A set of formulas for prime numbers

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Abstract

Here I present several formulas and conjectures on prime numbers. I'm interested in studying prime numbers, Euler's totient function and sum of the divisors of natural numbers.

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Formula 1

Let $\sigma(n)$ denotes the divisor function which sums the divisors of n, an integer ≥ 1. We introduce the function f such that:

$$f(n) = 1 + (n!)^2 - \sigma(n!)(n!)^2 + 2 \sum_{k=1}^{-1+\sigma(n!)} \left\lfloor \frac{k(1+(n!)^2)}{\sigma(n!)} \right\rfloor$$

When $f(n) = 2n + 1$, is $2n + 1$ always prime?

And for example for $n = 6$ we have the prime number 13 that is of the form $2(6) + 1$.

The first examples are given by the following sequence:

Sequence

1, 1, 1, 1, 1, 13, 1, 17, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 61, 1, 1, 1, 1, 1, 1, 1, 1, 61, 1, 1, 1, 193, 1, 1, 1, 757, 61, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 109, 1, 1, 1, 181, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 113...

Carl Schildkraut proved this property [1].

Let $\{x\} = x - \lfloor x \rfloor$, let $m = n!$, and let $t = \sigma(m)$. Then

$$2\sum_{k=1}^{t-1}\left\lfloor \frac{k(1+m^2)}{t} \right\rfloor = \sum_{k=0}^{t-1} \frac{2k(1+m^2)}{t} - 2\sum_{k=0}^{t-1}\left\{\frac{k(1+m^2)}{t}\right\};$$

the first sum is $(1+m^2)(t-1)$, and so

$$\begin{aligned} f(n) &= 1 + m^2 - m^2 t + (1+m^2)(t-1) - 2\sum_{k=0}^{t-1}\left\{\frac{k(1+m^2)}{t}\right\} \\ &= t - 2\sum_{k=0}^{t-1}\left\{\frac{k(1+m^2)}{t}\right\}. \end{aligned}$$

Let $u = \gcd(1+m^2, t)$. The t values $\{0, 1+m^2, 2(1+m^2), \ldots, (t-1)(1+m^2)\}$ modulo t consist of u copies of each multiple of u in $[0, t)$, and so

$$\sum_{k=0}^{t-1}\left\{\frac{k(1+m^2)}{t}\right\} = u\sum_{j=0}^{\frac{t}{u}-1} \frac{uj}{t} = \frac{t-u}{2}.$$

This means

$$f(n) = t - 2\frac{t-u}{2} = \gcd(1+(n!)^2, \sigma(n!)).$$

(In particular, if $1+(n!)^2$ and $\sigma(n!)$ are coprime, $f(n) = 1$.)

With this knowledge about $f(n)$, we can tackle the problem at hand. If $f(n) = 2n+1$, then, in particular, $2n+1$ divides $1+(n!)^2$. So, $2n+1$ is relatively prime to $n!$. This means that $2n+1$ cannot have any factors in the set $\{2, \ldots, n\}$. However, every number in $\{n+1, \ldots, 2n\}$ is too large to be a factor of $2n+1$. So, $2n+1$ cannot have any factors strictly between 1 and $2n+1$, and must be prime.

Formula 2

Let x denotes an integer such that $x > 1$. We define the function f such that:

$$f(x) = \frac{1}{\pi}\arctan(x)$$

We have:

$$f(x) = \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + ...}}}}$$

$(a, b, c, d$ are integers $\geq 1)$ We have:

$$\lim_{x\to\infty} \frac{x}{b} = \frac{4}{\pi}$$

Formula 3

Let k be a positive integer.

Let n be an integer such that $n = 6k - 1$

Let r be the remainder of the division of $(n-1)! - n$ by $(n+2)$

Property: if $6k+1$ is prime $r = 3k + 2$

We define the prime $6k+1$ such that $6k+1 = r(n) + r(n-1)$ where $r(n)$ is the sequence of the successive remainders with $r(1) = 5$ and $n \geq 2$. We suppose $r(n) \neq 2$ and $r(n-1) \neq 2$.

For example the first 25 values of r are:

$5, 8, 11, 2, 17, 20, 23, 2, 2, 32, 35, 38, 41, 2, 2, 50, 53, 56, 2, 2, 65, 2, 71, 2, 77$

And we have:

$8 + 5 = 13 = 6(2) + 1$

$11 + 8 = 19 = 6(3) + 1$

$20 + 17 = 37 = 6(6) + 1$

$23 + 20 = 43 = 6(7) + 1$

$35 + 32 = 67 = 6(11) + 1$

$38 + 35 = 73 = 6(12) + 1$

$41 + 38 = 79 = 6(13) + 1$

$53 + 50 = 103 = 6(17) + 1$

$56 + 53 = 109 = 6(18) + 1$

Formula 4

Let a to be a natural number ($a \geq 1$), $n = 4 * m$ where m is a natural number ≥ 1) and ϕ is the Euler's totient function such as:

$$\phi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right)$$

Prove that if $\phi(a^n - 2) + 1 \equiv n - 1 \pmod{n}$ then $\phi(a^n - 2) + 1$ is always a prime number.

Max Alekseyev studied this conjecture but no proof has been found [2].

Formula 5

(a, b) is a couple of twin primes such that $b = a + 2$ and $a > 29$. Let $N = 4^b$ and q the quotient which results from the division of N by a and r is the remainder. We calculate $P = (q \bmod b)a + r - 1$ Below we prove that $P = 3(10b + 1)$ using Fermat's little theorem. $N = 16 \cdot 4^a \equiv 64 \pmod{a}$ then $r = 64$ $(a > 64)$ $4^b = (b-2)q + 64$ then $4 \equiv 64 - 2q \pmod{b}$ and $q \equiv 30 \pmod{b}$ Finally we have $P = 30a + 63 = 3(10b + 1)$

Formula 6

n is a natural number > 1, $\varphi(n)$ denotes the Euler's totient function, P_n is the n^{th} prime number and $\sigma(n)$ is the sum of the divisors of n. Consider the expression:

$$F(n) = \varphi(|P_{n+2} - \sigma(n)|) + 1$$

Conjecture: when $F(n) \equiv 3 \pmod{20}$ then this number is a prime or not. When the number is not a prime it can be a power of prime by calculating $|P_{n+2} - \sigma(n)| = p^k$ (p prime, k a natural number > 1).

Examples:

We have $n = 680$:

$$F(680) = \varphi(|P_{682} - \sigma(680)|) + 1 = \varphi(5101 - 1620) + 1 = 3423$$

which is not prime but we have $P_{n+2} - \sigma(n) = p^2$, more precisely it is the square of 59.

Interestingly for $n \leq 526388126$ (calculations with PARI/GP) all counterexamples are the power of prime.

Another example is found for $k = 6$, this is $n = 526388126$. In this case, we have:

$$F(n) = 10549870323$$

which is not prime and $|P_{n+2} - \sigma(n)| = 47^6$ (here $k = 6$).

The question is: "Are there only these two solutions? 1. A power of prime if the result is not a prime 2. Or the result is prime

Formula 7

Definitions:

Here I present a novel conjecture using basic mathematical tools like the sum of the divisors of an integer n called $\sigma(n)$, the sum of the squares of the positive divisors of n called $\sigma_2(n)$. I also use the prime-counting function which is the function counting the number of prime numbers less than or equal to some real number n. The prime-counting function is called $\pi(n)$.

Conjecture:

We introduce the following expression called A:

$$A = \sigma_2(\pi(n) - \sigma((n + 2))))$$

We focus on numbers ends with 2. I calculate $A - 1$ and so the new number ends with 1. Then I calculate the square root of this number ends with 1. When the number is an integer, it is always prime.

Example:

Let $n = 100547$, we have $A = \sigma_2(9639 - \sigma(100549)) = 8264809922$ We have $A - 1 = 8264809921$ We calculate the square root of 8264809921 and we have $A - 1 = \sqrt{8264809921} = 90911$ and 90911 is prime.

References

[1] Carl Schildkraut (https://math.stackexchange.com/users/253966/carl-schildkraut), Primes of the form $2n+1$, URL (version: 2022-06-26): https://math.stackexchange.com/q/4480961

[2] Max Alekseyev (https://math.stackexchange.com/users/147470/max-alekseyev), Euler's totient function and primes, URL (version: 2022-06-23): https://math.stackexchange.com/q/4478910