^{\omega}_R\left(æ,z>0\right)=b_{\mathrm{P}}\cdot e^{-æ\cdot q_{\mathrm{P}}\cdot z}+b_{\mathrm{S}}\cdot q_{\mathrm{S}}\cdot e^{-æ\cdot q_{\mathrm{S}}\cdot z}$, $\Lambda^{\omega}_z\left(æ,z>0\right)=b_{\mathrm{P}}\cdot q_{\mathrm{P}}\cdot e^{-æ\cdot q_{\mathrm{P}}\cdot z}+b_{\mathrm{S}}\cdot e^{-æ\cdot q_{\mathrm{S}}\cdot z}$ we come to $\ell^{\omega}_{R,z}\left(R,z>0\right)$:

$$\ell^{\omega}_R\left(R,z>0\right)=\int_0^\infty dæ\cdot æ\cdot J_1\left(æ\cdot R\right)\cdot\left(b_{\mathrm{P}}\cdot e^{-æ\cdot q_{\mathrm{P}}\cdot z}+b_{\mathrm{S}}\cdot q_{\mathrm{S}}\cdot e^{-æ\cdot q_{\mathrm{S}}\cdot z}\right)\qquad(36.\mathrm{R})$$

$$\ell^{\omega}_z\left(R,z>0\right)=\int_0^\infty dæ\cdot æ\cdot J_0\left(æ\cdot R\right)\cdot\left(b_{\mathrm{P}}\cdot q_{\mathrm{P}}\cdot e^{-æ\cdot q_{\mathrm{P}}\cdot z}+b_{\mathrm{S}}\cdot e^{-æ\cdot q_{\mathrm{S}}\cdot z}\right)\qquad(36.\mathrm{z})$$

where $b_{\mathrm{P,S}}$ are defined in (35.P,S).

Analytical Solutions

Here for the most important case $z=0^+$ the limit transfer of (36.R,z) gives: (36.0.R,z)

$$\ell_R^\omega(R)\Big|_{z=0^+} = \frac{P_\omega R_0}{\omega^2\rho}\int_0^\infty d\text{æ}\frac{\text{æ}}{\blacktriangle}J_1(\text{æ}R)J_1(\text{æ}R_0)\begin{pmatrix} h_0\text{æ}\left(q_{0\text{P}}q_\text{S}-(1-q_\text{P}q_\text{S})\rho_0/\rho\right)+ \\ +\,q_{0\text{P}}\cdot\left(1-\bar{\beta}\cdot(1-q_\text{P}\cdot q_\text{S})\right)\end{pmatrix}$$

$$\ell_z^\omega(R)\Big|_{z=0^+} = \frac{P_\omega R_0}{\omega^2\rho}\int_0^\infty d\text{æ}\frac{\text{æ}}{\blacktriangle}J_0(\text{æ}R)J_1(\text{æ}R_0)\,q_{0\text{P}}\left(h_0\text{æ}\left(1-\beta(1-q_\text{P}q_\text{S})\right)+q_\text{P}\right)$$

Now we build an analytical solution for $\mathbf{u}_{R,z}^\omega$ (26.R,z) via $\mathbf{U}_{R,z}^\omega$ (28+33) with (13,17). Here for symmetry we replace (0) in RHS of (28.z) with $\left(-\hat{\rho}\cdot\mathbf{\Phi}_z^\omega\right)$ and apply $\mathbf{\Phi}_z^\omega\equiv 0$ after this solution. First, we use representation (35.R,z) for $\mathbf{U}_{R,z}^\omega$, similar to $\Lambda_{R,z}^\omega$, but with a new set $\left(\hat{a}_\text{P,S},\ \hat{b}_\text{P,S}\right)$, allowing a method of undetermined coefficients usage. One can verify that (35.R,z) (for $\mathbf{U}_{R,z}^\omega$ instead of $\Lambda_{R,z}^\omega$) with $\left(\hat{a}_\text{P,S},\ \hat{b}_\text{P,S}\right)=\text{const}$ satisfy uniform version of (28.R,z) (when $\mathbf{\Phi}_{R,z}^\omega\equiv 0$ in RHS). For $\mathbf{\Phi}_{R,z}^\omega\neq 0$ we get: (37)

$$\hat{a}_\text{P}=\hat{\alpha}_\text{P}-\frac{\text{æ}/2}{\omega^2}\int_0^z dz'e^{-\hat{q}_\text{P}\cdot\text{æ}\cdot z'}\left(\mathbf{\Phi}_R^\omega/\hat{q}_\text{P}+\mathbf{\Phi}_z^\omega\right),\quad \hat{a}_\text{S}=\hat{\alpha}_\text{S}-\frac{\text{æ}/2}{\omega^2}\int_0^z dz'e^{-\hat{q}_\text{S}\cdot\text{æ}\cdot z'}\left(\mathbf{\Phi}_R^\omega+\mathbf{\Phi}_z^\omega/\hat{q}_\text{S}\right)$$

$$\hat{b}_\text{P}=\hat{\beta}_\text{P}+\frac{\text{æ}/2}{\omega^2}\int_0^z dz'e^{+\hat{q}_\text{P}\cdot\text{æ}\cdot z'}\left(\mathbf{\Phi}_R^\omega/\hat{q}_\text{P}-\mathbf{\Phi}_z^\omega\right),\quad \hat{b}_\text{S}=\hat{\beta}_\text{S}-\frac{\text{æ}/2}{\omega^2}\int_0^z dz'e^{+\hat{q}_\text{S}\cdot\text{æ}\cdot z'}\left(\mathbf{\Phi}_R^\omega-\mathbf{\Phi}_z^\omega/\hat{q}_\text{S}\right)$$

with unknown $\left(\hat{\alpha},\hat{\beta}\right)_\text{P,S}^{z<0}=(\alpha,\beta)_\text{0P,0S}$, $\left(\hat{\alpha},\hat{\beta}\right)_\text{P,S}^{z>0}=(\alpha,\beta)_\text{P,S}$ instead of $\left(\hat{a}_\text{P,S},\hat{b}_\text{P,S}\right)$, see notations after (35.0). The decay $\mathbf{U}_{R,z}^\omega(\text{æ},z)_{|z|=\infty}=0$ in (35.R,z) gives $\left(a_\text{P,S}=b_\text{0P,0S}=0\right)$, which means: for $z<0$ the unknowns are $\alpha_\text{0P, 0S}$; for $z>0$ the unknowns are $\beta_\text{P,S}$.

Note: here and below we use (35.0) for $\hat{q}(z)$, ignoring the Anomaly. Indeed, if we apply $\hat{q}(z')$ instead of $\hat{q}(z)$, we actually take into account the ignored second degree of Anomaly decomposition terms (small due to (27.R,z): $\mathbf{\Phi}_{R,z}^\omega\sim\Theta_{R,z}^\omega$).

In (37) (and (38) below) for generality we assume, that Anomalies can occupy UHS also, so $\mathbf{\Phi}_{R,z}^\omega\Big|_{z<0}\neq 0$ (without touching the boundary $z=0$, so $\mathbf{\Phi}_{R,z}^\omega\Big|_{z=0_-^+}=0$).

Thus, omitting bulky calculations, for $\mathbf{U}_{R,z}^{\omega}\Big|_{\text{any } z}$ via $\mathbf{\Phi}_{R,z}^{\omega}(æ, z')$ we have:

$$\mathbf{U}_{R}^{\omega}\Big|_{z<0} = \alpha_{0\mathrm{P}} \cdot e^{+æ\cdot q_{0\mathrm{P}}\cdot z} - \alpha_{0\mathrm{S}} \cdot q_{0\mathrm{S}} \cdot e^{+æ\cdot q_{0\mathrm{S}}\cdot z} + \frac{æ/2}{\omega^2} \cdot \tag{38}$$

$$\cdot \left(\begin{array}{l} \int\limits_{-\infty}^{z<0} dz' \cdot \left(e^{-æ\cdot q_{0\mathrm{P}}\cdot(z-z')} \cdot \left(\mathbf{\Phi}_R^{\omega} / q_{0\mathrm{P}} - \mathbf{\Phi}_z^{\omega} \right) - e^{-æ\cdot q_{0\mathrm{S}}(z-z')} \cdot \left(\mathbf{\Phi}_R^{\omega} \cdot q_{0\mathrm{S}} - \mathbf{\Phi}_z^{\omega} \right) \right) + \\ + \int\limits_{z<0}^{0} dz' \cdot \left(e^{-æ\cdot q_{0\mathrm{P}}\cdot(z'-z)} \left(\mathbf{\Phi}_R^{\omega} / q_{0\mathrm{P}} + \mathbf{\Phi}_z^{\omega} \right) - e^{-æ\cdot q_{0\mathrm{S}}(z'-z)} \cdot \left(\mathbf{\Phi}_R^{\omega} \cdot q_{0\mathrm{S}} + \mathbf{\Phi}_z^{\omega} \right) \right) \end{array} \right)$$

$$\mathbf{U}_{R}^{\omega}\Big|_{z>0} = \beta_{\mathrm{P}} \cdot e^{-æ\cdot q_{\mathrm{P}}\cdot z} + \beta_{\mathrm{S}} \cdot q_{\mathrm{S}} \cdot e^{-æ\cdot q_{\mathrm{S}}\cdot z} + \frac{æ/2}{\omega^2} \cdot$$

$$\cdot \left(\begin{array}{l} \int\limits_{z>0}^{+\infty} dz' \cdot \left(e^{-æ\cdot q_{\mathrm{P}}\cdot(z'-z)} \cdot \left(\mathbf{\Phi}_R^{\omega} / q_{\mathrm{P}} + \mathbf{\Phi}_z^{\omega} \right) - e^{-æ\cdot q_{\mathrm{S}}(z'-z)} \cdot \left(\mathbf{\Phi}_R^{\omega} \cdot q_{\mathrm{S}} + \mathbf{\Phi}_z^{\omega} \right) \right) + \\ + \int\limits_{0}^{z>0} dz' \cdot \left(e^{-æ\cdot q_{\mathrm{P}}\cdot(z-z')} \left(\mathbf{\Phi}_R^{\omega} / q_{\mathrm{P}} - \mathbf{\Phi}_z^{\omega} \right) - e^{-æ\cdot q_{\mathrm{S}}(z-z')} \cdot \left(\mathbf{\Phi}_R^{\omega} \cdot q_{\mathrm{S}} - \mathbf{\Phi}_z^{\omega} \right) \right) \end{array} \right)$$

$$\mathbf{U}_{z}^{\omega}\Big|_{z<0} = -\alpha_{0\mathrm{P}} \cdot q_{0\mathrm{P}} \cdot e^{+æ\cdot q_{0\mathrm{P}}\cdot z} + \alpha_{0\mathrm{S}} \cdot e^{+æ\cdot q_{0\mathrm{S}}\cdot z} + \frac{æ/2}{\omega^2} \cdot$$

$$\cdot \left(\begin{array}{l} \int\limits_{-\infty}^{z<0} dz' \cdot \left(e^{-æ\cdot q_{0\mathrm{P}}\cdot(z-z')} \cdot \left(\mathbf{\Phi}_R^{\omega} - \mathbf{\Phi}_z^{\omega} \cdot q_{0\mathrm{P}} \right) - e^{-æ\cdot q_{0\mathrm{S}}(z-z')} \cdot \left(\mathbf{\Phi}_R^{\omega} - \mathbf{\Phi}_z^{\omega} / q_{0\mathrm{S}} \right) \right) + \\ + \int\limits_{z<0}^{0} dz' \left(-e^{-æ\cdot q_{0\mathrm{P}}\cdot(z'-z)} \left(\mathbf{\Phi}_R^{\omega} + \mathbf{\Phi}_z^{\omega} \cdot q_{0\mathrm{P}} \right) + e^{-æ\cdot q_{0\mathrm{S}}(z'-z)} \left(\mathbf{\Phi}_R^{\omega} + \mathbf{\Phi}_z^{\omega} / q_{0\mathrm{S}} \right) \right) \end{array} \right)$$

$$\mathbf{U}_{z}^{\omega}\Big|_{z>0} = \beta_{\mathrm{P}} \cdot q_{\mathrm{P}} \cdot e^{-æ\cdot q_{\mathrm{P}}\cdot z} + \beta_{\mathrm{S}} \cdot e^{-æ\cdot q_{\mathrm{S}}\cdot z} + \frac{æ/2}{\omega^2} \cdot$$

$$\cdot \left(\begin{array}{l} \int\limits_{z>0}^{+\infty} dz' \cdot \left(-e^{-æ\cdot q_{\mathrm{P}}\cdot(z'-z)} \cdot \left(\mathbf{\Phi}_R^{\omega} + \mathbf{\Phi}_z^{\omega} \cdot q_{\mathrm{P}} \right) + e^{-æ\cdot q_{\mathrm{S}}(z'-z)} \left(\mathbf{\Phi}_R^{\omega} + \mathbf{\Phi}_z^{\omega} / q_{\mathrm{S}} \right) \right) + \\ + \int\limits_{0}^{z>0} dz' \left(e^{-æ\cdot q_{\mathrm{P}}\cdot(z-z')} \cdot \left(\mathbf{\Phi}_R^{\omega} - \mathbf{\Phi}_z^{\omega} \cdot q_{\mathrm{P}} \right) - e^{-æ\cdot q_{\mathrm{S}}(z-z')} \cdot \left(\mathbf{\Phi}_R^{\omega} - \mathbf{\Phi}_z^{\omega} / q_{\mathrm{S}} \right) \right) \end{array} \right)$$

Now we return to the initial case, assuming that Anomaly occupies LM only (not UHS), so $\mathbf{\Phi}^{\omega}_{R,z}\big|_{z<0}=0$. Also, for simplicity we use UHS = NVF, making limit transfer $\mu_0\to 0$ for a final results (same as for $b_{\mathrm{P,S}}$ (35.P,S), where $q^2_{0\mathrm{P}}\to 1-\omega^2\cdot\rho_0/\left(æ^2\cdot\lambda_0\right)$, $q^2_{0\mathrm{S}}\to 1-i\cdot\infty$). Thus, using (38), we have four BC (33.0,R,z) with four unknowns $\left(\alpha_{0\mathrm{P}},\alpha_{0\mathrm{S}},\beta_{\mathrm{P}},\beta_{\mathrm{S}}\right)$. Solving such system & omitting bulky calculations, we get:

$$\mathbf{U}^{\omega}_{R}(æ)\Big|_{z=0^{\pm}}=\frac{æ/\blacktriangle}{\omega^2}\int_0^{\infty}dz'\begin{pmatrix} e^{-æ\cdot q_{\mathrm{P}}\cdot z'}\left(\mathbf{\Phi}^{\omega}_{R}+\mathbf{\Phi}^{\omega}_{z}\cdot q_{\mathrm{P}}\right)\left(\bar{\beta}\cdot q_{0\mathrm{P}}\cdot q_{\mathrm{S}}-\rho_0/\rho\right)+ \\ e^{-æ\cdot q_{\mathrm{S}}\cdot z'}\left(\mathbf{\Phi}^{\omega}_{R}\cdot q_{\mathrm{S}}+\mathbf{\Phi}^{\omega}_{z}\right)\left(\left(1-\bar{\beta}\right)q_{0\mathrm{P}}+q_{\mathrm{P}}\rho_0/\rho\right)\end{pmatrix} \quad (39.\mathrm{R})$$

$$\mathbf{U}^{\omega}_{z}(æ)\Big|_{z=0^{\pm}}=\frac{æ/\blacktriangle}{\omega^2}\int_0^{\infty}dz'\begin{pmatrix} e^{-æ\cdot q_{\mathrm{P}}\cdot z'}\left(\mathbf{\Phi}^{\omega}_{R}+\mathbf{\Phi}^{\omega}_{z}\cdot q_{\mathrm{P}}\right)q_{0\mathrm{P}}\left(1-\beta\right)+ \\ e^{-æ\cdot q_{\mathrm{S}}\cdot z'}\left(\mathbf{\Phi}^{\omega}_{R}\cdot q_{\mathrm{S}}+\mathbf{\Phi}^{\omega}_{z}\right)\cdot q_{0\mathrm{P}}\cdot q_{\mathrm{P}}\cdot\beta\end{pmatrix} \quad (39.\mathrm{z})$$

Here $\left(q_{0\mathrm{P}},q_{\mathrm{P}},q_{\mathrm{S}},\blacktriangle,\beta,\bar{\beta}\right)(æ)$ are given in (35.0,1). Now we apply $\mathbf{\Phi}^{\omega}_{z}\equiv 0$ in the result (39), see remark below (36.0.R,z). Here, using (26.R,z), we have:

$$\mathbf{u}^{\omega}_{R}(R)\Big|_{z=0^{\pm}}=\int_0^{\infty}\frac{dæ\cdot æ^2}{\omega^2\cdot\blacktriangle}\cdot J_1(æ\cdot R)\cdot\int_0^{\infty}dz'\cdot\begin{pmatrix} +e^{-æ\cdot q_{\mathrm{P}}\cdot z'}\cdot\left(\bar{\beta}\cdot q_{0\mathrm{P}}\cdot q_{\mathrm{S}}-\rho_0/\rho\right)+ \\ +e^{-æ\cdot q_{\mathrm{S}}\cdot z'}q_{\mathrm{S}}\left(\left(1-\bar{\beta}\right)q_{0\mathrm{P}}+q_{\mathrm{P}}\rho_0/\rho\right)\end{pmatrix}\cdot\mathbf{\Phi}^{\omega}_{R} \quad (40.\mathrm{R})$$

$$\mathbf{u}^{\omega}_{z}(R)\Big|_{z=0^{\pm}}=\int_0^{\infty}\frac{dæ\cdot æ^2}{\omega^2\cdot\blacktriangle}\cdot J_0(æ\cdot R)\cdot\int_0^{\infty}dz'\cdot\begin{pmatrix} +e^{-æ\cdot q_{\mathrm{P}}\cdot z'}\cdot q_{0\mathrm{P}}\cdot\left(1-\beta\right)+ \\ +e^{-æ\cdot q_{\mathrm{S}}\cdot z'}\cdot q_{0\mathrm{P}}\cdot q_{\mathrm{P}}\cdot q_{\mathrm{S}}\cdot\beta\end{pmatrix}\cdot\mathbf{\Phi}^{\omega}_{R}(æ,z') \quad (40.\mathrm{z})$$

Note: the explicit dependency $\mathbf{\Phi}^{\omega}_{R}(æ,z)$ on $\ell^{\omega}_{R,z}\left(r,z=z_{\mathrm{A}}>0\right)$ is clear from (27.R):

$$\mathbf{\Phi}^{\omega}_{R}(æ,z)=\int_0^{\infty}dr\cdot r\cdot J_1(æ\cdot r)\cdot\int_0^{2\pi}\frac{d\psi}{2\pi}\cdot\theta^{\omega}_{R}(r,\psi,z) \quad (40.0)$$

and from (13.R), where θ^{ω}_{R} explicitly depends on $\ell^{\omega}_{R,z}$ (computed in (36.R,z)).

Note: in (38-40) we use (35.0) for $\hat{q}(æ,z)$, not $\hat{q}(æ,z')$ (see Note after (37)).

Now from (40), using (13.R) with $\theta^{\omega}_{R}(r,\psi,z')$, and applying (36.R,z), we get:

$$\mathbf{u}_R^{\omega}(R)\Big|_{z=0^{\pm}} = \int_0^{\infty} \frac{dæ \cdot æ^2}{\omega^2 \cdot \blacktriangle} \frac{J_1(æ \cdot R)}{2\pi \cdot \rho} \left(A_{\mathrm{P}} \left(\bar{\beta} \cdot q_{0\mathrm{P}} \cdot q_{\mathrm{S}} - \frac{\rho_0}{\rho} \right) + A_{\mathrm{S}} \cdot q_{\mathrm{S}} \left((1-\bar{\beta}) q_{0\mathrm{P}} + q_{\mathrm{P}} \frac{\rho_0}{\rho} \right) \right)$$

$$\mathbf{u}_z^{\omega}(R)\Big|_{z=0^{\pm}} = \int_0^{\infty} \frac{dæ \cdot æ^2}{\omega^2 \cdot \blacktriangle} \frac{J_0(æ \cdot R)}{2\pi \cdot \rho} \left(A_{\mathrm{P}} \cdot q_{0\mathrm{P}} \cdot (1-\beta) + A_{\mathrm{S}} \cdot q_{0\mathrm{P}} \cdot q_{\mathrm{P}} \cdot q_{\mathrm{S}} \cdot \beta \right), \text{ where} \quad (41.\mathrm{R.z})$$

$$A_{\mathrm{P,S}} = \int_0^{\infty} dr \cdot r \cdot J_1(æ \cdot r) \cdot \int_0^{\infty} dz' \cdot e^{-æ \cdot q_{\mathrm{P,S}} \cdot z'} \int_0^{2\pi} d\psi \cdot \rho \cdot \theta_R^{\omega}(r, \psi, z') \quad (41.0)$$

Note: localized Anomaly's thicknesses are small: $\Delta\xi_{\mathrm{A}} \sim 0 \left(\Delta\xi_{\mathrm{A}} = \text{any of} \left(\Delta R_{\mathrm{A}},\ R_{\mathrm{A}}\Delta\varphi_{\mathrm{A}},\ \Delta z_{\mathrm{A}} \right) \right)$. Here we remind: $\Delta\vartheta_{\mathrm{A}} = 0^{\text{outside}}_{\text{Anom.}}$ or any of $\left(\Delta\rho_{\mathrm{A}}, \Delta\lambda_{\mathrm{A}}, \Delta\mu_{\mathrm{A}}\right)^{\text{inside}}_{\text{Anom.}}$; $\left(\Delta\Omega_{\mathrm{A}} = \Delta R_{\mathrm{A}} \cdot R_{\mathrm{A}} \cdot \Delta\varphi_{\mathrm{A}} \cdot \Delta z_{\mathrm{A}}\right)$, and $\vec{r}(R_{\mathrm{A}}, \varphi_{\mathrm{A}}) \equiv \vec{R}_{\mathrm{A}}$. Besides, replacing θ_R^{ω} in (41.0) with θ_R^{ω} (13.R), we can use integration by parts, applying $\Delta\vartheta^{\text{outside}}_{\text{Anom.}} = 0$. It allows to express $A_{\mathrm{P,S}}$ (41.0) via $\Delta\vartheta_{\mathrm{A}}$ (not via unknown derivatives $\partial\Delta\vartheta_{\mathrm{A}} / \partial(r, z')$ in (13.R)). Also: the dependency θ_R^{ω} in (13.R) on ψ comes only from terms $\Delta\vartheta_{\mathrm{A}}(\psi)$, there are no even $\partial\Delta\vartheta_{\mathrm{A}} / \partial\psi$ -terms. Thus (41.1)

$$A_{\mathrm{P,S}} \cdot \frac{e^{+æ \cdot q_{\mathrm{P,S}} \cdot z_{\mathrm{A}}}}{\Delta\Omega_{\mathrm{A}}} = -J_0(æ \cdot R_{\mathrm{A}}) \cdot æ \cdot \left(\Delta\lambda_{\mathrm{A}} \cdot \left(\frac{\partial \ell_R^{\omega}}{\partial R} + \frac{\ell_R^{\omega}}{R} + \frac{\partial \ell_z^{\omega}}{\partial Z} \right)_{\mathrm{A}} + 2 \cdot \Delta\mu_{\mathrm{A}} \cdot \frac{\partial \ell_R^{\omega}}{\partial R}\bigg|_{\mathrm{A}} \right) +$$

$$+ J_1(æ \cdot R_{\mathrm{A}}) \left(\omega^2 \cdot \Delta\rho_{\mathrm{A}} \cdot \ell_R^{\omega}\big|_{\mathrm{A}} + 2 \frac{\Delta\mu_{\mathrm{A}}}{R_{\mathrm{A}}} \left(\frac{\partial \ell_R^{\omega}}{\partial R} - \frac{\ell_R^{\omega}}{R} \right)_{\mathrm{A}} + æ \cdot q_{\mathrm{P,S}} \Delta\mu_{\mathrm{A}} \left(\frac{\partial \ell_z^{\omega}}{\partial R} + \frac{\partial \ell_R^{\omega}}{\partial Z} \right)_{\mathrm{A}} \right)$$

Note: if we need to integrate $A_{\mathrm{P,S}}$ by $d\Omega_{\mathrm{A}}$, we must allow $R_{\mathrm{A}}(\varphi_{\mathrm{A}}),\ z_{\mathrm{A}}(R_{\mathrm{A}}, \varphi_{\mathrm{A}})$. Simple example: Anomaly is an infinite tunnel (Fig.3), perpendicular to a virtual x-line, (so $R_{\mathrm{A}} = R_{\perp} / \cos\varphi_{\mathrm{A}}$, where $R_{\perp}$ is the min Anomaly's distance from Source Center), but $z_{\mathrm{A}} = \text{Const}$, so $\iiint \frac{A_{\mathrm{P,S}}}{\Delta\Omega_{\mathrm{A}}} d\Omega_{\mathrm{A}} = \int_{R^{\min}_{\mathrm{An.}}}^{R^{\max}_{\mathrm{An.}}} dR_{\mathrm{A}} \int_{-\pi/2}^{+\pi/2} R_{\mathrm{A}}(\varphi_{\mathrm{A}}) d\varphi_{\mathrm{A}} \int_{z^{\min}_{\mathrm{An.}}}^{z^{\max}_{\mathrm{An.}}} dz \frac{A_{\mathrm{P,S}}}{\Delta\Omega_{\mathrm{A}}} \approx \Delta R_{\mathrm{A}} R_{\perp} \Delta z_{\mathrm{A}} \int_{-\pi/2}^{+\pi/2} \frac{d\varphi_{\mathrm{A}} \cdot A_{\mathrm{P,S}}}{\cos\varphi_{\mathrm{A}} \Delta\Omega_{\mathrm{A}}}$. The last integral converges due to $J_{0,1}^{R_{\mathrm{A}} \to \infty} \sim \ell_{\omega,R,z}^{R_{\mathrm{A}} \to \infty} \sim R_{\mathrm{A}}^{-1/2}$ in (41.1) & (36.R,z) (where instead of R we should use R_{A}).

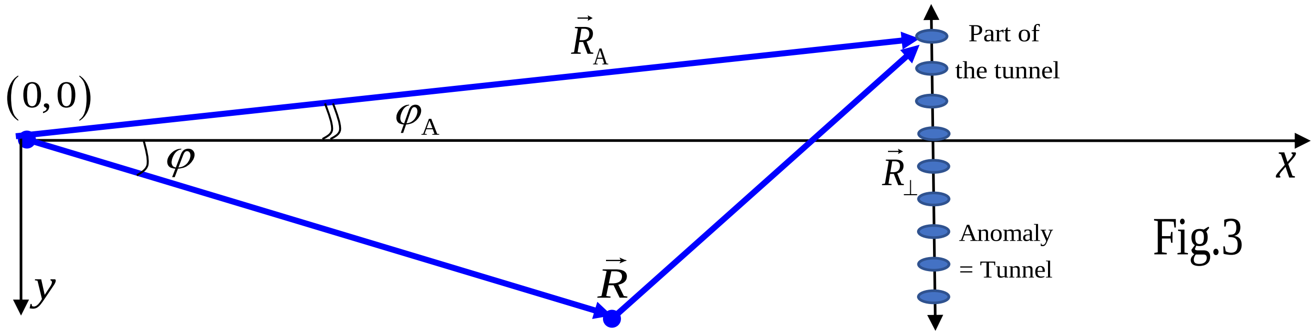

Fig.3

Now we build an analytical solution for u_z^ω (26.z.a) via U_z^ω (29.z) with (17.b,13.z) and BC (34) in a form: $U_z^\omega\Big|_{+\delta z_+} = U_z^\omega\Big|_{-\delta z_-}$ (42.0) & under UHS = NVF condition ($\mu_0 \to 0$):

$$(\lambda + 2\mu)\cdot \frac{\partial U_z^\omega}{\partial z}\Bigg|_{+\delta z^+} = \lambda_0 \cdot \frac{\partial U_z^\omega}{\partial z}\Bigg|_{-\delta z^-} \quad (42.z)$$

We will need a decay: $U_z^\omega\Big|_{|z|\to\infty} = 0$ (42.1) and also re-write (29.z) in a form:

$\frac{\partial^2 U_z^\omega}{\partial z^2} - \hat{Q}^2 \cdot U_z^\omega(\text{æ, any k, } z) = -\hat{\Pi}(\text{æ, any k, } z)$ (43.0), where (assuming $z_A > 0$):

$$\hat{Q}^2 = \begin{cases} Q_0^2 = -\omega^2 / V_{0P}^2 \ \ (V_{0S} \to 0) \text{ at } z < 0 \\ Q_{æ}^2 = \left(æ^2 \cdot V_S^2 - \omega^2\right)/V_P^2 \quad \text{at } z > 0 \end{cases}, \quad \hat{\Pi}(z) = \begin{cases} 0 & \text{at } z < 0 \\ \Phi_z^\omega(\text{æ, any k, } z)/V_P^2 & \text{at } z > 0 \end{cases} \quad (43.1)$$

Using method of undetermined coefficients, we can represent U_z^ω (43.0):

$$2\cdot U_z^\omega(æ,\mathrm{k})\Big|_{z\neq 0} = e^{\hat{Q}\cdot z}\left(2\hat{\alpha} - \int_0^z dz' \cdot e^{-\hat{Q}\cdot z'}\hat{\Pi}(z')/\hat{Q}\right) + e^{-\hat{Q}\cdot z}\left(2\hat{\beta} + \int_0^z dz' \cdot e^{\hat{Q}\cdot z'}\hat{\Pi}(z')/\hat{Q}\right) \quad (44)$$

where $(\hat{\alpha},\hat{\beta})_{z<0} = (\alpha_0, \beta_0)$, $(\hat{\alpha},\hat{\beta})_{z>0} = (\alpha, \beta)$.

Note: here and below we use $\hat{Q}(z)$ (not $\hat{Q}(z')$) under integrals, see Note after (37).

Taking into account (43.1), we have $\left(Q_0^{\mathrm{Re}} > 0,\ Q_0^{\mathrm{Im}} < 0;\ \ Q_{æ}^{\mathrm{Re}} > 0,\ Q_{æ}^{\mathrm{Im}} < 0\right)$:

$$U_z^\omega\Big|_{z<0} = \alpha_0 \cdot e^{+Q_0\cdot z} + \beta_0 \cdot e^{-Q_0 \cdot z} \quad (44.z-)$$

$$U_z^\omega\Big|_{z>0} = e^{Q_{æ}\cdot z}\left(\alpha - \int_0^z dz' \cdot e^{-Q_{æ}\cdot z'} \cdot \Phi_z^\omega(æ,\mathrm{k},z')/\left(2Q_{æ}\cdot V_P^2\right)\right) + $$
$$+ e^{-Q_{æ}\cdot z}\left(\beta + \int_0^z dz' \cdot e^{+Q_{æ}\cdot z'} \cdot \Phi_z^\omega(æ,\mathrm{k},z')/\left(2Q_{æ}\cdot V_P^2\right)\right) \quad (44.z+)$$

Now, using decay (42.1), we have: $\beta_0 = 0$, so $U_z^\omega\Big|_{z<0} = \alpha_0 \cdot e^{+Q_0 \cdot z}$ (45.z -)

Also, introducing $\gamma = \int_0^\infty dz' \cdot e^{-Q_{æ}\cdot z'} \cdot \Phi_z^\omega(æ,\mathrm{k},z')$, we get: $\alpha \cdot 2Q_{æ}\cdot V_P^2 = \gamma$, so

$$U_z^\omega\Big|_{z>0} = \beta\cdot e^{-Q_{æ}\cdot z} + e^{-Q_{æ}\cdot z}\cdot \int_0^z dz' \cdot e^{+Q_{æ}\cdot z'} \cdot \Phi_z^\omega(æ,\mathrm{k},z')/\left(2Q_{æ}\cdot V_P^2\right) + $$
$$+ e^{+Q_{æ}\cdot z}\cdot \int_z^\infty dz' \cdot e^{-Q_{æ}\cdot z'} \cdot \Phi_z^\omega(æ,\mathrm{k},z')/\left(2Q_{æ}\cdot V_P^2\right) \quad (45.z+)$$

Note: indeed $e^{+Q_{æ}\cdot z}\cdot \int_z^\infty dz' \cdot e^{-Q_{æ}\cdot z'}\Phi_z^\omega(z') \sim \Phi_z^\omega(z\sim\infty)/Q_{æ} \sim 0$ (Anomaly is localized)

Thus, from BC (42) with $\Phi_z^\omega\big|_{z=0^\pm}=0$

$$U_z^\omega\big|_{z=-\delta z^-}=\alpha_0=U_z^\omega\big|_{z=+\delta z^+}=\beta+\gamma/\left(2Q_æ\cdot V_P^2\right) \quad (46.z-)$$

$$\lambda_0\cdot\frac{\partial U_z^\omega}{\partial z}\bigg|_{-\delta z^-}=\alpha_0\cdot Q_0\cdot\lambda_0=\left(\lambda+2\mu\right)\cdot\frac{\partial U_z^\omega}{\partial z}\bigg|_{+\delta z^+}=\left(\lambda+2\mu\right)\cdot\left(\frac{\gamma/2}{V_P^2}-\beta\cdot Q_æ\right) \quad (46.z+)$$

Now (46) gives:

$$U_z^\omega\big|_{z=0^\pm}=\alpha_0=\frac{\gamma}{V_P^2}\cdot\frac{\left(\lambda+2\mu\right)}{\left(\lambda+2\mu\right)\cdot Q_æ+\lambda_0\cdot Q_0},\quad \beta=\frac{\gamma/2}{Q_æ\cdot V_P^2}\left(\frac{\left(\lambda+2\mu\right)\cdot Q_æ-\lambda_0\cdot Q_0}{\left(\lambda+2\mu\right)\cdot Q_æ+\lambda_0\cdot Q_0}\right) \quad (47)$$

According to (3.a,b), for UHS = NVF: $V_F\equiv V_{P0}$, and $\lambda_0=\rho_0\cdot V_{P0}^2$, so $\dfrac{\lambda_0}{\lambda+2\mu}=\dfrac{\rho_0\cdot V_{P0}^2}{\rho\cdot V_P^2}$.

Thus, using γ via Φ_z^ω and (26.z.a),(17.b), we have:

$$u_z^\omega\left(R,\varphi\right)\big|_{z=0^\pm}=\int_0^\infty dæ\cdotæ\cdot\int_0^\infty dr\cdot r\cdot\int_0^\infty dz'\cdot e^{-Q_æ\cdot z'}\cdot\int_0^{2\pi}\frac{d\psi}{2\pi}\cdot\rho\cdot\theta_z^\omega\left(r,\psi,z'\right)\cdot$$

$$\cdot\sum_{k=-\infty}^{\infty}e^{i\cdot k\cdot(\varphi-\psi)}\cdot J_k\left(æ\cdot R\right)\cdot J_k\left(æ\cdot r\right)/\left(\rho_0\cdot V_{P0}^2\cdot Q_0+\rho\cdot V_P^2\cdot Q_æ\right) \quad (48)$$

The achieved (48) allows BAT usage (see [6-8.530.2] and Fig.3):

$$\sum_{k=-\infty}^{\infty}J_k\left(æ\cdot r\right)\cdot J_k\left(æ\cdot R\right)\cdot e^{i\cdot k\cdot(\varphi-\psi)}=J_0\left(æ\cdot\left|\vec r-\vec R\right|\right),\quad \left|\vec r-\vec R\right|^2=r^2+R^2-2r\cdot R\cdot\cos\left(\psi-\varphi\right) \quad (49)$$

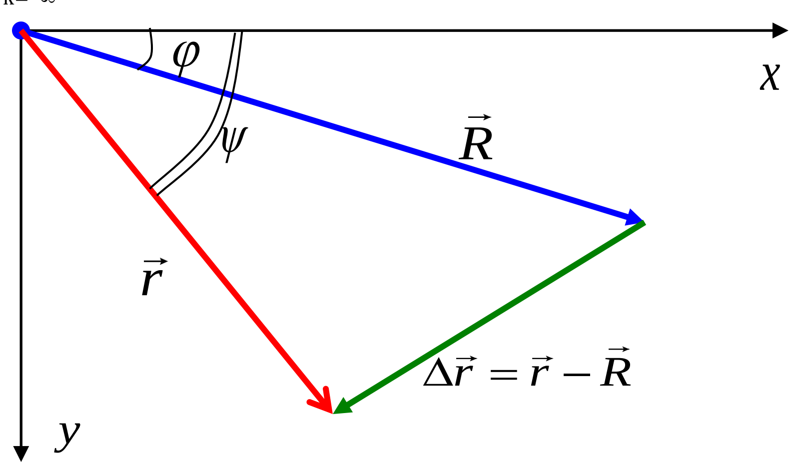

Fig.4. Geometry notations for "Bessel Addition Theorem".

Thus (48) can be re-written as:

$$u_z^\omega\left(R,\varphi\right)\Big|_{z=0^\pm}=\int_0^\infty \frac{d\text{æ}\cdot\text{æ}\cdot C_\text{æ}^\omega/\left(2\pi\right)}{\rho_0\cdot V_{0\text{P}}^2\cdot Q_0+\rho\cdot V_\text{P}^2\cdot Q_\text{æ}}\text{, where} \tag{50}$$

$$C_\text{æ}^\omega=\int_0^\infty dr\cdot r\cdot\int_0^\infty dz'\cdot e^{-Q_\text{æ}\cdot z'}\cdot\int_0^{2\pi}d\psi\cdot J_0\left(\text{æ}\cdot\left|\vec{r}\left(r,\psi\right)-\vec{R}\right|\right)\cdot\rho\cdot\theta_z^\omega\left(r,\psi,z'\right) \tag{50.0}$$

Here we refer to Note after (41.0), replacing $\left(\theta_R^\omega,A_{\text{P,S}},\left(41.0,13.R\right)\right)$ with $\left(\theta_z^\omega,C_\text{æ}^\omega,\left(50.0,13.z\right)\right)$.
Now, using (13.z) with $\theta_z^\omega\left(r,\psi,z'\right)$ and (36.R,z), we get:

$$C_\text{æ}^\omega\cdot\frac{e^{+Q_\text{æ}\cdot z_\text{A}}}{\Delta\Omega_\text{A}}=J_1\left(\text{æ}\cdot\Delta r_\text{A}\right)\cdot\frac{\partial\Delta r_\text{A}}{\partial R_\text{A}}\cdot\Delta\mu_\text{A}\cdot\text{æ}\cdot\left(\frac{\partial\ell_z^\omega}{\partial R}+\frac{\partial\ell_R^\omega}{\partial z}\right)_\text{A}+J_0\left(\text{æ}\cdot\Delta r_\text{A}\right)\cdot$$
$$\cdot\left(\omega^2\cdot\Delta\rho_\text{A}\cdot\ell_z^\omega\Big|_\text{A}+Q_\text{æ}\cdot\left(\Delta\lambda_\text{A}\cdot\left(\frac{\partial\ell_R^\omega}{\partial R}+\frac{\ell_R^\omega}{R}\right)_\text{A}+\left(\Delta\lambda_\text{A}+2\cdot\Delta\mu_\text{A}\right)\cdot\frac{\partial\ell_z^\omega}{\partial z}\Big|_\text{A}\right)\right) \tag{50.1}$$

$$\Delta r_\text{A}^2=\left|\vec{R}_\text{A}-\vec{R}\right|^2=R_\text{A}^2+R^2-2R_\text{A}\cdot R\cdot\cos\left(\varphi_\text{A}-\varphi\right),\quad\frac{\partial\Delta r_\text{A}}{\partial R_\text{A}}=\frac{R_\text{A}-R\cdot\cos\left(\varphi_\text{A}-\varphi\right)}{\Delta r_\text{A}} \tag{50.2}$$

Note: we allow $R_\text{A}\left(\varphi_\text{A}\right),\ z_\text{A}\left(R_\text{A},\varphi_\text{A}\right)$, if we integrate by $d\Omega_\text{A}$ (see text after (41.1) and Fig.3). Here we use the same example: Anomaly is tunnel, perpendicular to the virtual x - line with $z_\text{A}=\text{Const}$. Now the convergence of $C_\text{æ}^\omega$ is provided in view of: (50.1) contains diminishing $J_{0,1}^{R_\text{A}\to\infty}\left(\text{æ}\cdot\Delta r_\text{A}\right)\sim\ell_{\omega,R,z}^{R_\text{A}\to\infty}\sim R_\text{A}^{-1/2}$, due to $\Delta r_\text{A}\big|_{R_\text{A}\to\infty}\sim R_\text{A}$ and (36.R,z)(where R is replaced with R_A), so $\iiint\frac{C_\text{æ}^\omega}{\Delta\Omega_\text{A}}d\Omega_\text{A}\approx\Delta R_\text{A}\cdot R_\perp\cdot\Delta z_\text{A}\int_{-\pi/2}^{+\pi/2}\frac{d\varphi_\text{A}\cdot C_\text{æ}^\omega}{\cos\varphi_\text{A}\cdot\Delta\Omega_\text{A}}$.

Summarizing, the total solution for $l_{R,z}^\omega\left(R,\varphi\right)_{z=0^\pm}$ in (25) can be found as superposition of "NO Anomaly case" (36.0.R,z) for $\ell_{R,z}^\omega\left(R\right)\Big|_{z=0^\pm}$ & "Axis-Symmetrical Anomaly" (41) for $\mathbf{u}_{R,z}^\omega\left(R\right)\Big|_{z=0^\pm}$ & z-response (50) from "Real Anomaly" for $u_z^\omega\left(R,\varphi\right)\Big|_{z=0^\pm}$. Each of them relates to axis-symmetrical source ($f_\varphi^\omega=0$), located at origin $x,y,z=0$ & usual media: uniform UHS = NVF and LM, which consists of uniform SHS with localized Anomaly.

Conclusion

In this article we built some seismic solutions for the mentioned above Media and Sources. In order to validate such solutions, one can compare $\Delta \ell_{R,z}^{\omega}(R)\Big|_{z=0^{\pm}}^{\text{Layer}}$ (spectral numerical result for a thin Layer in uniform LM, which is much easier to achieve) vs. artificial Layer, combined from many small parallelepipeds (Anomalies) at the same mid-depth as z_{Layer} with thickness $\Delta z_{\text{Anom.}} = \Delta z_{\text{Layer}}$ (ignoring the small effects from the absence of numerical infinities: $(x,y)_{\max} \neq \infty,\ \Delta z \neq 0$ & imperfectness of the models). If Anomaly is deep, its recognition usually is concealed by multi-layer structure, requiring impractically high measurement accuracy, but the ω-effects of shallow Anomaly looks radically different from a potential shallow layer.

Here we emphasize that the obtained solutions represent just Direct Tasks (seismic responses from given Media and Sources), while the practical applications usually require Inverse Task: identification of Anomaly's factors: content (Air or Water or…), its form (karst or cave or tunnel or...) and its center $(R_{\text{A}}, \varphi_{\text{A}}, z_{\text{A}})$, its volume $\Omega_{\text{A}} = \iiint d\Omega_{\text{A}}$ (if such center or Ω_{A} do exist). Such Tasks require numerical integration by $d\Omega_{\text{A}}$, making pattern for each content & form, which can be coded with Splines' factors (convergence of the obtained solutions was showed above, see texts after (41.1) & (50.2)). Here is a similar example: given content $(\Delta\rho, \Delta\lambda, \Delta\mu)_{\text{A}}$ inside tunnel, so within proper pattern (Fig.3) we need to identify only 3 parameters: $(R_{\perp}, \varphi, z_{\text{A}}) = ?$ Moreover, if φ is given (we should use at least two geophones and the axis of such tunnel is supposed to be perpendicular to x, which is the axis of symmetry between the two), so we have just 2 factors $(R_{\perp}, z_{\text{A}})$ to identify.

Based on common sense & EM-computer-modeling, we expect $\mathbb{u}_{R,z}^{\omega}(R,\varphi)_{z=0^{\pm}}$ (11-14) to have a bell-like spectral shape around *big* central frequency ω_{C} (Anomaly is assumed to be shallow). This gives an opportunity of using many ω-equations for each pattern, enough to identify the mentioned above parameters in the Inverse Task (in case of $(R_{\perp}, z_{\text{A}}) = ?$ we need just two points of such curve). Required for such scenario the Uniform solid Half-Space content (ρ, λ, μ) and Source Signature (SS) can be calculated in advance, using neighboring measurements (free from Anomalies). These tasks deserve a separate article.

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