The Continuum Theorem II

David Selke

August 2024

Abstract

An injection from the countable ordinals to the paths in the binary tree leads to a bijection between all paths in the binary tree and only those in the $\aleph_1$-sized injection. Since there are $2^{\aleph_0}$ many paths in the binary tree, this proves the Continuum Hypothesis, $2^{\aleph_0} = \aleph_1$.

1 Introduction

The Continuum Hypothesis (CH), posed by Georg Cantor, remains the most prominent problem in set theory. It asks whether the next infinite size after that of the integers is identical with the size of the points on an interval. David Hilbert listed it as the first of 23 mathematical problems for the 20th century in 1900. Since the work of Gödel and Cohen, it is known that the Continuum Hypothesis can not be settled using Zermelo-Fraenkel set theory with the Axiom of Choice (ZFC), the most common set-theoretic foundation of mathematics. It is consistent with ZFC whether true or false. However, this work may lead to a natural strengthening of ZFC into something that would have been accepted if this proof had been known soon after 1900.

2 Proof

We will compare a "canonical" tree, the binary tree with all its nodes and paths considered equally, with "arbitrary" trees, binary trees in which there is an injection from the countable ordinals to the paths, which selects some paths and nodes of interest.

An injection from ω_1 to the paths in the binary tree exists because $\aleph_1$ is less than or equal to the continuum.

We will call any paths in an arbitrary tree, regardless of whether they map to any ordinal in the injection, "tree paths." Paths that have an ordinal mapped to them by the injection we call "ordinal paths." By $\aleph_n$-node we mean a node with $\aleph_n$ ordinal paths passing through it. An $\aleph_n$-node-path is a path whose every node is an $\aleph_n$-node.

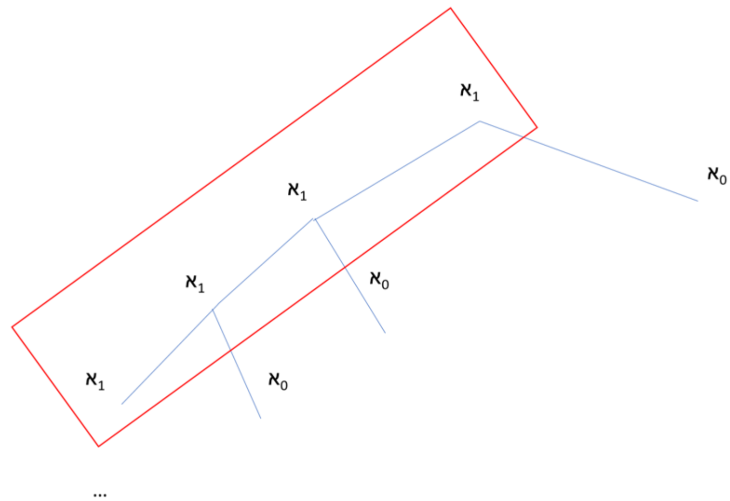

Figure 1: This (sub) tree is impossible because removing $\aleph_0$ paths, $\aleph_0$ times, does not produce distinct paths as the injection requires.

Each $\aleph_1$-node has two $\aleph_1$-node children or has a descendant that has two $\aleph_1$-node children. For suppose not. Then some $\aleph_1$-node has only one $\aleph_1$-node descendant at every lower level of the tree. In that subtree, there are $\aleph_1$ many ordinal paths corresponding to only one tree path, as shown in Figure 1.

The lack of distinctness in the mapping of ordinals to tree paths contradicts the definition of an injection.

Label each pair of $\aleph_1$-nodes that share a parent with finite binary strings of increasing length as follows: the first such left child is labelled "0" and the first such right child is labelled "1." The first such left child that descends from the "0" node is labelled "00." The corresponding right child is labelled "01." The similar descendants of "1" are labelled "10" and "11." Some of these labels are shown in Figure 2. The infinite binary strings built up from these finite binary strings label the paths.

In the canonical tree, every node is called an $\aleph_n$ node to denote that con-

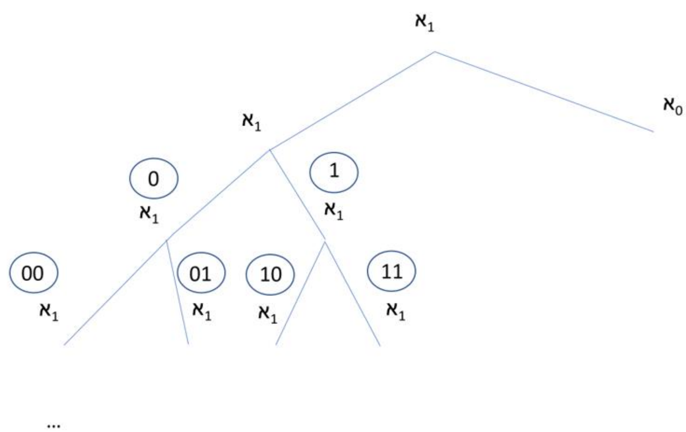

Figure 2: A tree like this remains to satisfy the requirement of an injection from ω_1 to the tree paths. $\aleph_0$-nodes may occur in many, but not "too many" places, as shown in Figure 1.

tinuum paths pass through it. Also every node below the root contains a label and every row contains all the labels having the same length, for instance, row three contains all eight length-three labels. Figure 3 shows the canonical tree and Figure 4 shows an arbitrary tree in which some of the length-two labels occur on level three of the tree. However, note that for any n, all of the length-n labels appear before some finite level of the tree. This is a consequence of the condition that each $\aleph_1$-node has two $\aleph_1$-node children or has a descendant that has two $\aleph_1$-node children. Those children are nodes that have labels, a pair each of their descendants have labels (of one greater length), and so on. Figure 5 and Figure 6 show the 1-to-1 correspondence between paths in an arbitrary tree and paths in the canonical tree. They are related by having the same label, an infinite string of ones and zeros which is a limit of the labels of the nodes each path contains.

Since the labelled paths (ordinal paths) in the arbitrary trees are $\aleph_1$-many (they descend from an $\aleph_1$-node at the root and are identified by an injection from the countable ordinals), this correspondence is between $\aleph_1$ and continuum.

3 Conclusion

The Continuum Hypothesis is demonstrated with remarkably few assumptions. The existence of the binary tree and the labelling scheme for paths and nodes seem uncontroversial. However the argument consists of little else. Axioms that go beyond ZFC may be regarded as "strong," and despite appearances this argument may harbor an interesting root, as happened with the Well-Ordering Theorem and the Axiom of Choice which was refined out of it.

It seems clear that a proof along these lines would have been accepted by Hilbert and the broader community if it had appeared in 1900, before the development of ZFC. What is more interesting is to consider whether what developed in place of ZFC would have needed to formalize this proof in such a scenario, as the Continuum Hypothesis was already a cornerstone of set theory and a formalization that excluded it, if CH had been as widely believed as Cantor's Theorem, would have been just as unacceptable as a formalization that could not prove Cantor's Theorem.

Smullyan and Fitting [1], after discussing formalism, have this to say about Platonism:

> "The so-called *mathematical realist* or *Platonist* (and this seems to include a large number of working mathematicians) looks upon the matter very differently. We can describe the realist viewpoint as follows. There is a well defined mathematical reality of sets, and in this reality, the continuum hypothesis is definitely true or false. The axioms of *ZF* give a true but incomplete description of this reality. The independence results cast no light on the truth or falsity of the

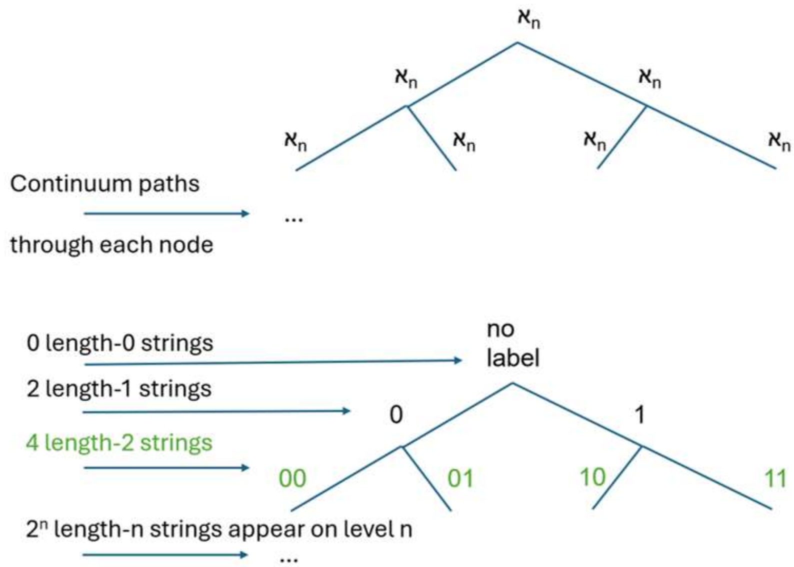

Figure 3: The canonical tree. Continuum paths pass through every node.

> continuum hypothesis, nor do they in any way indicate that it is neither true nor false. Rather they highlight the inadequacy of our present day axiom system *ZF*. But it is perfectly possible that new principles of set theory may be found which, though not derivable from the present axioms, are nevertheless self-evident (as the axiom of choice is to most mathematicians) and which might settle the continuum hypothesis one way or the other."

If there is a Platonist "project" to find such a way to settle CH, this proof is offered in support of it.

References

[1] Smullyan, Raymond and Fitting, Melvin, *Set Theory and the Continuum Problem* (Dover Publications, Inc., Mineola, New York, 2010). pp. 10

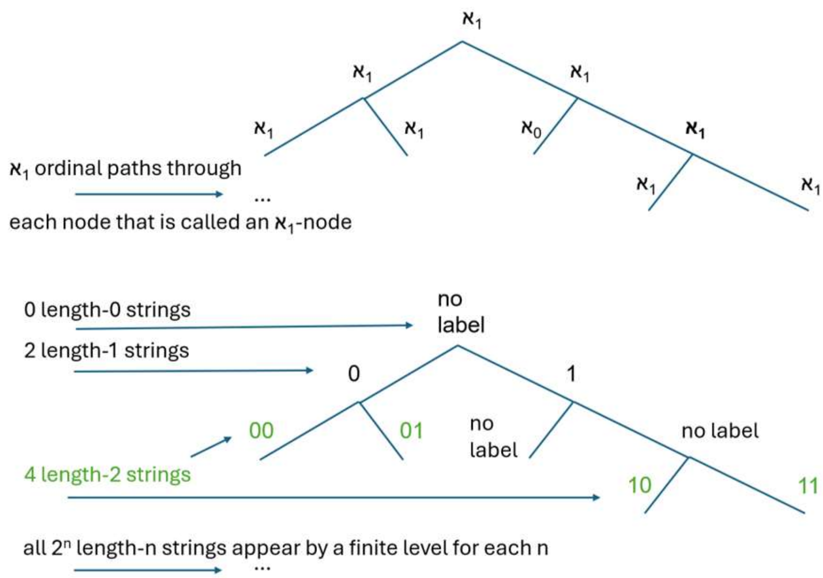

Figure 4: An arbitrary tree. $\aleph_1$ ordinal paths pass through the root and the $\aleph_1$-nodes. All of the strings of a given length which label a set of nodes appear before some finite level of the tree.

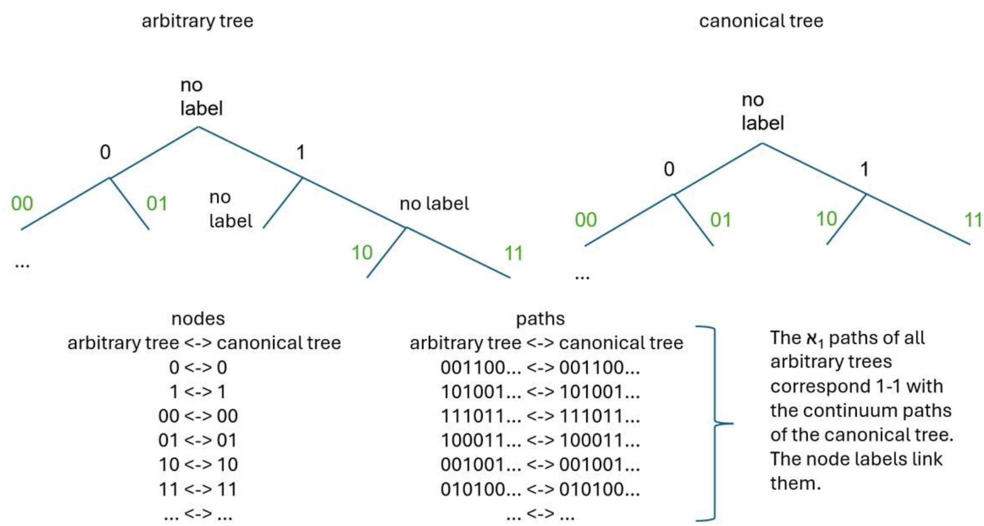

Figure 5: The one-to-one correspondence between the nodes in any arbitrary tree and the nodes with the same label in the canonical tree causes a one-to-one correspondence between the paths based on the identity of the path labels.

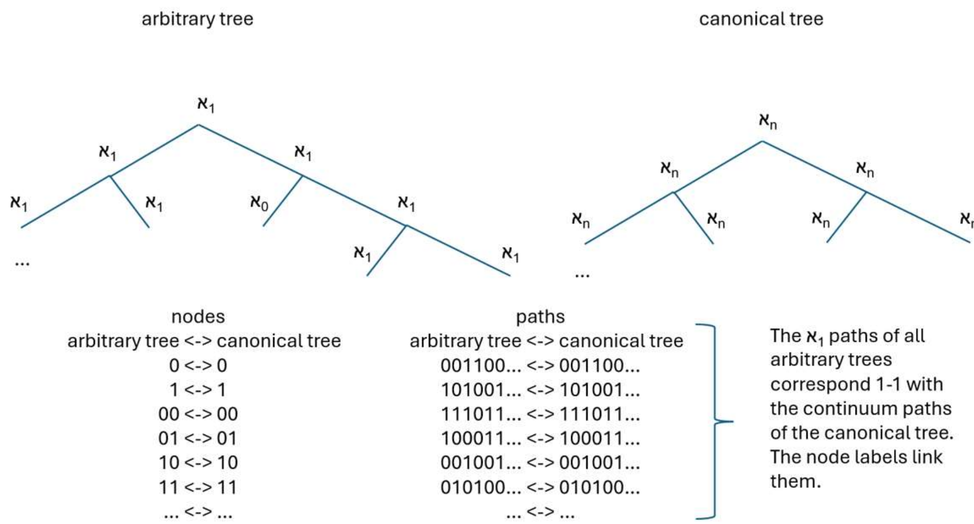

Figure 6: The path correspondence is despite the fact that in an arbitrary tree, only ordinal paths (forming part of the injection from countable ordinals to paths) have labels. The path correspondence outlined is thus a one-to-one correspondence between $\aleph_1$ and continuum.