The Sine-Gordon breather in an infinite potential well

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abstract:

In this paper we want to solve the motion of a breather soliton of the Sine-Gordon equation in an infinite potential well. This problem can be solved analytically for a well whose width L is far greater than the size of the soliton d, using the two breather solution of the Sine-Gordon equation. We show that this solution exhibits discrete energy levels with a quantisation condition equivalent to that obtained from quantum mechanics. They do arise in a similar way as standing waves give rise to discrete modes, with a wave and a reflected wave superimposed. The energy levels are given by the same formula as obtained from the Klein-Gordon equation of relativistic quantum mechanics for the same problem, but with a quantum constant h derived from the theory itself.

I. Introduction:

For a breather soliton model of elementary particles [1] it is necessary to show how the discrete nature of bound quantum states emerges from such a theory. Even though no complete soliton theory of elementary particles exists yet, it is nevertheless desirable, to show in toymodels, that the basic phenomena of relativity and quantum mechanics come out of such theories. Of particular interest are models based on non-linear, relativistic field equations, which posses breather soliton solutions. As was shown in [1] those solutions naturally give rise to all relativistic phenomena, like length contraction, time dilation and the relativistic energy-momentum relation, as well as basic quantum phenomena, like the uncertainty relations, the wave-particle duality and the De Broglie relations, including an expression for the quantum constant h. Another important property which has to be shown, is that such a model gives rise to discrete energy levels of bound particle states. That this can indeed be the case, and how it works in principle, we want to demonstrate in this paper with the Sine-Gordon model, a 1 dimensional toymodel [2,3]. It is based on the Sine-Gordon equation

$$\Box\varphi=-\frac{1}{d^2}\sin(\varphi) \tag{1.1}$$

which is of the type mentioned above. Here, d is a length parameter and $\Box$ the d'Alembert operator. This equation has N-Soliton and N-breather solutions [4]. Each of the breathers then represents a 1 dimensional particle in the framework of this model. For this we want to solve the problem of a particle (a breather) moving in an infinite potential well and show, that it indeed exhibits discrete energy levels. We will show, that the obtained solution leads to the same quantisation condition

$$k\,L=n\,\pi \tag{1.2}$$

as obtained from quantum mechanics and that the obtained energy levels agree with those obtained from the Klein-Gordon equation of relativistic quantum theory for the same problem. The quantum constant h is derived from the model itself, as was already elaborated in more detail in [2,1]. The way the discreteness of the energy levels arises is the same as for modes of standing waves. Due to the reflection of the breather at the walls, the full solution is a superposition of two breathers, one travelling left and one travelling right. The boundary conditions at the walls of the well will then lead to the quantisation condition (1.2).

II. The breather in an infinite potential well:

We now want to solve the motion of a breather in an infinite potential well. The solution has to satisfy the Sine-Gordon equation (1.1) in the region $x \in [0, L]$. Outside of this region, it has to be zero due to the infinite potential. Thus, the solution has to fulfill the boundary conditions

$$\varphi(0,t)=\varphi(L,t)=0 \tag{2.1}$$

The solution is a breather moving with a velocity v, which is reflected at the walls, resulting in another, identical breather, moving into the opposite direction with a velocity v and phase shifted by π. If the condition L>>d is fulfilled, this can be described by the two breather solution of the Sine-Gordon equation inside the walls. This we want to show in the following. We will first outline the idea for the solution qualitatively, and then construct the solution quantitatively from the two breather solution derived in the Appendix.

If we, for example, let the left breather of the solution φ (moving right with v) start at x=L/2 at t=0, then, if we put the right breather (moving left with -v) at x=3L/2 at t=0, it will describe the reflected breather after the collision of the left one with the wall at x=L. We will see, that then the two breather solution fulfills the boundary condition (2.1) at x=L for all times t if and only if

$$k L = n \pi$$

holds. Now, if the walls are far enough away, and L>>d holds, the boundary condition at the opposite wall is fulfilled to a very good approximation while the two breathers are in the region $x \in [L/2, 3L/2]$, since the solution decays exponentially away from the “position” of the breathers. At some time t=T/2 the reflected breather will reach x=L/2. At this point, we will describe the solution with a new two breather solution φ' with the right breather starting at x=L/2 and the left at x=-L/2. This solution will then fulfill the boundary condition at x=0 by construction. Also, due to the same argument as before, it will approximately fulfill them at x=L while both breathers are in the region $x \in [-L/2, L/2]$. Now, in the solution φ', the breather travelling right describes the reflected breather from the collision with the wall at x=0. After a time t=T, this reflected breather is again at x=L/2, and thus we have a periodic solution. All further periods can therefore be described with a function

$$\varphi(x,t)=\varphi_0(x, t-T n, n\eta) \tag{2.2}$$

$$n=\left\lfloor \frac{t}{T} \right\rfloor$$

η here is a phase shift in the periodic function of the two breather solution (A.19), which will accumulate during the reflections.

Now, we will construct the function φ_0. We already established, that it will consist of two different two breather solutions according to

$$\varphi_0(x,\tau)=\left\{\begin{matrix} \varphi_0(x,\tau) & \tau\in[0,\frac{T}{2}] \\ \varphi_0'(x,\tau) & \tau\in[\frac{T}{2},T] \end{matrix}\right\} \tag{2.3}$$

The functions φ and φ' are two breather solutions derived in the appendix, each with two identical breathers, “+” moving right with a velocity v, “-” moving left with a velocity -v and phase shifted by π. In the following we will just have to determine the constants $b^{(0)}$ and $a_\pm$ in (A.19 & A.20) of those solutions, as well as the shift η from the requirement of continuity of the solutions at t=T/2 and t=T. Demanding that the left breather of φ starts at x=L/2 at t=0 we find with the help of the asymptotic formula (A.13)[1])

$$a_+(\frac{L}{2},0)+\epsilon=0$$

and thus

$$a_+^{(0)}=\frac{-L}{2}-\frac{d\epsilon}{\gamma\cos(q)} \tag{2.4}$$

Consequently, we get for the right one:

$$a_-^{(0)}=\frac{-3L}{2}+\frac{d\epsilon}{\gamma\cos(q)} \tag{2.5}$$

Without loss of generality, we can set $b^{(0)}=0$. Now, to fulfill the boundary condition (2.1) at x=L, since arctan (0)=0, we have to have

$$f_i=0 \tag{2.6}$$

with f_i given by equation (A.17). Since, with (2.4 & 2.5) and (A.19), we have $a_+(L)=-a_-(L)$, and $\cosh(x)=\cosh(-x)$ as well as $\sinh(x)=-\sinh(-x)$, condition (2.6) is equivalent to

$$\sin(b_+)-\sin(b_-)=\cos(b_+)-\cos(b_-)=0 \tag{2.7}$$

With $b_\pm$ given by equation (A.20), it can easily be seen that (2.7) is fulfilled if and only if

$$kL=n\pi \tag{2.8}$$

holds. This is the claimed quantisation condition. At the same time, the boundary condition at x=0 will be fulfilled to a very good approximation, since during the time interval $t\in[0,T/2]$, both breathers are within the region $x\in[L/2,3L/2]$. Indeed, as can be seen from the asymptotics (A.13-16), the solutions decay as

$$\varphi\propto\exp(-r/d)$$

far away from the current position of each of the two breathers, with r being the distance to the breathers current position. Therefore, as long as the condition L>>d holds, the boundary condition at x=0 will be fulfilled up to a quantity of order $\exp(-L/d)$. Now, by definition, at t=T/2, the reflected breather is at the position x=L/2. From this demand, using the asymptotic (A.16), we get for the period T

$$T=\frac{2L}{v}-\frac{4d\epsilon}{v\gamma\cos(q)} \tag{2.9}$$

Demanding continuity between the two solutions φ,φ' at t=T/2, we have to have $\varphi_-^{\text{after}}=(\varphi_-')^{\text{before}}$; with the asymptotics (A.14 & A.16) this leads to

$$a_-^{'(0)}=\frac{-3L}{2}+\frac{3d\epsilon}{\gamma\cos(q)} \tag{2.10}$$

$$b^{'(0)}=-2\delta \tag{2.11}$$

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1) The “position” of a breather is defined as the point where the envelope function has its maximum value. Since, in this case, this function is $Sech(x)$, it is the point where x=0.

The left breather this time has to start on the opposite of x=0, and thus we get

$$a_{+}^{'(0)}=\frac{3L}{2}-\frac{3d\epsilon}{\gamma\cos(q)} \tag{2.12}$$

One can easily check, that the condition $\varphi^{'}(0,t)=0$ is satisfied by construction. Further, it can be seen that the reflected breather $(\varphi_{+}^{'})^{after}$ is at x=L/2 again at t=T, using the asymptotic (A.15). Thus, we have constructed a periodic solution, and can use (2.2) to describe all further periods. However, the periodic function in the solution (A.20) acquires a phase shift with each period, which is found to be

$$\eta=-4(\delta+\frac{\tan(q)}{\beta})+\frac{2\gamma L\sin(q)}{\beta d} \tag{2.13}$$

using the same asymptotic (A.16), this time with the phase of the periodic function evaluated at t=T. Thus, for the n th period, we have

$$b_n^{(0)}=n\eta \tag{2.14}$$

Now, we can write the complete solution as

$$\varphi(x,t)=\begin{cases}0 & x\notin[0,L]\\ \varphi_n(x,t-Tn) & x\in[0,L]\end{cases} \tag{2.15}$$

$$\varphi_n(x,\tau)=\begin{cases}\varphi_n(x,\tau) & \tau\in[0,\frac{T}{2}]\\ \varphi_n^{'}(x,\tau) & \tau\in[\frac{T}{2},T]\end{cases}$$

with $\varphi_n,\varphi_n^{'}$ the two breather solutions with constants $a_{\pm}^{(0)}$ and $a_{\pm}^{'(0)}$ given by (2.4, 2.5, 2.10, 2.12) for each n, and b given by

$$b_n^{(0)}=n\eta$$

$$b_n^{'(0)}=n\eta-2\delta$$

The obtained solution is plotted below

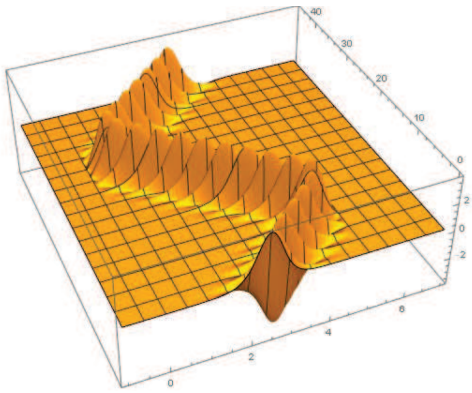

Figure 1. The solution (2.15) for the values d=0.2, L=5, β=0.2, $q=\arcsin(\pi/5\sqrt{1-0.2^2})\approx 0.66$. The x-axis is plotted in the range $x\in[-1,7]$, the t-axis in $t\in[0,45]$. The period is by (2.9) $T\approx 40.87$

The value of q was chosen so that the quantisation condition (2.8) is fulfilled. Indeed, by plugging (2.19) into it and solving for q one easily verifies the condition for q as

$$\sin(q)=\frac{n\pi d}{L\beta\gamma}$$

From the quantisation condition (2.8), we can also find the energy levels of the solution. In [5] it was shown, that a Sine-Gordon breather exhibits De Broglie relations

$$E=\hbar\omega \tag{2.16}$$

$$p=\hbar k \tag{2.17}$$

where ω and k and angular frequency and wavenumber of the periodic function in the one breather solution

$$\omega=\frac{\sin(q)c}{d}\gamma \tag{2.18}$$

$$k=\frac{\sin(q)}{dc}v\gamma \tag{2.19}$$

$$\hbar=\frac{E_0 d}{\sin(q)c} \tag{2.20}$$

E and p are the total energy and momentum of the breather, E_0 its rest energy, $m_0=E_0/c^2$. Further, it was shown (and can easily be verified) that the relativistic energy momentum relation

$$E=\sqrt{m_0^2c^4+c^2p^2} \tag{2.21}$$

holds. If we plug in the quantisation condition into (2.17) and this into (2.21), we obtain for the energy levels

$$E_n=\sqrt{m_0^2c^4+c^2\hbar^2\left(\frac{n\pi}{L}\right)^2} \tag{2.22}$$

which agrees with the expression obtained for the same potential with the Klein-Gordon equation of relativistic quantum mechanics.

III. Conclusions:

We have solved the motion of a breather soliton of the Sine-Gordon equation in an infinite potential well for a well of width L>>d. We have shown, that this problem can be solved analytically using the two breather solution of the Sine-Gordon equation. We proved that this solution exhibits discrete energy levels with a quantisation condition equivalent to that obtained from quantum mechanics. The energy levels are given by the same formula as obtained from the Klein-Gordon equation for the same problem, but with a quantum constant h derived from the theory itself. Thus, we have demonstrated exemplarily that in a soliton model of elementary particles, bound particle states indeed can exhibit discrete energy levels. They do arise similarly as standing waves give rise to discrete modes, with a wave and a reflected wave superimposed. To show that this is the case for arbitrary potentials, especially in the full theory, remains a task to be done.

Appendix A (The one and two breather solutions):

The N-Soliton solution to the Sine Gordon equation

$$\Box\varphi=-\frac{1}{d^2}\sin(\varphi)$$

is given by [5]

$$\varphi=4\arctan(\frac{f_i}{f_r}) \tag{A.1}$$

$$f=f_r+if_i=W(\psi_1,\ldots,\psi_N)(X) \tag{A.2}$$

with

$$\zeta_k=\alpha_k X+\frac{1}{\alpha_k}T+\zeta_k^{(0)} \qquad \psi_k=\exp(\frac{\zeta_k}{2})+\exp(\frac{-\zeta_k}{2}+\delta_k)$$

$$X=\frac{x+ct}{2d} \qquad T=\frac{x-ct}{2d}$$

Here, W denotes the Wronskian with derivatives performed after X. f_r,f_i are the real and imaginary part of f; X, T are the so called light cone coordinates, α is a complex parameter and δ a complex phase. N is the number of solitons the solution has.
From this, the one breather solution is obtained by choosing N=2, since one breather is composed of two solitons, and requiring

$$\psi_1^*=\psi_2 \quad\Leftrightarrow\quad \alpha_1^*=\alpha_2=:\alpha$$

This yields for the solution

$$f_r=2c\exp(\delta_r)\cos(b-\delta_i) \tag{A.3}$$

$$f_i=-2d\exp(\delta_r)\cosh(a-\delta_r) \tag{A.4}$$

with

$$a=\zeta_r=\frac{\gamma\cos(q)}{d}(x-vt+a^{(0)}) \qquad b=\zeta_i=\frac{\gamma\sin(q)}{d}(ct-\beta x)+b^{(0)} \tag{A.5}$$

with $a^{(0)}=\zeta_r^{(0)}$ and $b^{(0)}=\zeta_i^{(0)}$. v is the velocity of the breather, $\beta=v/c$ and $\gamma=1/\sqrt{1-\beta^2}$. Further, we have

$$c=|\alpha|\cos(q) \qquad d=|\alpha|\sin(q)$$

$$|\alpha|^2=\frac{1-\beta}{1+\beta}$$

with q a real parameter. Thus, we can write the 1 breather solution with (A.1) as

$$\varphi(x,t)=4\arctan(ctg(q)\frac{\cos(\frac{\gamma\sin(q)}{d}(ct-\frac{v}{c}x)+b^{(0)})}{\cosh(\frac{\gamma\cos(q)}{d}(x-vt+a^{(0)}))}) \tag{A.6}$$

setting δ=0 since it can be absorbed in the phases of (A.5) and using the formula $\arctan(1/x)=\pi/2-\arctan(x)$.

To obtain a two breather solution we chose N=4, and demand:

$$\psi_1^* = \psi_2 \;\wedge\; \psi_3^* = \psi_4 \;\Leftrightarrow\; \alpha_1^* = \alpha_2 =: \alpha_+ \;\wedge\; \alpha_3^* = \alpha_4 =: \alpha_-$$

which is the condition for the pairs of solitons 1 & 2 and 3 & 4 each forming one breather. Again, we can set $\delta_i = 0$ since we saw it just yields a trivial phase which can be absorbed in the phases of the corresponding expressions of (A.9 & A.10) in the final solution. Performing the Wronskian with Mathematica, one obtains the two breather solution as:

$$f_i = \frac{1}{4}(-d_+ c_- A\cosh(a_+)\cos(b_-) - c_+ d_- D\cos(b_+)\cosh(a_-)) \\ + c_+ c_- d_+ d_- F(\sin(b_-)\sinh(a_+) - \sin(b_+)\sinh(a_-)) \tag{A.7}$$

$$f_r = \frac{1}{4}(-d_+ d_- B\cosh(a_+)\cosh(a_-) + c_+ c_- C\cos(b_+)\cos(b_-)) \\ + c_+ c_- d_+ d_- E(\sin(b_-)\sin(b_+) + \sinh(a_+)\sinh(a_-)) \tag{A.8}$$

with

$$a_\pm = (\zeta_\pm)_r = \frac{\gamma_\pm \cos(q_\pm)}{d}(x - v_\pm t + a_\pm^{(0)}) \tag{A.9}$$

$$b_\pm = (\zeta_\pm)_i = \frac{\gamma_\pm \sin(q_\pm)}{d}(ct - \beta_\pm x) + b_\pm^{(0)} \tag{A.10}$$

with $a_\pm^{(0)} = (\zeta_\pm^{(0)})_r$ and $b_\pm^{(0)} = (\zeta_\pm^{(0)})_i$. $v_\pm$ are the velocities of the breathers “+” and “-”, $\beta_\pm = v_\pm / c$ and $\gamma_\pm = 1/\sqrt{1-\beta_\pm^2}$. Further, we have

$$c_\pm = |\alpha_\pm| \cos(q_\pm) \qquad\qquad d_\pm = |\alpha_\pm| \sin(q_\pm)$$

$$|\alpha_\pm|^2 = \frac{1-\beta_\pm}{1+\beta_\pm}$$

and

$$A = (|\alpha_+|^2 - |\alpha_-|^2)^2 + 2|\alpha_+|^2|\alpha_-|^2(\cos(2q_-) - \cos(2q_+))$$

$$B = (|\alpha_+|^2 + |\alpha_-|^2)^2 + 2|\alpha_+|^2|\alpha_-|^2(\cos(2q_-) + \cos(2q_+))$$

$$C = (|\alpha_+|^2 + |\alpha_-|^2)^2 - 2|\alpha_+|^2|\alpha_-|^2(\cos(2q_-) + \cos(2q_+))$$

$$D = (|\alpha_+|^2 - |\alpha_-|^2)^2 + 2|\alpha_+|^2|\alpha_-|^2(\cos(2q_+) - \cos(2q_-))$$

$$E = |\alpha_+|^2 + |\alpha_-|^2 \qquad\qquad F = -|\alpha_+|^2 + |\alpha_-|^2$$

We can now obtain the two breather solution again with by plugging (A.7 & A.8) into (A.1). It is best expressed in the form

$$\varphi = 4\arctan\left(\frac{-d_+ c_- A z_- - c_+ d_- D z_+ + 4 c_+ c_- d_+ d_- F(y_- \tanh(a_+) - y_+ \tanh(a_-))}{-d_+ d_- B + c_+ c_- C z_+ z_- + 4 c_+ c_- d_+ d_- E(y_+ y_- + \tanh(a_+)\tanh(a_-))}\right) \tag{A.11}$$

with

$$z_\pm = \frac{\cos(b_\pm)}{\cosh(a_\pm)} \qquad\qquad y_\pm = \frac{\sin(b_\pm)}{\cosh(a_\pm)}$$

which is obtained by dividing denominator and nominator by $\cosh(a_+)\cosh(a_-)$.
This expression also allows one to easily find the asymptotics and the phase shifts of the two breathers. Without loss of generality, we‘ll assume that $v_+ > v_-$. That means, that before the collision, breather “+” is left of breather “-”, and vice versa after the collision. Now we can easily find the asymptotics by first looking at, for example, breather “+” before the collison. Here, breather “-” is far enough away, so that $z_- \approx y_- \approx 0$ holds in the vicinity of “+”. In addition, since breather “-” is right of breather “+”, we have $\tanh(a_-) \approx -1$. This leaves us with:

$$\varphi \approx 4\arctan\left(\frac{-c_+ d_- D z_+ + 4 c_+ c_- d_+ d_- F y_+}{-d_+ d_- B + 4 c_+ c_- d_+ d_- E(-\tanh(a_+))}\right) =: \varphi_+^{\text{before}}$$

or equivalently

$$\varphi_+^{\text{before}} = 4\arctan\left(\frac{c_- d_+ A\cos(b_+) - 4 c_+ c_- d_+ d_- F\sin(b_+)}{d_+ d_- B\cosh(a_+) + 4 c_+ c_- d_+ d_- E\sinh(a_+)}\right) \tag{A.12}$$

By setting:

$$\sin(\delta) = 4 c_+ c_- d_+ d_- F \qquad \cos(\delta) = c_- d_+ A$$

$$\sinh(\epsilon) = 4 c_+ c_- d_+ d_- E \qquad \cosh(\epsilon) = d_+ d_- B$$

we can use the addition theorems for trigonometric and hyperbolic functions to write (A.12) as

$$\varphi_+^{\text{before}} = 4\arctan\left(\frac{A_+ \cos(b_+ + \delta_+)}{B\cosh(a_+ + \epsilon)}\right) \tag{A.13}$$

with the phase shifts given by

$$\tan(\delta_+) = \frac{4 c_+ d_- F}{A} \qquad \tanh(\epsilon) = \frac{4 c_+ c_- E}{B}$$

$$A_+ = \sqrt{(d_+ c_- A)^2 + (4 c_+ c_- d_+ d_- F)^2} \qquad B = \sqrt{(d_- d_+ B)^2 - (4 c_+ c_- d_+ d_- E)^2}$$

One can see, that (A.13) has the form of the one breather solution of the Sine-Gordon equation, shifted by the phases (A.14). In the same way we can obtain the other three asymptotics as

$$\varphi_-^{\text{before}} = 4\arctan\left(\frac{A_- \cos(b_- + \delta_-)}{B\cosh(a_- - \epsilon)}\right) \tag{A.14}$$

$$\varphi_+^{\text{after}} = 4\arctan\left(\frac{A_+ \cos(b_+ - \delta_+)}{B\cosh(a_+ - \epsilon)}\right) \tag{A.15}$$

$$\varphi_-^{\text{after}} = 4\arctan\left(\frac{A_- \cos(b_- - \delta_-)}{\cosh(a_- + \epsilon)}\right) \tag{A.16}$$

with

$$\tan(\delta_-) = \frac{4 c_- d_+ F}{D}$$

$$A_- = \sqrt{(d_- c_+ D)^2 + (4 c_+ c_- d_+ d_- F)^2}$$

Now, we can apply to above formulas to two identical breathers moving with the same speed into opposite direction and phase shifted by π. In this case we have $q:=q_+=q_-$, $v:=v_+=-v_-$ and $b^{(0)}:=b_+^{(0)}=b_-^{(0)}-\pi$. This implies $|\alpha_-|=1/|\alpha_+|$. With this we obtain

$$A=D=16\beta^2\gamma^4$$

$$B=4(1+\beta^2)^2\gamma^4+4\cos(2q) \qquad C=4(1+\beta^2)^2\gamma^4-4\cos(2q)$$

$$E=2\gamma^2(1+\beta^2) \qquad F=4\beta\gamma^2$$

and thus for the solution

$$\begin{aligned} f_i=&4\sin(q)\cos(q)\beta^2\gamma^4(\cosh(a_+)\cos(b_-)-\cos(b_+)\cosh(a_-))\\ &-4\sin(q)^2\cos(q)^2\beta\gamma^2(\sin(b_-)\sinh(a_+)+\sin(b_+)\sinh(a_-)) \end{aligned} \tag{A.17}$$

$$\begin{aligned} f_r=&-\sin^2(q)(\gamma^4(1+\beta^2)^2+\cos(2q))\cosh(a_+)\cosh(a_-)\\ &-\cos^2(q)(\gamma^4(1+\beta^2)^2-\cos(2q))\cos(b_+)\cos(b_-)\\ &+2\gamma^2(1+\beta^2)\sin^2(q)\cos^2(q)(-\sin(b_-)\sin(b_+)+\sinh(a_+)\sinh(a_-)) \end{aligned} \tag{A.18}$$

with

$$a_\pm=\frac{\gamma\cos(q)}{d}(x\mp vt+a_\pm^{(0)}) \tag{A.19}$$

$$b_\pm=\frac{\gamma\cos(q)}{d}(ct\mp\beta x)+b^{(0)} \tag{A.20}$$

The phase shifts are given by

$$\tan(\delta_+)=\tan(\delta_-)=\frac{\sin(q)\cos(q)}{\beta\gamma^2} \tag{A.21}$$

$$\tanh(\epsilon)=\frac{2\gamma^2(1+\beta^2)\cos^2(q)}{\gamma^4(1+\beta^2)^2+\cos(2q)} \tag{A.22}$$

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