On-Shell Electromagnetism As A Step Towards QM *Classification*

Gilad Laredo*

(Dated: 24/09/2024)

This study extends the classical theory of electromagnetism to include quantum phenomena, thus creating a bridge between classical EM theory and quantum mechanics. This bridge is facilitated through a mathematical structure that covers both Maxwell equations, Dirac equation and the Proca equation. One outcome of this study is a new Lagrangian that maintains the same fermionic field dynamics generated by the QED Lagrangian while being more compact and symmetric. Additionally, this work introduces a 'fermionic' stress-energy tensor that can be integrated into Einstein's field equations as source of spacetime curvature, demonstrating compatibility with general relativity.

Introduction

Quantum mechanics departs classical mechanics by its mathematical formulation. The wave function formulation speared first in the creation of wave mechanics by Erwin Schrodinger seminal work in 1926 and used in various flavors in different quantum fields. In 1928, Paul Dirac introduced his renowned equation [1] as the special relativity generalization of the Schrödinger equation. Dirac formulated his equation by hypothesizing a matrix-based solution for the mass shell condition, represented by quantum operators and used the non-commutative property of square matrices.

This study addresses the same problem using a similar 'guessing' approach but instead of using matrix non-commutative properties and wave functions (bi-spinors), the formalism adopted in this work is grounded on a coordinate-independent symmetry, identified in Maxwell's equations. This approach manifest quantum mechanics in the language of classical electrodynamics, replacing the wave function with 'classical' fields. Another important aspect is that to maintain local U(1) symmetry, the Dirac Lagrangian requires an addition of a gauge field which happens to be the electromagnetic field while In this work, a local U(1) compatible Lagrangian is derived in a single blow with no need in an additional gauge field.

I. ON-SHELL ELECTROMAGNETISM

Consider the following operator matrix eigenvalue equation:

$$\begin{pmatrix} \frac{1}{c}\partial_t & 0 & 0 & j\nabla\cdot() \\ 0 & \frac{1}{c}\partial_t & j\nabla() & j\nabla\times() \\ 0 & j\nabla\cdot() & -\frac{1}{c}\partial_t & 0 \\ j\nabla() & -j\nabla\times() & 0 & -\frac{1}{c}\partial_t \end{pmatrix} \begin{pmatrix} S_0^+ \\ \boldsymbol{S}^+ \\ S_0^- \\ \boldsymbol{S}^- \end{pmatrix} = j\frac{mc}{\hbar} \begin{pmatrix} S_0^+ \\ \boldsymbol{S}^+ \\ S_0^- \\ \boldsymbol{S}^- \end{pmatrix} \tag{1}$$

The operator matrix is populated with first derivative, coordinate-independent differential operators. The state vector is composed of $S_0^+$and S_0^- which are scalar fields and $\boldsymbol{S}^+$and $\boldsymbol{S}^-$ which are complex vector fields, $'j'$ - the imaginary unit, $'m'$ - the mass of the *'particle'* field, $'c'$ - the speed of light and $\hbar$ - the reduced Plank constant.

Applying the same operator matrix to the left-hand side of eq.1 and correspondingly multiplying the right-hand side by $j\frac{mc}{\hbar}$ yields a set of scalar and vector Klein-Gordon equations:

$$\left(\frac{1}{c^2}\partial_{tt} - \nabla^2\right) \begin{pmatrix} S_0^+ \\ \boldsymbol{S}^+ \\ S_0^- \\ \boldsymbol{S}^- \end{pmatrix} = -(\frac{mc}{\hbar})^2 \begin{pmatrix} S_0^+ \\ \boldsymbol{S}^+ \\ S_0^- \\ \boldsymbol{S}^- \end{pmatrix} \tag{2}$$

By selecting solutions of the form $S \propto e^{-j(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})}$ for all state vector components and identifying

$$\omega = \frac{E}{\hbar} \ , \ \boldsymbol{k} = \frac{\boldsymbol{p}}{\hbar}$$

Hence, the mass-energy shell condition $E^2 = |\boldsymbol{p}|^2 + m^2c^4$ is simultaneously satisfied for all rows of eq.2, similarly to the Dirac equation case.

Let's write eq.1 in its non-matrix form:

$$\begin{aligned} j\nabla\cdot\boldsymbol{S}^- &= \left(j\frac{mc}{\hbar} - \frac{1}{c}\partial_t\right) S_0^+ \\ j\nabla\times\boldsymbol{S}^- &= -j\nabla S_0^- + \left(j\frac{mc}{\hbar} - \frac{1}{c}\partial_t\right)\boldsymbol{S}^+ \\ j\nabla\cdot\boldsymbol{S}^+ &= \left(j\frac{mc}{\hbar} + \frac{1}{c}\partial_t\right) S_0^- \\ j\nabla\times\boldsymbol{S}^+ &= j\nabla S_0^+ - \left(j\frac{mc}{\hbar} + \frac{1}{c}\partial_t\right)\boldsymbol{S}^- \end{aligned} \tag{3}$$

Considering $\boldsymbol{S}^\pm = c\boldsymbol{B}^\pm - j\boldsymbol{E}^\pm$ where $\boldsymbol{E}^\pm$ and $\boldsymbol{B}^\pm$ can be identified as an *electric-like* and a *magnetic-like* fields correspondingly, that have real amplitude coefficients. eq.3 can be expanded by real/imaginary separation to:

* GiladLaredo@gmail.com

$$\begin{aligned}
&\nabla \cdot \boldsymbol{B}^{+} = 0 \\
&\nabla \cdot \boldsymbol{E}^{-} = \left(j\frac{mc}{\hbar} - \frac{1}{c}\partial_t\right) S_0^{+} \\
&c\nabla \times \boldsymbol{B}^{+} = \nabla S_0^{+} + \left(j\frac{mc}{\hbar} + \frac{1}{c}\partial_t\right) \boldsymbol{E}^{-} \\
&\nabla \times \boldsymbol{E}^{-} = \left(j\frac{mc}{\hbar} - \frac{1}{c}\partial_t\right) c\boldsymbol{B}^{+} \\
&\overline{\overline{\nabla \cdot \boldsymbol{B}^{-} = 0}} \\
&\nabla \cdot \boldsymbol{E}^{+} = \left(j\frac{mc}{\hbar} + \frac{1}{c}\partial_t\right) S_0^{-} \\
&c\nabla \times \boldsymbol{B}^{-} = -\nabla S_0^{-} - \left(j\frac{mc}{\hbar} - \frac{1}{c}\partial_t\right) \boldsymbol{E}^{+} \\
&\nabla \times \boldsymbol{E}^{+} = -\left(j\frac{mc}{\hbar} + \frac{1}{c}\partial_t\right) c\boldsymbol{B}^{-}
\end{aligned} \tag{4}$$

The reason the term $j\frac{mc}{\hbar}$ retains its 'j' factor is to align with the derivative of the complex exponent $e^{-j(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})}$. The similarity of the top and bottom of eq.4 to Maxwell equations is evident. Furthermore, if $\boldsymbol{E}^{\pm}$ are consider to be electric fields, then the units of the scalar fields $S_0^{\pm}$ are identified to be similar to electrical field by units comparison. It can be demonstrated that all vector and scalar fields in eq.4 uphold Klein-Gordon equation structure.

II. POTENTIALS AND GAUGE CONDITIONS

By following the same procedure used to derive the electric and magnetic potentials from Maxwell's equations, one can obtain the corresponding potentials:

$$\boldsymbol{B}^{-} = \nabla \times \boldsymbol{A}^{-} \tag{5}$$

$$\boldsymbol{B}^{+} = \nabla \times \boldsymbol{A}^{+} \tag{6}$$

$$\boldsymbol{E}^{-} = -\nabla\phi^{+} + \left(j\frac{mc}{\hbar} - \frac{1}{c}\partial_t\right) c\boldsymbol{A}^{+} \tag{7}$$

$$\boldsymbol{E}^{+} = -\nabla\phi^{-} - \left(j\frac{mc}{\hbar} + \frac{1}{c}\partial_t\right) c\boldsymbol{A}^{-} \tag{8}$$

Where the sign indices over the the scalar potential ϕ where arbitrarily chosen to align with the signs indices of the vector potentials. This will prove useful in the following sections.

To derive the gauge conditions one can start by substitute eq.6 and eq.7 in the third row of eq.4 :

$$\begin{aligned}
&c\nabla \times \nabla \times \boldsymbol{A}^{+} = \\
&= \nabla S_0^{+} + \left(j\frac{mc}{\hbar} + \frac{1}{c}\partial_t\right)\left[-\nabla\phi^{+} + \left(j\frac{mc}{\hbar} - \frac{1}{c}\partial_t\right) c\boldsymbol{A}^{+}\right]
\end{aligned}$$

Using the identity $\nabla \times \nabla \times \boldsymbol{A} = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A}$:

$$\begin{aligned}
&c\nabla\left(\nabla \cdot \boldsymbol{A}^{+}\right) - c\nabla^2\boldsymbol{A}^{+} = \\
&= \nabla S_0^{+} - \nabla\left(j\frac{mc}{\hbar} + \frac{1}{c}\partial_t\right)\phi^{+} - \left[\left(\frac{mc}{\hbar}\right)^2 + \frac{1}{c^2}\partial_{tt}\right] c\boldsymbol{A}^{+}
\end{aligned}$$

reordering the terms:

$$\begin{aligned}
&\left[\left(\frac{mc}{\hbar}\right)^2 + \frac{1}{c^2}\partial_{tt} - \nabla^2\right] c\boldsymbol{A}^{+} = \\
&= \nabla\left[S_0^{+} - \left(j\frac{mc}{\hbar} + \frac{1}{c}\partial_t\right)\phi^{+} - c\nabla \cdot \boldsymbol{A}^{+}\right]
\end{aligned} \tag{9}$$

Since $\boldsymbol{B}^{\pm} \propto e^{-j(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})}$ and $\boldsymbol{B}^{\pm} = \nabla \times \boldsymbol{A}^{\pm}$, it follows that $\boldsymbol{A}^{\pm}$ also has the same exponential dependency. Consequently, due to the mass shell condition, the left-hand side of eq.9 becomes null:

$$\left[\left(\frac{mc}{\hbar}\right)^2 + \frac{1}{c^2}\partial_{tt} - \nabla^2\right]\boldsymbol{A}^{\pm} = 0 \tag{10}$$

Therefore, the right hand side of eq.9 is null. Thus, the first condition is:

$$c\nabla \cdot \boldsymbol{A}^{+} - S_0^{+} + \left(j\frac{mc}{\hbar} + \frac{1}{c}\partial_t\right)\phi^{+} = 0 \tag{11}$$

By applying a similar derivation to the seventh row eq.4, a second condition can be obtained:

$$c\nabla \cdot \boldsymbol{A}^{-} + S_0^{-} - \left(j\frac{mc}{\hbar} - \frac{1}{c}\partial_t\right)\phi^{-} = 0 \tag{12}$$

By substituting eq.7 and eq.8 in the divergence of the electric fields in eq.4, it can be shown that also the scalar potentials $\phi^{\pm}$ satisfy the mass shell condition:

$$\left[\left(\frac{mc}{\hbar}\right)^2 + \frac{1}{c^2}\partial_{tt} - \nabla^2\right]\phi^{\pm} = 0. \tag{13}$$

From eq.11 and eq.12 one can express the scalar fields in terms of the derivatives of the potentials.

$$S_0^{+} = c\nabla \cdot \boldsymbol{A}^{+} + \left(j\frac{mc}{\hbar} + \frac{1}{c}\partial_t\right)\phi^{+} \tag{14}$$

$$S_0^{-} = -c\nabla \cdot \boldsymbol{A}^{-} + \left(j\frac{mc}{\hbar} - \frac{1}{c}\partial_t\right)\phi^{-} \tag{15}$$

A. Gauge conditions

1. Strong gauge condition

Consider the transformation:

$$\acute{\boldsymbol{A}}^{\pm} \rightarrow \boldsymbol{A}^{\pm} + \nabla\chi \tag{16}$$

$$\acute{\phi}^{\mp} \rightarrow \phi^{\mp} - \left(\tfrac{1}{c}\partial_t \pm j\tfrac{mc}{\hbar}\right)\chi \tag{17}$$

where $\chi = \chi(\boldsymbol{r}, t)$.

Applying this transformation on eq.14 and eq.15 :

$$\begin{aligned}
\acute{S}_0^{\pm} &= \pm c\nabla\cdot\acute{\boldsymbol{A}}^{\pm} + \left(j\frac{mc}{\hbar} \pm \frac{1}{c}\partial_t\right)\acute{\phi}^{\pm} \\
\acute{S}_0^{\pm} &= \pm c\nabla\cdot\left(\boldsymbol{A}^{\pm} + \nabla\chi\right) + \\
&\quad + \left(j\frac{mc}{\hbar} \pm \frac{1}{c}\partial_t\right)\left[\phi^{\pm} - \left(\frac{1}{c}\partial_t \mp j\frac{mc}{\hbar}\right)\chi\right] \\
\acute{S}_0^{\pm} &= \pm c\nabla\cdot\boldsymbol{A}^{\pm} \pm c\nabla^2\chi + \left(j\frac{mc}{\hbar} \pm \frac{1}{c}\partial_t\right)\phi^{\pm} - \\
&\quad - \left(j\frac{mc}{\hbar} \pm \frac{1}{c}\partial_t\right)\left(\frac{1}{c}\partial_t \mp j\frac{mc}{\hbar}\right)\chi \\
\acute{S}_0^{\pm} &= S_0^{\pm} \pm c\nabla^2\chi \mp \left(\frac{1}{c}\partial_t \pm j\frac{mc}{\hbar}\right)\left(\frac{1}{c}\partial_t \mp j\frac{mc}{\hbar}\right)\chi \\
\acute{S}_0^{\pm} &= S_0^{\pm} \pm c\nabla^2\chi \mp \left[\frac{1}{c^2}\partial_{tt} + \left(\frac{mc}{\hbar}\right)^2\right]\chi \\
\acute{S}_0^{\pm} &= S_0^{\pm} \mp \left[-c\nabla^2 + \frac{1}{c^2}\partial_{tt} + \left(\frac{mc}{\hbar}\right)^2\right]\chi \\
\acute{S}_0^{\pm} &= S_0^{\pm} \mp \left[\left(\frac{\boldsymbol{p}}{\hbar}\right)^2 - \frac{E^2}{c^2\hbar^2} + \left(\frac{mc}{\hbar}\right)^2\right]\chi \qquad (18) \\
\acute{S}_0^{\pm} &= S_0^{\pm} \qquad (19)
\end{aligned}$$

Where the transition to eq.18 is based on the assumption that the field χ has of the form $\chi_0 e^{-j(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})}$ with $E = \omega\hbar$ and $\boldsymbol{p} = \boldsymbol{k}\hbar$. The transition to eq.19 is valid only if χ is a massive field that satisfies the mass shell condition. It can be similarly demonstrated that the fields $\boldsymbol{E}^{\pm}, \boldsymbol{B}^{\pm}$ are also conserved under the transformation described in eq.16 and eq.17. Therefore, eq.16 and eq.17 constitute a gauge condition.

It's worth noting that, unlike the Lorentz gauge condition in which the gauge field χ is (only) required to have second derivatives in time and space, in this gauge condition, χ must be of the form $\chi_0 e^{j(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})}$with a mass m which is identical to the mass term of the transformed fields.

2. weak gauge condition

Consider the standard Lorentz gauge from classical electromagnetism:

$$\acute{\boldsymbol{A}}^{\pm} \rightarrow \boldsymbol{A}^{\pm} + \nabla\chi \tag{20}$$

$$\acute{\phi}^{\pm} \rightarrow \phi^{\pm} - \tfrac{1}{c}\partial_t\chi \tag{21}$$

Applying this on eq.14 and eq.15 :

$$\begin{aligned}
\acute{S}_0^{\pm} &= \pm c\nabla\cdot\acute{\boldsymbol{A}}^{\pm} + \left(j\frac{mc}{\hbar} \pm \frac{1}{c}\partial_t\right)\acute{\phi}^{\pm} \\
\acute{S}_0^{\pm} &= \pm c\nabla\cdot\left(\boldsymbol{A}^{\pm} + \nabla\chi\right) + \left(j\frac{mc}{\hbar} \pm \frac{1}{c}\partial_t\right)\left(\phi^{\pm} - \frac{1}{c}\partial_t\chi\right) \\
\acute{S}_0^{\pm} &= \pm c\nabla\cdot\boldsymbol{A}^{\pm} \pm c\nabla^2\chi + \left(j\frac{mc}{\hbar} \pm \frac{1}{c}\partial_t\right)\phi^{\pm} - \\
&\quad - \frac{1}{c}\left(j\frac{mc}{\hbar} \pm \frac{1}{c}\partial_t\right)\partial_t\chi \qquad (22) \\
\acute{S}_0^{\pm} &= S_0^{\pm} \pm c\nabla^2\chi - \frac{1}{c}\left(j\frac{mc}{\hbar} \pm \frac{1}{c}\partial_t\right)\partial_t\chi \qquad (23)
\end{aligned}$$

Using the same transformation on eq.7 and eq.8:

$$\begin{aligned}
\acute{\boldsymbol{E}}^{\pm} &= -\nabla\acute{\phi}^{\mp} \mp \left(j\frac{mc}{\hbar} + \frac{1}{c}\partial_t\right)c\acute{\boldsymbol{A}}^{\mp} \\
\acute{\boldsymbol{E}}^{\pm} &= -\nabla\left(\phi^{\mp} - \frac{1}{c}\partial_t\chi\right) \mp \left(j\frac{mc}{\hbar} + \frac{1}{c}\partial_t\right)\left(c\boldsymbol{A}^{\pm} + c\nabla\chi\right) \\
\acute{\boldsymbol{E}}^{\pm} &= -\nabla\acute{\phi}^{\mp} + \frac{1}{c}\nabla\partial_t\chi \mp \left(j\frac{mc}{\hbar} + \frac{1}{c}\partial_t\right)c\boldsymbol{A}^{\pm} \mp \\
&\quad \mp c\left(j\frac{mc}{\hbar} + \frac{1}{c}\partial_t\right)\nabla\chi \qquad (24) \\
\acute{\boldsymbol{E}}^{\pm} &= \boldsymbol{E}^{\pm} + \frac{1}{c}\nabla\partial_t\chi \mp c\left(j\frac{mc}{\hbar} + \frac{1}{c}\partial_t\right)\nabla\chi \qquad (25)
\end{aligned}$$

Writing the equivalent to Gauss law in eq.4 as follows:

$$\nabla\cdot\boldsymbol{E}^{\pm} = \left(j\frac{mc}{\hbar} \pm \frac{1}{c}\partial_t\right)S_0^{\mp} \tag{26}$$

Substitute the transformed expressions in eq.23 and eq.25:

$$\begin{aligned}
\nabla\cdot\acute{\boldsymbol{E}}^{\pm} &= \left(j\frac{mc}{\hbar} \pm \frac{1}{c}\partial_t\right)\acute{S}_0^{\mp} \\
\nabla\cdot\left[\boldsymbol{E}^{\pm} + \frac{1}{c}\nabla\partial_t\chi \mp c\left(j\frac{mc}{\hbar} + \frac{1}{c}\partial_t\right)\nabla\chi\right] &= \\
= \left(j\frac{mc}{\hbar} \pm \frac{1}{c}\partial_t\right)&\left[S_0^{\mp} \mp c\nabla^2\chi - \frac{1}{c}\left(j\frac{mc}{\hbar} \mp \frac{1}{c}\partial_t\right)\partial_t\chi\right] \\
\nabla\cdot\boldsymbol{E}^{\pm} + \frac{1}{c}\nabla^2\partial_t\chi \mp \underline{c\left(j\frac{mc}{\hbar} + \frac{1}{c}\partial_t\right)\nabla^2\chi} &= \\
= \left(j\frac{mc}{\hbar} \pm \frac{1}{c}\partial_t\right)S_0^{\mp} &\mp \underline{c\left(j\frac{mc}{\hbar} \pm \frac{1}{c}\partial_t\right)\nabla^2\chi} - \\
- \frac{1}{c}&\left(j\frac{mc}{\hbar} \pm \frac{1}{c}\partial_t\right)\left(j\frac{mc}{\hbar} \mp \frac{1}{c}\partial_t\right)\partial_t\chi \\
\nabla\cdot\boldsymbol{E}^{\pm} &= \left(j\frac{mc}{\hbar} \pm \frac{1}{c}\partial_t\right)S_0^{\mp} + \left[\left(\frac{mc}{\hbar}\right)^2 + \frac{1}{c^2}\partial_{tt} - \nabla^2\right]\frac{\partial_t\chi}{c}
\end{aligned}$$

Hence eq.26 is invariant if the right term is null, hence:

$$\left[\left(\frac{mc}{\hbar}\right)^2 + \frac{1}{c^2}\partial_{tt} - \nabla^2\right]\partial_t \chi = 0 \tag{27}$$

This result reiterates the previous constraint on χ to be a massive scalar field of the form $\chi_0 e^{j(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})}$with the same mass. This invariance can be demonstrated for the rest of eq.4 using the same process. Therefore, under the transformation described by eq.20 and eq.21, the fields are not conserved (nor the Lagrangian in Sec.III E) but the 'equations of motion' of the fields remain invariant.

III. TOWARDS CLASSIFICATION OF QED

A. Connection to Dirac Equation

Formulating eq.1 in Cartesian coordinate system results:

$$\begin{pmatrix}
\frac{1}{c}\partial_t & 0 & 0 & 0 & 0 & j\partial_x & j\partial_y & j\partial_z \\
0 & \frac{1}{c}\partial_t & 0 & 0 & j\partial_x & 0 & -j\partial_z & j\partial_y \\
0 & 0 & \frac{1}{c}\partial_t & 0 & j\partial_y & j\partial_z & 0 & -j\partial_x \\
0 & 0 & 0 & \frac{1}{c}\partial_t & j\partial_z & -j\partial_y & j\partial_x & 0 \\
0 & j\partial_x & j\partial_y & j\partial_z & -\frac{1}{c}\partial_t & 0 & 0 & 0 \\
j\partial_x & 0 & j\partial_z & -j\partial_y & 0 & -\frac{1}{c}\partial_t & 0 & 0 \\
j\partial_y & -j\partial_z & 0 & j\partial_x & 0 & 0 & -\frac{1}{c}\partial_t & 0 \\
j\partial_z & j\partial_y & -j\partial_x & 0 & 0 & 0 & 0 & -\frac{1}{c}\partial_t
\end{pmatrix}
\begin{pmatrix} S_0^+ \\ S_x^+ \\ S_y^+ \\ S_z^+ \\ S_0^- \\ S_x^- \\ S_y^- \\ S_z^- \end{pmatrix}
= j\frac{mc}{\hbar}
\begin{pmatrix} S_0^+ \\ S_x^+ \\ S_y^+ \\ S_z^+ \\ S_0^- \\ S_x^- \\ S_y^- \\ S_z^- \end{pmatrix} \tag{28}$$

Eq.28 can be manipulated to align with the Dirac equation structure. This could provide a more direct comparison between the suggested equation and the well-established Dirac equation:

$$\left(j\gamma^\mu \partial_\mu - \frac{mc}{\hbar}\right)\psi = 0 \tag{29}$$

To achieve the correct sign of the mass term, eq.28 needs to be multiplied by $'-j'$:

$$j\begin{pmatrix}
-\frac{1}{c}\partial_t & 0 & 0 & 0 & 0 & -j\partial_x & -j\partial_y & -j\partial_z \\
0 & -\frac{1}{c}\partial_t & 0 & 0 & -j\partial_x & 0 & j\partial_z & -j\partial_y \\
0 & 0 & -\frac{1}{c}\partial_t & 0 & -j\partial_y & -j\partial_z & 0 & j\partial_x \\
0 & 0 & 0 & -\frac{1}{c}\partial_t & -j\partial_z & j\partial_y & -j\partial_x & 0 \\
0 & -j\partial_x & -j\partial_y & -j\partial_z & \frac{1}{c}\partial_t & 0 & 0 & 0 \\
-j\partial_x & 0 & -j\partial_z & j\partial_y & 0 & \frac{1}{c}\partial_t & 0 & 0 \\
-j\partial_y & j\partial_z & 0 & -j\partial_x & 0 & 0 & \frac{1}{c}\partial_t & 0 \\
-j\partial_z & -j\partial_y & j\partial_x & 0 & 0 & 0 & 0 & \frac{1}{c}\partial_t
\end{pmatrix}
\begin{pmatrix} S_0^+ \\ S_x^+ \\ S_y^+ \\ S_z^+ \\ S_0^- \\ S_x^- \\ S_y^- \\ S_z^- \end{pmatrix}
- \frac{mc}{\hbar}
\begin{pmatrix} S_0^+ \\ S_x^+ \\ S_y^+ \\ S_z^+ \\ S_0^- \\ S_x^- \\ S_y^- \\ S_z^- \end{pmatrix} = 0 \tag{30}$$

The new gamma matrices can be identified as:

$$\gamma^0 = \begin{pmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\quad
\gamma^1 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & -j & 0 & 0 \\
0 & 0 & 0 & 0 & -j & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & j \\
0 & 0 & 0 & 0 & 0 & 0 & -j & 0 \\
0 & -j & 0 & 0 & 0 & 0 & 0 & 0 \\
-j & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -j & 0 & 0 & 0 & 0 \\
0 & 0 & j & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

$$\gamma^2 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & -j & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -j \\
0 & 0 & 0 & 0 & -j & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & j & 0 & 0 \\
0 & 0 & -j & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & j & 0 & 0 & 0 & 0 \\
-j & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -j & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\quad
\gamma^3 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & -j \\
0 & 0 & 0 & 0 & 0 & 0 & j & 0 \\
0 & 0 & 0 & 0 & 0 & -j & 0 & 0 \\
0 & 0 & 0 & 0 & -j & 0 & 0 & 0 \\
0 & 0 & 0 & -j & 0 & 0 & 0 & 0 \\
0 & 0 & -j & 0 & 0 & 0 & 0 & 0 \\
0 & j & 0 & 0 & 0 & 0 & 0 & 0 \\
-j & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \tag{31}$$

Therefore, eq.1 can be written as an *'extended'* 8×8 Dirac equation:

$$\left(j\gamma^\mu \partial_\mu - \frac{mc}{\hbar}\right) \boldsymbol{S} = 0 \qquad (32)$$

Where the 4x4 Dirac gamma matrices were replaced by 8x8 gamma matrices and the bi-spinor wave function ψ was replaced by 8 component complex vector noted as $\boldsymbol{S}$ (to differ it from the traditional bi-spinor), composed of 2 sets of scalar and vector 'electromagnetic-like' fields.

the fifth gamma matrix can calculate by $\gamma^5 = j\gamma^0\gamma^1\gamma^2\gamma^3$ to be:

$$\gamma^5 = \begin{pmatrix} 0 & \eta \\ \eta & 0 \end{pmatrix} \qquad (33)$$

where

$$\eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (34)$$

is the Minkowski metric. Furthermore, it can be checked that similarly to the Dirac equation case, the gamma matrices generate a Clifford algebra, characterized by the following anti-commutation relations:

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = -2\eta^{\mu\nu} I_4 \qquad (35)$$

Note that the sign at the right hand side of the anti-commutation relation is dependent on the metric signature definition. Here the signature was chosen to be (-1,1,1,1).

Additionally, γ^5 anti-commutes with the four gamma matrices-

$$\{\gamma^5, \gamma^\nu\} = \gamma^5\gamma^\nu + \gamma^\nu\gamma^5 = 0 \qquad (36)$$

Using the definitions of left and right chirality projection operators from quantum mechanics (Dirac formalism), their corresponding definitions can be written as:

$$P_L = \frac{I-\gamma^5}{2} = \frac{1}{2}\begin{pmatrix} I_4 & -\eta \\ -\eta & I_4 \end{pmatrix}$$
$$P_R = \frac{I+\gamma^5}{2} = \frac{1}{2}\begin{pmatrix} I_4 & \eta \\ \eta & I_4 \end{pmatrix} \qquad (37)$$

where I_4 is the 4×4 identity matrix. These operators are singular matrices as expected, and it is interesting to find how the difference between the left and right operators is related to the metric signature.

It can be shown that the non-zero eigenvalues are all equal to 1 and the general eigenvectors of the parity projection operators have the form (using the notation of eq.1)

$$\psi_L = \begin{pmatrix} S_0 \\ \boldsymbol{F} \\ S_0 \\ -\boldsymbol{F} \end{pmatrix}, \ \psi_R = \begin{pmatrix} S_0 \\ \boldsymbol{F} \\ -S_0 \\ \boldsymbol{F} \end{pmatrix} \qquad (38)$$

Though $\psi_L$and ψ_R translation to EM-like fields and their symmetries are of interest, it will not be covered by this work and left to the inquisitive reader.

B. Spin

To investigate the spin property, we shall take the familiar path of defining the Hamiltonian and the angular momentum operator within the Dirac formalism and then evaluate their commutation relations.

The structure of the Hamiltonian is given by $\mathcal{H} = \alpha_i p^i + \beta mc^2$ [2] and can be derived from the extended Dirac equation (eq.32) by multiplying it by γ^0, and using the relation $\gamma^0\gamma^0 = I$:

$$\left(j\partial_0 + j\gamma^0\gamma^1\partial_1 + j\gamma^0\gamma^2\partial_2 + j\gamma^0\gamma^3\partial_3 - \gamma^0\frac{mc}{\hbar}\right) S = 0$$

Thus, by using the assumption that $S \propto e^{-j(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})}$ for all state vector components and identifying $\omega = E/\hbar$ and $\boldsymbol{k} = \boldsymbol{p}/\hbar$, the Hamiltonian can be expressed as:

$$\mathcal{H} = c\gamma^0\gamma^1 p^1 + c\gamma^0\gamma^2 p^2 + c\gamma^0\gamma^3 p^3 + \gamma^0 mc^2$$

Defining α^i as $\alpha^i = c\gamma^0\gamma^i \ , \ i = 1,2,3$, the Hamiltonian gets the form (using Einstein summation convention):

$$\mathcal{H} = \alpha^i p^i + \gamma^0 mc^2 \qquad (39)$$

To check the conservation of angular momentum, the standard QM angular momentum operator definition will be used:

$$L_i = c\epsilon_{ijk} x_j p_k$$

Where ϵ_{ijk} is the Levi-Civita symbol.

The standard procedure to assess angular momentum conservation in the $'i'$ direction is by checking if L_i commutes with the Hamiltonian. Since this calculation is identical to the standard QM case (mainly using position and momentum commutation relation) it will not be articulated here and only the result of it is given:

$$[L_i, \mathcal{H}] = jc\epsilon_{ijk}\alpha_i \hbar p_k \qquad (40)$$

As expected, since the matrix algebra is identical to the Dirac equation, there is an additional intrinsic angular momentum term.

The spin generators are to be calculated similarly to the Dirac equation case:

$$\widetilde{\alpha}_i \equiv -\gamma^5\alpha_i = -c\gamma^5\gamma^0\gamma^i \quad , \; i = 1,2,3$$

Explicitly:

$$\widetilde{\alpha}_1 = -c\begin{pmatrix} 0 & j & 0 & 0 & 0 & 0 & 0 & 0 \\ -j & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -j & 0 & 0 & 0 & 0 \\ 0 & 0 & j & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -j & 0 & 0 \\ 0 & 0 & 0 & 0 & j & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -j \\ 0 & 0 & 0 & 0 & 0 & 0 & j & 0 \end{pmatrix} \tag{41}$$

$$\widetilde{\alpha}_2 = -c\begin{pmatrix} 0 & 0 & j & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & j & 0 & 0 & 0 & 0 \\ -j & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -j & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -j & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & j \\ 0 & 0 & 0 & 0 & j & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -j & 0 & 0 \end{pmatrix} \tag{42}$$

$$\widetilde{\alpha}_3 = -c\begin{pmatrix} 0 & 0 & 0 & j & 0 & 0 & 0 & 0 \\ 0 & 0 & -j & 0 & 0 & 0 & 0 & 0 \\ 0 & j & 0 & 0 & 0 & 0 & 0 & 0 \\ -j & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -j \\ 0 & 0 & 0 & 0 & 0 & 0 & -j & 0 \\ 0 & 0 & 0 & 0 & 0 & j & 0 & 0 \\ 0 & 0 & 0 & 0 & j & 0 & 0 & 0 \end{pmatrix} \tag{43}$$

In an identical way to the Dirac equation case, the commutation relation of the above matrices with the Hamiltonian yields:

$$[\widetilde{\alpha}_i, \mathcal{H}] = -2jc\epsilon_{ijk}\alpha_k p_k \tag{44}$$

Consequently, upon combining eq.44 with eq.40:

$$[L_i + \frac{\hbar}{2}\widetilde{\alpha}_i, \mathcal{H}] = 0 \tag{45}$$

Hence, as in the Dirac equation formalism, the internal spin angular momentum in direction $i \in [1, 2, 3]$ is:

$$s_i = \frac{\hbar}{2}\widetilde{\alpha}_i = -\frac{\hbar}{2}c\gamma^5\gamma^0\gamma^i \tag{46}$$

The eigenvalues and eigenvectors corresponding to the spin in the 'z' direction are delineated in Table 1. It is evident that in contrast to the Dirac equation, this formulation introduces an additional (independent) spin eigenvector for each spin eigenvalue.

λ_i:	$\frac{1}{2}\hbar$	$-\frac{1}{2}\hbar$	$\frac{1}{2}\hbar$	$-\frac{1}{2}\hbar$	$\frac{1}{2}\hbar$	$-\frac{1}{2}\hbar$	$\frac{1}{2}\hbar$	$-\frac{1}{2}\hbar$
S_0^+:	$-j\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	0	0	0	0	0	0
S_x^+:	0	0	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	0	0	0	0
S_y^+:	0	0	$-j\frac{1}{\sqrt{2}}$	$j\frac{1}{\sqrt{2}}$	0	0	0	0
S_z^+:	$\frac{1}{\sqrt{2}}$	$-j\frac{1}{\sqrt{2}}$	0	0	0	0	0	0
S_0^-:	0	0	0	0	$j\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	0	0
S_x^-:	0	0	0	0	0	0	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
S_y^-:	0	0	0	0	0	0	$-j\frac{1}{\sqrt{2}}$	$j\frac{1}{\sqrt{2}}$
S_z^-:	0	0	0	0	$\frac{1}{\sqrt{2}}$	$j\frac{1}{\sqrt{2}}$	0	0

TABLE I. Eigenvalues and eigenvectors for the spin operator at 'z' direction $\frac{\hbar}{2}\widetilde{\alpha}_3$

To discern between these two spin states (which can be summed together), an examination of the resulted field equations are provided. Consider the first eigenvector in Table 1 while recalling that $S_i^\pm = cB_i^\pm - jE_i^\pm$ and assuming all field amplitude coefficients are real, it is evident that-

$$E_z^+ = S_0^+ \tag{47}$$

All remaining fields are null. Incorporating the relation from eq.47 into eq.4 yields:

$$\begin{aligned} &\nabla E_z^+ = 0 \\ &\nabla \cdot (E_z^+\hat{z}) = 0 \\ &\left(j\frac{mc}{\hbar} - \frac{1}{c}\partial_t\right)E_z^+ = 0 \\ &\nabla \times (E_z^+\hat{z}) = 0 \end{aligned} \tag{48}$$

The third row in eq.48 indicates that E_z^+(and S_0^+ per eq.47) fluctuates over time without external momentum while the mass is the sole contributor to the (negative) energy. Moreover, all spatial derivatives of E_z^+(and correspondingly S_0^+) are zero. Consequently, E_z^+ and S_0^+ exhibit no spatial variations, as could be anticipated per the uncertainty principle (they are dispersed throughout all space).

A comparable examination for the second eigenvector in Table 1 yields a similar outcome, with the distinction that in this instance, $E_z^+ = -S_0^+$. Similarly, the fifth and sixth eigenvectors yield identical results, albeit with a positive energy. Consequently, one can identify that $S_0^\pm$ is a scalar field that carries angular momentum in the $'i'$ direction when combined with a corresponding $E_i^\pm$ field component.

The third eigenvector in Table 1 is simplified to the following relations:

$$cB_y^+ = E_x^+ \quad , \quad cB_x^+ = -E_y^+ \tag{49}$$

Setting these relations in eq.4 and removing null and redundant rows the remaining equations is as follows:

$$\begin{aligned}
&\nabla \cdot \left(-E_y^+ \hat{x} + E_x^+ \hat{y}\right) = 0 \\
&\nabla \times \left(-E_y^+ \hat{x} + E_x^+ \hat{y}\right) = 0 \\
&\nabla \cdot \left(E_x^+ \hat{x} + E_y^+ \hat{y}\right) = 0 \\
&\left(j\frac{mc}{\hbar} - \frac{1}{c}\partial_t\right)\left(E_x^+ \hat{x} + E_y^+ \hat{y}\right) = 0 \\
&\nabla \times \left(E_x^+ \hat{x} + E_y^+ \hat{y}\right) = 0
\end{aligned} \tag{50}$$

expanding the nabla operators yields

$$\begin{aligned}
&\partial_x E_y^+ - \partial_y E_x^+ = 0 \\
&\partial_z E_x^+ = \partial_z E_y^+ = 0 \\
&\partial_x E_x^+ + \partial_y E_y^+ = 0 \\
&\left(j\frac{mc}{\hbar} - \frac{1}{c}\partial_t\right)\left(E_x^+ \hat{x} + E_y^+ \hat{y}\right) = 0
\end{aligned} \tag{51}$$

refining these equations :

$$E_x^+ = E_y^+ e^{j\frac{\pi}{2}} \tag{52}$$

$$cB_x^+ = cB_y^+ e^{j\frac{\pi}{2}} \tag{53}$$

$$\boldsymbol{E} \perp \boldsymbol{B} \tag{54}$$

$$cB_y^+ = E_x^+ \tag{55}$$

Thus, $\boldsymbol{E}^+$ and $\boldsymbol{B}^+$ are orthogonal and rotate in the (x, y) plains all over space in a circular polarization manner with no spatial variation. Additionally, the associated Poyniting vector is pointing at the $+\hat{z}$ direction and similarly to the previous cases, the mass is the sole contributor to the (negative) energy.

In contrast to the previous case where the angular momentum was carried by the scalar field S_0(with no apparent rotation in the equations), here the rotated vector fields are identified to carry angular momentum.

An intriguing observation here is that the eigenvectors of the spin operator couple fields which are not coupled by eq.4. Hence, the mathematical description of spin enforces coupling of two fields from separate sets (next to be marked by $'+'$ and $'-'$ signs). A single 'quazi-Maxwell' equation set is insufficient to describe the spin phenomenon.

C. Constructing a Lagrangian

While the Dirac equation formalism describes fermionic fields (matter) and aligns with special relativity, it cannot be fully integrated in Einstein general theory of relativity (GR) as the source of the space-time curvature. On the other hand, Maxwell's electromagnetism formalism can be incorporate in GR as a source via the electromagnetic stress-energy tensor, but it cannot describe matter fields.

Given that the formalism suggested in this work is somewhat an expansion of Maxwell equations towards the Dirac equation (or vice versa) it is enticing to construct a quantum-Maxwell-like Lagrangian that will describe *quantum-matter* and also be of a form that can be incorporated in GR (describing *quantum-matter* as a space-time curvature source).

The differential eigenvalues equation, eq.28, is the starting point for the derivation of the Lagrangian and can be written in an 'inverse' form by interchanging the roles of the state vector components and the derivatives:

$$\begin{pmatrix}
S_0^- & S_x^+ & S_y^+ & S_z^+ & 0 & 0 & 0 & 0 \\
-S_x^+ & S_0^- & S_z^- & -S_y^- & 0 & 0 & 0 & 0 \\
-S_y^+ & -S_z^- & S_0^- & S_x^- & 0 & 0 & 0 & 0 \\
-S_z^+ & S_y^- & -S_x^- & S_0^- & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & S_0^+ & S_x^- & S_y^- & S_z^- \\
0 & 0 & 0 & 0 & -S_x^- & S_0^+ & -S_z^+ & S_y^+ \\
0 & 0 & 0 & 0 & -S_y^- & S_z^+ & S_0^+ & -S_x^+ \\
0 & 0 & 0 & 0 & -S_z^- & -S_y^+ & S_x^+ & S_0^+
\end{pmatrix}
\begin{pmatrix}
-\frac{1}{c}\partial_t \\ j\partial_x \\ j\partial_y \\ j\partial_z \\ \frac{1}{c}\partial_t \\ j\partial_x \\ j\partial_y \\ j\partial_z
\end{pmatrix}
= j\frac{mc}{\hbar}
\begin{pmatrix}
S_0^- \\ S_x^+ \\ S_y^+ \\ S_z^+ \\ S_0^+ \\ S_x^- \\ S_y^- \\ S_z^-
\end{pmatrix} \tag{56}$$

Utilizing the expansion $S_i^\pm = cB_i^\pm - jE_i^\pm$ and separating real and imaginary terms, eq.56 splits to two matrix equations:

For the real part:

$$\begin{pmatrix} S_0^- & E_x^+ & E_y^+ & E_z^+ & 0 & 0 & 0 & 0 \\ -cB_x^+ & 0 & E_z^- & -E_y^- & 0 & 0 & 0 & 0 \\ -cB_y^+ & -E_z^- & 0 & E_x^- & 0 & 0 & 0 & 0 \\ -cB_z^+ & E_y^- & -E_x^- & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & S_0^+ & E_x^- & E_y^- & E_z^- \\ 0 & 0 & 0 & 0 & -cB_x^- & 0 & -E_z^+ & E_y^+ \\ 0 & 0 & 0 & 0 & -cB_y^- & E_z^+ & 0 & -E_x^+ \\ 0 & 0 & 0 & 0 & -cB_z^- & -E_y^+ & S_x^+ & 0 \end{pmatrix} \begin{pmatrix} -\frac{1}{c}\partial_t \\ \partial_x \\ \partial_y \\ \partial_z \\ \frac{1}{c}\partial_t \\ \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} = j\frac{mc}{\hbar} \begin{pmatrix} S_0^- \\ cB_x^+ \\ cB_y^+ \\ cB_z^+ \\ S_0^+ \\ cB_x^- \\ cB_y^- \\ cB_z^- \end{pmatrix} \tag{57}$$

For the Imaginary part:

$$\begin{pmatrix} 0 & cB_x^+ & cB_y^+ & cB_z^+ & 0 & 0 & 0 & 0 \\ E_x^+ & S_0^- & B_z^- & -B_y^- & 0 & 0 & 0 & 0 \\ E_y^+ & -B_z^- & S_0^- & B_x^- & 0 & 0 & 0 & 0 \\ E_z^+ & B_y^- & -B_x^- & S_0^- & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & cB_x^- & cB_y^- & cB_z^- \\ 0 & 0 & 0 & 0 & E_x^- & S_0^+ & -B_z^+ & B_y^+ \\ 0 & 0 & 0 & 0 & E_y^- & B_z^+ & S_0^+ & -B_x^+ \\ 0 & 0 & 0 & 0 & E_z^- & -B_y^+ & B_x^+ & S_0^+ \end{pmatrix} \begin{pmatrix} -\frac{1}{c}\partial_t \\ \partial_x \\ \partial_y \\ \partial_z \\ \frac{1}{c}\partial_t \\ \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} = j\frac{mc}{\hbar} \begin{pmatrix} 0 \\ -E_x^+ \\ -E_y^+ \\ -E_z^+ \\ 0 \\ -E_x^- \\ -E_y^- \\ -E_z^- \end{pmatrix} \tag{58}$$

In the two equations above, half of the rows were used to define the potentials and thus will add no information once replacing the fields with the potentials. Hence, these rows can be omitted without any loss of information. The rows of the equations that do contain information are the first and fifth rows in eq.57 and the second, third, fourth, sixth, seventh and eighth rows in eq.58. Therefore, all these lines in eq.57 and eq.58 can be unified to a single 8×8 matrix equation as follow-

$$\begin{pmatrix} S_0^- & E_x^+ & E_y^+ & E_z^+ & 0 & 0 & 0 & 0 \\ -E_x^+ & -S_0^- & -cB_z^- & cB_y^- & 0 & 0 & 0 & 0 \\ -E_y^+ & cB_z^- & -S_0^- & -cB_x^- & 0 & 0 & 0 & 0 \\ -E_z^+ & -cB_y^- & cB_x^- & -S_0^- & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & S_0^+ & E_x^- & E_y^- & E_z^- \\ 0 & 0 & 0 & 0 & -E_x^- & -S_0^+ & cB_z^+ & -cB_y^+ \\ 0 & 0 & 0 & 0 & -E_y^- & -cB_z^+ & -S_0^+ & cB_x^+ \\ 0 & 0 & 0 & 0 & -E_z^- & cB_y^+ & -cB_x^+ & -S_0^+ \end{pmatrix} \begin{pmatrix} -\frac{1}{c}\partial_t \\ \partial_x \\ \partial_y \\ \partial_z \\ \frac{1}{c}\partial_t \\ \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} = j\frac{mc}{\hbar} \begin{pmatrix} S_0^- \\ E_x^+ \\ E_y^+ \\ E_z^+ \\ S_0^+ \\ E_x^- \\ E_y^- \\ E_z^- \end{pmatrix} \tag{59}$$

Changing the arrangement of the four bottom rows, such that the components of $\boldsymbol{B}^+$ align with the signs of the $\boldsymbol{B}^-$ components:

$$\begin{pmatrix} S_0^- & E_x^+ & E_y^+ & E_z^+ & 0 & 0 & 0 & 0 \\ -E_x^+ & -S_0^- & -cB_z^- & cB_y^- & 0 & 0 & 0 & 0 \\ -E_y^+ & cB_z^- & -S_0^- & -cB_x^- & 0 & 0 & 0 & 0 \\ -E_z^+ & -cB_y^- & cB_x^- & -S_0^- & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -S_0^+ & E_x^- & E_y^- & E_z^- \\ 0 & 0 & 0 & 0 & -E_x^- & S_0^+ & -cB_z^+ & cB_y^+ \\ 0 & 0 & 0 & 0 & -E_y^- & cB_z^+ & S_0^+ & -cB_x^+ \\ 0 & 0 & 0 & 0 & -E_z^- & -cB_y^+ & cB_x^+ & S_0^+ \end{pmatrix} \begin{pmatrix} -\frac{1}{c}\partial_t \\ \partial_x \\ \partial_y \\ \partial_z \\ \frac{1}{c}\partial_t \\ -\partial_x \\ -\partial_y \\ -\partial_z \end{pmatrix} = j\frac{mc}{\hbar} \begin{pmatrix} S_0^- \\ E_x^+ \\ E_y^+ \\ E_z^+ \\ -S_0^+ \\ E_x^- \\ E_y^- \\ E_z^- \end{pmatrix} \tag{60}$$

Notice the intriguing combination of spacetime signatures in the derivative vector in eq.60.

Excluding the diagonal, the main blocks of the above matrix has the form of the electromagnetic tensor[3]:

$$F_{ik} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -cB_z & cB_y \\ -E_y & cB_z & 0 & -cB_x \\ -E_z & -cB_y & cB_x & 0 \end{pmatrix}$$

Consequently, the matrix in eq.60 can similarly be defined as an *extended* electromagnetic tensor $F^{\mp}_{\mu\nu}$ and the field vector be defined E^{μ} such that the fields dynamics can be compactly written as:

$$\partial^{\nu}_{\mp} F^{\mp}_{\mu\nu} = j\frac{mc}{\hbar} E^{\mu}_{\pm} \quad , \ \mu_{\pm}, \nu_{\pm} \in [0,1,2,3] \tag{61}$$

It can be easily checked that the scalar fields S_0^+, S_0^- are invariant to Lorentz transformation by performing a separate transformation on each matrix block in eq.60.

The inversion of the $\pm$ signs on the left hand side of eq.61 is attributed to the definition of the signs of the potentials definitions which will be useful in following equations. The classical electromagnetic (EM) tensor can be written by the 4-potential as $F^{ik} = A_{k,i} - A_{i,k}$ (using the notation $A_{k,i} = \partial_i A^k = \frac{\partial A^k}{\partial x^i}$). Eq.60 is slightly more complex due to the sign inversion over the $\boldsymbol{B}^+$components and the mixing of 'positive and negative energy' potential terms in the definition of the fields. Utilizing the potential construction of the fields in eq.5, eq.6, eq.7, eq.8, eq.14 and eq.15, the extended electromagnetic tensor F can be expressed as follow (while using the natural units $c = \hbar = 1$):

$$F =$$

$$\left(\begin{array}{cccccccc}
-\left(A^-_{i,i} + \partial_0\phi^- - jm\phi^-\right) & -\partial_1\phi^- - A^-_{1,0} - jmA^-_1 & -\partial_2\phi^- - A^-_{2,0} - jmA^-_2 & -\partial_3\phi^- - A^-_{3,0} - jmA^-_3 & 0 & 0 & 0 & 0 \\
\partial_1\phi^- + A^-_{1,0} + jmA^-_1 & A^-_{i,i} + \partial_0\phi^- - jm\phi^- & A^-_{2,1} - A^-_{1,2} & A^-_{3,1} - A^-_{1,3} & 0 & 0 & 0 & 0 \\
\partial_2\phi^- + A^-_{2,0} + jmA^-_2 & A^-_{1,2} - A^-_{2,1} & A^-_{i,i} + \partial_0\phi^- - jm\phi^- & A^-_{3,2} - A^-_{2,3} & 0 & 0 & 0 & 0 \\
\partial_3\phi^- + A^-_{3,0} + jmA^-_3 & A^-_{1,3} - A^-_{3,1} & A^-_{2,3} - A^-_{3,2} & A^-_{i,i} + \partial_0\phi^- - jm\phi^- & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & -\left(A^+_{i,i} + \partial_0\phi^+ + jm\phi^+\right) & -\partial_1\phi^+ - A^+_{1,0} + jmA^+_1 & -\partial_2\phi^+ - A^+_{2,0} + jmA^+_2 & -\partial_3\phi^+ - A^+_{3,0} + jmA^+_3 \\
0 & 0 & 0 & 0 & \partial_1\phi^+ + A^+_{1,0} - jmA^+_1 & A^+_{i,i} + \partial_0\phi^+ + jm\phi^+ & A^+_{2,1} - A^+_{1,2} & A^+_{3,1} - A^+_{1,3} \\
0 & 0 & 0 & 0 & \partial_2\phi^+ + A^+_{2,0} - jmA^+_2 & A^+_{1,2} - A^+_{2,1} & A^+_{i,i} + \partial_0\phi^+ + jm\phi^+ & A^+_{3,2} - A^+_{2,3} \\
0 & 0 & 0 & 0 & \partial_3\phi^+ + A^+_{3,0} - jmA^+_3 & A^+_{1,3} - A^+_{3,1} & A^+_{2,3} - A^+_{3,2} & A^+_{i,i} + \partial_0\phi^+ + jm\phi^+
\end{array}\right) \tag{62}$$

Until this point, the $\pm$ superscripts and subscripts were introduced to track the terms origin in the Dirac equation formulation as positive and negative energy stationary solutions. Now, the same $\pm$ indexing can be incorporated to differentiate between the top and bottom blocks of eq.62 which are decoupled (they can be coupled by the spin operator as was shown in the previous section).

Identifying the 4-potentials:

$$A^{\mu}_{+} = \left(\phi^+, A^+_1, A^+_2, A^+_3\right) \tag{63}$$

$$A^{\mu}_{-} = \left(\phi^-, A^-_1, A^-_2, A^-_3\right) \tag{64}$$

Eq.62 can be expressed as $F_{\mp} = \begin{pmatrix} F_- & \\ & F_+ \end{pmatrix}$:

$$F_{\mp} =$$

$$\left(\begin{array}{cccccccc}
-\left(\partial_\lambda A^\lambda_- - jmA^0_-\right) & -\partial_1 A^0_- - \partial_0 A^1_- - jmA^1_- & -\partial_2 A^0_- - \partial_0 A^2_- - jmA^2_- & -\partial_3 A^0_- - \partial_0 A^3_- - jmA^3_- & 0 & 0 & 0 & 0 \\
\partial_1 A^0_- + \partial_0 A^1_- + jmA^1_- & \partial_\lambda A^\lambda_- - jmA^0_- & \partial_1 A^2_- - \partial_2 A^1_- & \partial_1 A^3_- - \partial_3 A^1_- & 0 & 0 & 0 & 0 \\
\partial_2 A^0_- + \partial_0 A^2_- + jmA^2_- & \partial_2 A^1_- - \partial_1 A^2_- & \partial_\lambda A^\lambda_- - jmA^0_- & \partial_2 A^3_- - \partial_3 A^2_- & 0 & 0 & 0 & 0 \\
\partial_3 A^0_- + \partial_0 A^3_- + jmA^3_- & \partial_3 A^1_- - \partial_1 A^3_- & \partial_3 A^2_- - \partial_2 A^3_- & \partial_\lambda A^\lambda_- - jmA^0_- & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & -\left(\partial_\lambda A^\lambda_+ + jmA^0_+\right) & -\partial_1 A^{0+} - \partial_0 A^1_+ + jmA^1_+ & -\partial_2 A^0_+ - \partial_0 A^2_+ + jmA^2_+ & -\partial_3 A^0_+ - \partial_0 A^3_+ + jmA^3_+ \\
0 & 0 & 0 & 0 & \partial_1 A^0_+ + \partial_0 A^1_+ - jmA^1_+ & \partial_\lambda A^\lambda_+ + jmA^0_+ & \partial_1 A^2_+ - \partial_2 A^1_+ & \partial_1 A^3_+ - \partial_3 A^1_+ \\
0 & 0 & 0 & 0 & \partial_2 A^0_+ + \partial_0 A^2_+ - jmA^2_+ & \partial_2 A^1_+ - \partial_1 A^2_+ & \partial_\lambda A^\lambda_+ + jmA^0_+ & \partial_2 A^3_+ - \partial_3 A^2_+ \\
0 & 0 & 0 & 0 & \partial_3 A^0_+ + \partial_0 A^3_+ - jmA^3_+ & \partial_3 A^1_+ - \partial_1 A^3_+ & \partial_3 A^2_+ - \partial_2 A^3_+ & \partial_\lambda A^\lambda_+ + jmA^0_+
\end{array}\right) \tag{65}$$

and it presents a block level separation between $'+'$ and $'-'$ notation. Recalling that the

$(-,+,+,+)_-$, $(+,-,-,-)_+$ signatures are used for the top and bottom blocks correspondingly, the partial derivatives in their contra-variant form satisfy:

$$\begin{aligned}\partial^\lambda A^\lambda_- &= -\partial^0 A^0_- + \partial^1 A^1_- + \partial^2 A^2_- + \partial^3 A^3_- \\ \partial^\lambda A^\lambda_+ &= +\partial^0 A^0_+ - \partial^1 A^1_+ - \partial^2 A^2_+ - \partial^3 A^3_+\end{aligned} \tag{66}$$

Additionally, a corresponding extended metric tensor can be defined as:

$$\eta_\mp = \begin{pmatrix} \eta_- & 0 \\ 0 & \eta_+ \end{pmatrix} \tag{67}$$

where $\eta_-$and η_+ are the two configurations of the metric tensor:

$$\eta_- = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \eta_+ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{68}$$

Combining the 4 derivative with the mass term in accordance with the mass-shell condition which holds true for all frames of reference:

$$c^2\nabla^2 - \partial_{tt} = c^2\left(\frac{mc}{\hbar}\right)^2$$

$$|\partial_{\boldsymbol{i}}|^2 - (\partial_0)^2 = \left(\frac{mc}{\hbar}\right)^2$$

$$-(\partial_0^\mp)^2 + (\partial_i^\mp)^2 = \left(\frac{mc}{\hbar}\right)^2$$

$$\begin{gathered}(\partial_0^-, \partial_1^-, \partial_2^-, \partial_3^-) \cdot (-\partial_0^-, \partial_1^-, \partial_2^-, \partial_3^-) - \left(\frac{mc}{\hbar}\right)^2 = 0 \\ \left(\partial_0^- + j\frac{mc}{\hbar}, \partial_1^-, \partial_2^-, \partial_3^-\right) \cdot \\ \cdot \left(-\partial_0^- + j\frac{mc}{\hbar}, \partial_1^-, \partial_2^-, \partial_3^-\right) = 0\end{gathered} \tag{69}$$

$$\begin{gathered}(\partial_0^+, \partial_1^+, \partial_2^+, \partial_3^+) \cdot (\partial_0^+, -\partial_1^+, -\partial_2^+, -\partial_3^+) + \left(\frac{mc}{\hbar}\right)^2 = 0 \\ \left(\partial_0^+ + j\frac{mc}{\hbar}, \partial_1^+, \partial_2^+, \partial_3^+\right) \cdot \\ \cdot \left(\partial_0^+ - j\frac{mc}{\hbar}, -\partial_1^+, -\partial_2^+, -\partial_3^+\right) = 0\end{gathered} \tag{70}$$

eq.69 and eq.70 illustrate that in order to maintain the invariance property for the $'+'$ superscript part, the mass term should alternate signs between the contra-variant and variant forms. Therefore, the mass term should be indexed. Here we'll use m_0 and m^0, where, $m^0 = -m_0$ under the $(+,-,-,-,)_+$ metric. Specifically, the mass term changes sign when multiplies by $\eta_+^{\mu\nu}$, together with the spatial derivatives ∂_i^+. Another view of these invariance conditions is considering them as a modification to the 4-gradient definition for a massive fermionic field. It is now possible to define the tensor $F_{\mu\nu}^\mp$as:

$$F_{\mu\nu}^\mp =$$

$$\left(\begin{array}{ccccc}
-\partial_0 A_0^- - \partial_i A_i^- + jm_0 A_0^- & -\partial_1 A_0^- - \partial_0 A_1^- - jm_0 A_1^- & -\partial_2 A_0^- - \partial_0 A_2^- - jm_0 A_2^- & -\partial_3 A_0^- - \partial_0 A_3^- - jm_0 A_3^- & 0\;0\;0\;0 \\
\partial_1 A_0^- + \partial_0 A_1^- + jm_0 A_1^- & \partial_0 A_0^- + \partial_i A_i^- - jm_0 A_0^- & \partial^1 A_2^- - \partial_2 A_1^- & \partial_1 A_3^- - \partial_3 A_1^- & 0\;0\;0\;0 \\
\partial_2 A_0^- + \partial_0 A_2^- + jm_0 A_2^- & \partial_2 A_1^- - \partial_1 A_2^- & \partial_0 A_0^- + \partial_i A_i^- - jm_0 A_0^- & \partial_2 A_3^- - \partial_3 A_2^- & 0\;0\;0\;0 \\
\partial_3 A_0^- + \partial_0 A_3^- + jm_0 A_-^3 & \partial_3 A_1^- - \partial_1 A_3^- & \partial_3 A_2^- - \partial_2 A_3^- & \partial_0 A_0^- + \partial_i A_i^- - jm_0 A_0^- & 0\;0\;0\;0 \\
\hline
0\;0\;0\;0 & -\partial_0 A_0^+ - \partial_i A_i^+ - jm_0 A_0^+ & -\partial_1 A_0^+ - \partial_0 A_1^+ + jm_0 A_1^+ & -\partial_2 A_0^+ - \partial_0 A_2^+ + jm_0 A_2^+ & -\partial_3 A_0^+ - \partial_0 A_3^+ + jm_0 A_3^+ \\
0\;0\;0\;0 & \partial_1 A_0^+ + \partial_0 A_1^+ - jm_0 A_1^+ & \partial_0 A_+^0 + \partial_i A_i^+ + jm_0 A_0^+ & \partial_1 A_2^+ - \partial_2 A_1^+ & \partial_1 A_3^+ - \partial_3 A_1^+ \\
0\;0\;0\;0 & \partial_2 A_0^+ + \partial_0 A_2^+ - jm_0 A_2^+ & \partial_2 A_1^+ - \partial_1 A_2^+ & \partial_0 A_+^0 + \partial_i A_i^+ + jm_0 A_0^+ & \partial_2 A_3^+ - \partial_3 A_2^+ \\
0\;0\;0\;0 & \partial_3 A_0^+ + \partial_0 A_3^+ - jm_0 A_3^+ & \partial_3 A_1^+ - \partial_1 A_3^+ & \partial_3 A_2^+ - \partial_2 A_3^+ & \partial_0 A_+^0 + \partial_i A_i^+ + jm_0 A_0^+
\end{array}\right) \tag{71}$$

where the positions of the indices on the matrix terms (up/bottom) are merely for tracking purposes and do not affect the signs. Note that the horizontal line in the middle of the matrix is used only as a visualization aid, separating the matrix to top and bottom blocks.

Similarly,

$F^{\mu\nu}_{\mp} =$

$$\left(\begin{array}{cccc|cccc}
-\partial^0 A^0_- - \partial^i A^i_- - jm^0 A^0_- & \partial^1 A^0_- + \partial^0 A^1_- - jm^0 A^1_- & \partial^2 A^0_- + \partial^0 A^2_- - jm^0 A^2_- & \partial^3 A^0_- + \partial^0 A^3_- - jm^0 A^3_- & 0 & 0 & 0 & 0 \\
-\partial^1 A^0_- - \partial^0 A^1_- + jm^0 A^1_- & \partial^0 A^0_- + \partial^i A^i_- + jm^0 A^0_- & \partial^1 A^2_- - \partial^2 A^1_- & \partial^1 A^3_- - \partial^3 A^1_- & 0 & 0 & 0 & 0 \\
-\partial^2 A^0_- - \partial^0 A^2_- + jm^0 A^2_- & \partial^2 A^1_- - \partial^1 A^2_- & \partial^0 A^0_- + \partial^i A^i_- + jm^0 A^0_- & \partial^2 A^3_- - \partial^3 A^2_- & 0 & 0 & 0 & 0 \\
-\partial^3 A^0_- - \partial^0 A^3_- + jm^0 A^3_- & \partial^3 A^1_- - \partial^1 A^3_- & \partial^3 A^2_- - \partial^2 A^3_- & \partial^0 A^0_- + \partial^i A^i_- + jm^0 A^0_- & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & -\partial^0 A^0_+ - \partial^i A^i_+ + jm^0 A^0_+ & \partial^1 A^0_+ + \partial^0 A^1_+ + jm^0 A^1_+ & \partial^2 A^0_+ + \partial^0 A^2_+ + jm^0 A^2_+ & \partial^3 A^0_+ + \partial^0 A^3_+ + jm^0 A^3_+ \\
0 & 0 & 0 & 0 & -\partial^1 A^0_+ - \partial^0 A^1_+ - jm^0 A^1_+ & \partial^0 A^0_+ + \partial^i A^i_+ - jm^0 A^0_+ & \partial^1 A^2_+ - \partial^2 A^1_+ & \partial^1 A^3_+ - \partial^3 A^1_+ \\
0 & 0 & 0 & 0 & -\partial^2 A^0_+ - \partial^0 A^2_+ - jm^0 A^2_+ & \partial^2 A^1_+ - \partial^1 A^2_+ & \partial^0 A^0_+ + \partial^i A^i_+ - jm^0 A^0_+ & \partial^2 A^3_+ - \partial^3 A^2_+ \\
0 & 0 & 0 & 0 & -\partial^3 A^0_+ - \partial^0 A^3_+ - jm^0 A^3_+ & \partial^3 A^1_+ - \partial^1 A^3_+ & \partial^3 A^2_+ - \partial^2 A^3_+ & \partial^0 A^0_+ + \partial^i A^i_+ - jm^0 A^0_+
\end{array}\right) \tag{72}$$

which can be written as:

$$F^{\mu\nu}_{\mp} = \left(\begin{array}{c|c} \partial^\mu A^\nu_- - \partial^\nu A^\mu_- & 0_{4\times4} \\ \hline 0_{4\times4} & \partial^\mu A^\nu_+ - \partial^\nu A^\mu_+ \end{array}\right) \mp \eta^{\mu\nu}_- \left(\begin{array}{c|c} \left(\partial^0 A^0_- - \partial^i A^i_- + jm^0 A^0_-\right) & 0_{4\times4} \\ \hline 0_{4\times4} & \left(\partial^0 A^0_+ - \partial^i A^i_+ - jm^0 A^0_+\right) \end{array}\right)$$
$$+jm^0 \left(\begin{array}{c|c} -\delta^{\mu0} A^\nu_- + \delta^{0\nu} A^\mu_- & 0_{4\times4} \\ \hline 0_{4\times4} & -\delta^{\mu0} A^\nu_+ + \delta^{0\nu} A^\mu_+ \end{array}\right)$$

$F^{\mp}_{\mu\nu}$ looks similar to $F^{\mu\nu}_{\mp}$ (only with lowered indices). In a more compact formulation:

$$F^{\mu\nu}_{\mp} = \left(\partial^\mu A^\nu_\mp - \partial^\nu A^\mu_\mp\right) \mp \eta^{\mu\nu}_- \left(\partial^0 A^0_\mp - \partial^i A^i_\mp \pm jm^0 A^0_\mp\right) - jm^0 \left(\delta^{\mu0} A^\nu_\mp - \delta^{0\nu} A^\mu_\mp\right) \tag{73}$$
$$F^{\mp}_{\mu\nu} = \left(\partial_\mu A^\mp_\nu - \partial_\nu A^\mp_\mu\right) \mp \eta^{\mu\nu}_- \left(\partial_0 A^\mp_0 - \partial_i A^\mp_i \pm jm_0 A^\mp_0\right) - jm_0 \left(\delta^{\mu0} A^\mp_\nu - \delta^{0\nu} A^\mp_\mu\right) \tag{74}$$

The transform of $F^{\mu\nu}_{\mp}$ is $F^{\nu\mu}_{\mp}$:

$$\begin{aligned} F^{\nu\mu}_{\mp} &= -\left(\partial^\mu A^\nu_\mp - \partial^\nu A^\mu_\mp\right) + jm^0 \left(\delta^{\mu0} A^\nu_\mp - \delta^{0\nu} A^\mu_\mp\right) \pm \eta^{\mu\nu} \left(\partial^0 A^0_\mp - \partial^i A^i_\mp \pm jm^0 A^0_\mp\right) \\ &= -F^{\mu\nu}_{\mp} + 2Diag(F^{\mu\nu}_{\mp}) \\ &= (2Diag - 1)\, F^{\mu\nu}_{\mp} \end{aligned} \tag{75}$$

The proposed Lagrangian would be:

$$\mathcal{L} = -F^{\mp}_{\mu\nu} F^{\mu\nu}_{\mp} \tag{76}$$

such that

$$\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu A^\mp_\nu\right)} = -\frac{\partial\left(F^{\mp}_{\mu\nu} F^{\mu\nu}_{\mp}\right)}{\partial\left(\partial_\mu A^\mp_\nu\right)} = \frac{\partial \mathcal{L}}{\partial\left(F^{\mp}_{\alpha\beta}\right)} \times \frac{\partial\left(F^{\mp}_{\alpha\beta}\right)}{\partial\left(\partial_\mu A^\mp_\nu\right)} \tag{77}$$

Examining the $\partial\left(\partial_\mu A^\mp_\nu\right)$ derivative -

$$\frac{\partial\left(F^{\mp}_{\alpha\beta}\right)}{\partial\left(\partial_\mu A^\mp_\nu\right)} = \left(\delta^\mu_\alpha \delta^\nu_\beta - \delta^\nu_\alpha \delta^\mu_\beta\right) \mp \eta^{\lambda\rho}_- \delta^\lambda_\alpha \delta^\rho_\beta \delta^\mu_\lambda \delta^\nu_\rho \left(\delta^\mu_0 \delta^\nu_0 - \delta^\mu_i \delta^\nu_i\right) \tag{78}$$

where $i \in (1, 2, 3)$

Additionally, it can be shown (product rule) that-

$$-\frac{\partial \mathcal{L}}{\partial\left(F^{\mp}_{\alpha\beta}\right)} = \frac{\partial\left(F^{\mp}_{\mu\nu} F^{\mu\nu}_{\mp}\right)}{\partial\left(F^{\mp}_{\alpha\beta}\right)} = 2F^{\alpha\beta}_{\mp} \tag{79}$$

Unifying the two last results for eq.77:

$$\begin{aligned} \frac{\partial\left(F^{\mp}_{\mu\nu} F^{\mu\nu}_{\mp}\right)}{\partial\left(\partial_\mu A^\mp_\nu\right)} &= \frac{-\partial \mathcal{L}}{\partial\left(F^{\mp}_{\alpha\beta}\right)} \times \frac{\partial\left(F^{\mp}_{\alpha\beta}\right)}{\partial\left(\partial_\mu A^\mp_\nu\right)} \\ &= 2F^{\alpha\beta}_{\mp} \times \Big[\left(\delta^\mu_\alpha \delta^\nu_\beta - \delta^\nu_\alpha \delta^\mu_\beta\right) \mp \\ &\qquad \mp \eta^{\lambda\rho}_- \delta^\lambda_\alpha \delta^\rho_\beta \delta^\mu_\lambda \delta^\nu_\rho \left(\delta^\mu_0 \delta^\nu_0 - \delta^\mu_i \delta^\nu_i\right)\Big] \\ &= 2\left[\left(F^{\mu\nu}_{\mp} - F^{\nu\mu}_{\mp}\right) \mp \left\{\eta^{\mu\nu}_- F^{\mu\nu}_{\mp} \left(\delta^\mu_0 \delta^\nu_0 - \delta^\mu_i \delta^\nu_i\right)\right\}\right] \end{aligned} \tag{80}$$

$$\frac{\partial\left(F^{\mp}_{\mu\nu} F^{\mu\nu}_{\mp}\right)}{\partial\left(\partial_\mu A^\mp_\nu\right)} = 2\left[\left(F^{\mu\nu}_{\mp} - F^{\nu\mu}_{\mp}\right) + 2Diag(F^{\mu\nu}_{\mp})\right] \tag{81}$$

where the multiplication by factor 2 in the diagonal at the last transition is because $\left(\delta^\mu_0 \delta^\nu_0 - \delta^\mu_i \delta^\nu_i\right) = -Tr(\eta^{\mu\nu}_-) = -2$.

Using the expression for $F^{\nu\mu}_{\mp}$ from eq.75 -

$$\begin{aligned} \frac{\partial\left(F^{\mp}_{\mu\nu} F^{\mu\nu}_{\mp}\right)}{\partial\left(\partial_\mu A^\mp_\nu\right)} &= 2\left\{\left[F^{\mu\nu}_{\mp} - (2Diag - 1)\, F^{\mu\nu}_{\mp}\right] + 2Diag(F^{\mu\nu}_{\mp})\right\} \\ &= 2\left\{(2 - 2Diag)\, F^{\mu\nu}_{\mp} + 2Diag(F^{\mu\nu}_{\mp})\right\} = 4F^{\mu\nu}_{\mp} \end{aligned} \tag{82}$$

Proceeding to the second component of the Euler-Lagrange equation:

$$\frac{\partial F^{\mp}_{\alpha\beta}}{\partial A^{\mp}_{\nu}} = -jm_0\left(\delta^{\nu}_{\beta}\delta^{0}_{\alpha} - \delta^{0}_{\beta}\delta^{\nu}_{\alpha}\right) - jm_0\delta^{\alpha}_{\lambda}\delta^{\beta}_{\rho}\delta^{0}_{\nu}\eta^{-}_{\lambda\rho}$$
$$= -jm_0\left[\delta^{\nu}_{\beta}\delta^{0}_{\alpha} - \delta^{0}_{\beta}\delta^{\nu}_{\alpha} + \delta^{\lambda}_{\alpha}\delta^{\rho}_{\beta}\delta^{0}_{\nu}\eta^{-}_{\lambda\rho}\right] \quad (83)$$

$$\frac{\partial\left(F^{\mp}_{\mu\nu}F^{\mu\nu}_{\mp}\right)}{\partial\left(A^{\mp}_{\nu}\right)} = -\frac{\partial\mathcal{L}}{\partial F^{\mp}_{\alpha\beta}} \times \frac{\partial F^{\mp}_{\alpha\beta}}{\partial A^{\mp}_{\nu}}$$
$$= \quad -2jm_0 F^{\alpha\beta}_{\mp} \times \left[\delta^{\nu}_{\beta}\delta^{0}_{\alpha} - \delta^{0}_{\beta}\delta^{\nu}_{\alpha} + \delta^{\lambda}_{\alpha}\delta^{\rho}_{\beta}\delta^{0}_{\nu}\eta^{-}_{\lambda\rho}\right]$$
$$\frac{\partial\left(F^{\mp}_{\mu\nu}F^{\mu\nu}_{\mp}\right)}{\partial\left(A^{\mp}_{\nu}\right)} = -2jm_0\left[F^{0\nu}_{\mp} - F^{\nu 0}_{\mp} + F^{\alpha\beta}_{\mp}\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta}\delta^{0}_{\nu}\eta^{-}_{\mu\nu}\right] \quad (84)$$

for $\nu = 0$:

$$\frac{\partial\left(F^{\mp}_{\mu 0}F^{\mu 0}_{\mp}\right)}{\partial\left(A^{\mp}_{\nu}\right)} = -2jm_0 F^{\alpha\beta}_{\mp}\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta}\delta^{0}_{\nu}\eta_{\mu\nu}$$
$$= -2jm_0 F^{\mu\nu}_{\mp}\delta^{0}_{\nu}\eta_{\mu\nu}$$
$$\frac{\partial\left(F^{\mp}_{\mu 0}F^{\mu 0}_{\mp}\right)}{\partial\left(A^{\mp}_{\nu}\right)} = 4jm_0 F^{00}_{\mp} \quad (85)$$

and for $\nu = 1, 2, 3$:

$$\frac{\partial\left(F^{\mp}_{\mu\nu}F^{\mu\nu}_{\mp}\right)}{\partial\left(A^{\mp}_{\nu}\right)} = 2jm\left(F^{\nu 0}_{\mp} - F^{0\nu}_{\mp}\right)$$
$$\frac{\partial\left(F^{\mp}_{\mu\nu}F^{\mu\nu}_{\mp}\right)}{\partial\left(A^{\mp}_{\nu}\right)} = 4jm_0 F^{\nu 0}_{\mp} \quad (86)$$

In the final step, the tensor anti-symmetry $F^{\nu 0}_{\mp} = -F^{0\nu}_{\mp}$ was used.

Verifying that the selected Lagrangian satisfies the Euler-Lagrange equation $\partial_{\nu}\left(\frac{\partial\mathcal{L}}{\partial(A_{\mu,\nu})}\right) = \frac{\partial\mathcal{L}}{\partial A_{\mu}}$:

$$\frac{\partial\mathcal{L}}{\partial A_{\mu}} = 4jm_0\begin{pmatrix} F^{00}_{\mp} \\ F^{10}_{\mp} \\ F^{20}_{\mp} \\ F^{30}_{\mp}\end{pmatrix} = -4jm_0\begin{pmatrix} -S^{-}_{0} \\ E^{+}_{x} \\ E^{+}_{y} \\ E^{+}_{z} \\ S^{+}_{0} \\ E^{-}_{x} \\ E^{-}_{y} \\ E^{-}_{z}\end{pmatrix}$$
$$\partial^{\mp}_{\nu}\left(\frac{\partial\mathcal{L}}{\partial(A_{\mu,\nu})}\right) = -\,4\partial^{\mp}_{\nu}F^{\mu\nu}_{\mp} = \underbrace{-4jm_0\begin{pmatrix}\mp S^{\mp}_{0} \\ E^{\pm}_{1} \\ E^{\pm}_{1} \\ E^{\pm}_{1}\end{pmatrix}}\; = \frac{\partial\mathcal{L}}{\partial A_{\mu}} \quad (87)$$

the central equation represents the Euler-Lagrange equation, however, it is not identical to the original equation that had the derivative $\partial^{\nu}_{\mp}$. Making the transitions:

$$\partial^{\mp}_{\nu} \to \partial^{\nu}_{\mp}\ ,\ m_0 \to m^0$$

Also, it's important to note that changes in the mass index only affect the last four lines (the $'+'$ section):

$$\partial^{\nu}_{\mp}\left(\frac{\partial\mathcal{L}}{\partial\left(A_{\mu,\nu}\right)}\right) = -4\partial^{\nu}_{\mp}F^{\mu\nu}_{\mp} = -4j\eta^{\mu\nu}_{\mp}m_0\begin{pmatrix}\mp S^{\mp}_{0} \\ E^{\pm}_{1} \\ E^{\pm}_{1} \\ E^{\pm}_{1}\end{pmatrix}$$
$$-4\partial^{\nu}_{\mp}F^{\mu\nu}_{\mp} = -4jm^0\begin{pmatrix}\pm S^{\mp}_{0} \\ E^{\pm}_{1} \\ E^{\pm}_{1} \\ E^{\pm}_{1}\end{pmatrix} \quad (88)$$

subtracting the $'-4'$ factor from both sides of the equation:

$$\partial^{\nu}_{\mp}F^{\mu\nu}_{\mp} = jm^0\begin{pmatrix}\pm S^{\mp}_{0} \\ E^{\pm}_{1} \\ E^{\pm}_{1} \\ E^{\pm}_{1}\end{pmatrix}$$

which is equivalent to the original field equation (eq.60). Hence, the Euler-Lagrange equation is satisfied with the Lagrangian $L = -F^{\mp}_{\mu\nu}F^{\mu\nu}_{\mp}$. where

$$F^{\mu\nu}_{\mp} = \left(\partial^{\mu}A^{\nu}_{\mp} - \partial^{\nu}A^{\mu}_{\mp}\right) \mp \eta^{\mu\nu}_{-}\left(\partial^{0}A^{0}_{\mp} - \partial^{i}A^{i}_{\mp} \pm jm^0 A^{0}_{\mp}\right) - -jm^0\left(\delta^{\mu 0}A^{\nu}_{\mp} - \delta^{0\nu}A^{\mu}_{\mp}\right)$$

Therefore, using eq.60 and the conjugation relation between $F^{\mp}_{\mu\nu}$ and $F^{\mu\nu}_{\mp}$, the field expression for the Lagrangian is given by:

$$\mathcal{L} = \mathcal{L}^{-} + \mathcal{L}^{+} = -F^{-}_{\mu\nu}F^{\mu\nu}_{-} - F^{+}_{\mu\nu}F^{\mu\nu}_{+} \quad (89)$$
$$= 2\left(\underbrace{|E^{+}|^2 - |B^{-}|^2 + |S^{-}_{0}|^2}_{\frac{1}{2}\mathcal{L}^{-}} + \underbrace{|E^{-}|^2 - |B^{+}|^2 - |S^{+}_{0}|^2}_{\frac{1}{2}\mathcal{L}^{+}}\right)$$

Thus, the scalar fields S^{-}_{0} and S^{+}_{0} contribute the overall Lagrangian of system in anti-symmetric manner. Specifically, S^{-}_{0} makes a positive contribution to $\mathcal{L}^{-}$, while S^{+}_{0} has a negative contribution to $\mathcal{L}^{+}$.

Next, $F^{\mu\nu}_{\mp}$ is to be verified to transform as a tensor within the General Relativity framework [4]:

$$\overline{F}^{\alpha\beta}_{\mp} = \left(\frac{\partial \overline{A}^{\beta}_{\mp}}{\partial \overline{x}_{\alpha}} - \frac{\partial \overline{A}^{\alpha}_{\mp}}{\partial x_{\beta}}\right) \mp \overline{\eta}^{\alpha\beta}_{-}\left(\frac{\partial \overline{A}^{0}_{\mp}}{\partial \overline{x}_{0}} - \frac{\partial \overline{A}^{i}_{\mp}}{\partial \overline{x}_{i}} \pm jm^0\overline{A}^{0}_{\mp}\right) - jm^0\left(\overline{\delta}^{\alpha 0}\overline{A}^{\beta}_{\mp} - \overline{\delta}^{0\beta}\overline{A}^{\alpha}_{\mp}\right)$$

$$\overline{F}^{\alpha\beta}_{\mp} = \left[\frac{\partial}{\partial \overline{x}_{\alpha}}\left(\frac{\partial x_{\gamma}}{\partial \overline{x}_{\beta}}A^{\gamma}_{\mp}\right) - \frac{\partial}{\partial \overline{x}_{\beta}}\left(\frac{\partial \overline{x}_{\delta}}{\partial \overline{x}_{\alpha}}A^{\delta}_{\mp}\right)\right] \mp$$

$$\mp \eta^{\delta\gamma}_{-}\frac{\partial x_{\delta}}{\partial \overline{x}_{\alpha}}\frac{\partial x_{\gamma}}{\partial \overline{x}_{\beta}}\left[\frac{\partial \overline{A}^{0}_{\mp}}{\partial \overline{x}_{0}} - \frac{\partial \overline{A}^{i}_{\mp}}{\partial \overline{x}_{i}} \pm jm^0\overline{A}^{0}_{\mp}\right] - jm^0\left[\frac{\partial x_{\delta}}{\partial \overline{x}_{\alpha}}\delta^{\delta 0}\left(\frac{\partial x_{\gamma}}{\partial \overline{x}_{\beta}}A^{\gamma}_{\mp}\right) - \frac{\partial x_{\gamma}}{\partial \overline{x}_{\beta}}\delta^{0\gamma}\left(\frac{\partial x_{\delta}}{\partial \overline{x}_{\alpha}}A^{\delta}_{\mp}\right)\right]$$

$$= \left[\frac{\partial^2 x_{\gamma}}{\partial \overline{x}_{\alpha}\partial \overline{x}_{\beta}}A^{\gamma}_{\mp} + \frac{\partial x_{\gamma}}{\partial \overline{x}_{\beta}}\frac{\partial A^{\gamma}_{\mp}}{\partial \overline{x}_{\alpha}} - \frac{\partial^2 x_{\delta}}{\partial \overline{x}_{\beta}\partial \overline{x}_{\alpha}}A^{\delta}_{\mp} - \frac{\partial x_{\delta}}{\partial \overline{x}_{\alpha}}\frac{\partial A^{\delta}_{\mp}}{\partial \overline{x}_{\beta}}\right] \mp$$

$$\mp \eta^{\delta\gamma}_{-}\frac{\partial x_{\delta}}{\partial \overline{x}_{\alpha}}\frac{\partial x_{\gamma}}{\partial \overline{x}_{\beta}}\left[\left(\frac{\partial}{\partial \overline{x}_{0}} \pm jm^0\right)\overline{A}^{0}_{\mp} - \frac{\partial}{\partial \overline{x}_{i}}\overline{A}^{i}_{\mp}\right] - jm^0\left[\frac{\partial x_{\delta}}{\partial \overline{x}_{\alpha}}\delta^{\delta 0}\left(\frac{\partial x_{\gamma}}{\partial \overline{x}_{\beta}}A^{\gamma}_{\mp}\right) - \frac{\partial x_{\gamma}}{\partial \overline{x}_{\beta}}\delta^{0\gamma}\left(\frac{\partial x_{\delta}}{\partial \overline{x}_{\alpha}}A^{\delta}_{\mp}\right)\right]$$

$$= \left[\frac{\partial x_{\gamma}}{\partial \overline{x}_{\beta}}\frac{\partial x_{\delta}}{\partial \overline{x}_{\alpha}}\frac{\partial A^{\gamma}_{\mp}}{\partial x_{\delta}} - \frac{\partial x_{\delta}}{\partial \overline{x}_{\alpha}}\frac{\partial x_{\gamma}}{\partial \overline{x}_{\beta}}\frac{\partial A^{\delta}_{\mp}}{\partial x_{\gamma}}\right] \mp$$

$$\mp \eta^{\delta\gamma}_{-}\frac{\partial x_{\delta}}{\partial \overline{x}_{\alpha}}\frac{\partial x_{\gamma}}{\partial \overline{x}_{\beta}}\left[\left(\frac{\partial}{\partial \overline{x}_{0}} \pm jm^0,\ -\frac{\partial}{\partial \overline{x}_{i}}\right)\cdot\left(\overline{A}^{0}_{\mp},\ \overline{A}^{i}_{\mp}\right)\right] - jm^0\left[\frac{\partial x_{\delta}}{\partial \overline{x}_{\alpha}}\frac{\partial x_{\gamma}}{\partial \overline{x}_{\beta}}\delta^{\delta 0}\left(A^{\gamma}_{\mp}\right) - \frac{\partial x_{\gamma}}{\partial \overline{x}_{\beta}}\frac{\partial x_{\delta}}{\partial \overline{x}_{\alpha}}\delta^{0\gamma}\left(A^{\delta}_{\mp}\right)\right]$$

$$= \frac{\partial x_{\gamma}}{\partial \overline{x}_{\beta}}\frac{\partial x_{\delta}}{\partial \overline{x}_{\alpha}}\left[\frac{\partial A^{\gamma}_{\mp}}{\partial x_{\delta}} - \frac{\partial A^{\delta}_{\mp}}{\partial x_{\gamma}}\right] \mp$$

$$\mp \eta^{\delta\gamma}_{-}\frac{\partial x_{\delta}}{\partial \overline{x}_{\alpha}}\frac{\partial x_{\gamma}}{\partial \overline{x}_{\beta}}\left[\left(\frac{\partial}{\partial x_{0}} \pm jm^0,\ -\frac{\partial}{\partial x_{i}}\right)\cdot\left(A^{0}_{\mp},\ A^{i}_{\mp}\right)\right] - jm^0\frac{\partial x_{\delta}}{\partial \overline{x}_{\alpha}}\frac{\partial x_{\gamma}}{\partial \overline{x}_{\beta}}\left[\delta^{\delta 0}\left(A^{\gamma}_{\mp}\right) - \delta^{0\gamma}\left(A^{\delta}_{\mp}\right)\right]$$

$$\overline{F}^{\alpha\beta}_{\mp} = \frac{\partial x_{\gamma}}{\partial \overline{x}_{\beta}}\frac{\partial x_{\delta}}{\partial \overline{x}_{\alpha}}\left\{\left(\frac{\partial A^{\gamma}_{\mp}}{\partial x_{\delta}} - \frac{\partial A^{\delta}_{\mp}}{\partial x_{\gamma}}\right) \mp \eta^{\delta\gamma}_{-}\left[\left(\frac{\partial}{\partial x_{0}} \pm jm^0,\ -\frac{\partial}{\partial x_{i}}\right)\cdot\left(A^{0}_{\mp},\ A^{i}_{\mp}\right)\right] - jm^0\left[\delta^{\delta 0}\left(A^{\gamma}_{\mp}\right) - \delta^{0\gamma}\left(A^{\delta}_{\mp}\right)\right]\right\}$$

$$\overline{F}^{\alpha\beta}_{\mp} = \frac{\partial x_{\delta}}{\partial \overline{x}_{\alpha}}\frac{\partial x_{\gamma}}{\partial \overline{x}_{\beta}}F^{\delta\gamma}_{\mp} \tag{90}$$

Where the term $\mp\eta_{-}\left(\frac{\partial}{\partial x_0} \pm jm^0,\ -\frac{\partial}{\partial x_i}\right)\cdot\left(A^{0}_{\mp},\ A^{i}_{\mp}\right)$ is $S^{\mp}_{0}$ fields which are invariant under coordinate transform as they are scalar fields.

Consequently, $F^{\alpha\beta}_{\mp}$ transforms as a tensor under a coordinate change, indicating that $F^{\alpha\beta}_{\mp}$ aligns with General Relativity framework as a second rank tensor.

Another noteworthy point is that the expression $\frac{\partial \mathcal{L}}{\partial A_{\mu}}$ which is equivalent to the 4-current density (or the source of the fields) is essentially the fields themselves, multiplied by positive and negative 'jm' factors.

D. Stress-Energy Tensor

An extended stress-energy tensor can be defined as follows [5]:

$$T^{\mu\nu}_{\mp} = \pm F^{\mu\alpha}_{\mp}g^{\mp}_{\alpha\beta}F^{\nu\beta}_{\mp} \mp \frac{1}{4}g^{\mu\nu}_{\mp}F^{\mp}_{\lambda\rho}F^{\lambda\rho}_{\mp} \tag{91}$$

where the $\pm$ sign on the first term and $\mp$ on the second, arise due to the signature difference between the upper and lower blocks of $F_{\mp}$. Additionally, $g^{\mu\nu}_{\mp}$ is the curved spacetime metric tensor extended to $8x8$ tensor by $g^{\mu\nu}_{\mp} = \begin{pmatrix} g^{\mu\nu}_{-} & 0_{4\times 4} \\ 0_{4\times 4} & g^{\mu\nu}_{+} \end{pmatrix}$ where like in the case of $\eta^{\mu\nu}_{\mp}$, each matrix block describes the same curvature but in a different space-time signature.

Writing eq.91 in Minkowski spacetime:

$$T^{\mu\nu}_{\mp} = \pm F^{\mu\lambda}_{\mp}F^{\nu}_{\mp\lambda} \mp \frac{1}{4}\eta^{\mu\nu}_{\mp}F^{\mp}_{\lambda\rho}F^{\lambda\rho}_{\mp} \tag{92}$$

Using the symmetry of $F_{\mp}^{\mu\lambda}$, the left term $F_{\mp}^{\mu\lambda}F_{\mp\lambda}^{\nu}$ can be calculated by a matrix multiplication:

$$F_{\mp}^{\mu\lambda}F_{\mp\lambda}^{\nu}=\begin{pmatrix} S_0^- & -E_x^+ & -E_y^+ & -E_z^+ & 0 & 0 & 0 & 0 \\ E_x^+ & -S_0^- & -cB_z^- & cB_y^- & 0 & 0 & 0 & 0 \\ E_y^+ & cB_z^- & -S_0^- & -cB_x^- & 0 & 0 & 0 & 0 \\ E_z^+ & -cB_y^- & cB_x^- & -S_0^- & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -S_0^+ & -E_x^- & -E_y^- & -E_z^- \\ 0 & 0 & 0 & 0 & E_x^- & S_0^+ & -cB_z^+ & cB_y^+ \\ 0 & 0 & 0 & 0 & E_y^- & cB_z^+ & S_0^+ & -cB_x^+ \\ 0 & 0 & 0 & 0 & E_z^- & -cB_y^+ & cB_x^+ & S_0^+ \end{pmatrix}\begin{pmatrix} -S_0^- & -E_x^+ & -E_y^+ & -E_z^+ & 0 & 0 & 0 & 0 \\ -E_x^+ & -S_0^- & cB_z^- & -cB_y^- & 0 & 0 & 0 & 0 \\ -E_y^+ & -cB_z^- & -S_0^- & cB_x^- & 0 & 0 & 0 & 0 \\ -E_z^+ & cB_y^- & -cB_x^- & -S_0^- & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -S_0^+ & E_x^- & E_y^- & E_z^- \\ 0 & 0 & 0 & 0 & E_x^- & -S_0^+ & -cB_z^+ & cB_y^+ \\ 0 & 0 & 0 & 0 & E_y^- & cB_z^+ & -S_0^+ & -cB_x^+ \\ 0 & 0 & 0 & 0 & E_z^- & -cB_y^+ & cB_x^+ & -S_0^+ \end{pmatrix}^*$$

$$F_{\mp}^{\mu\lambda}F_{\mp\lambda}^{\nu}=$$

$$=\left(\begin{array}{cccc} \left|E^+\right|^2-\left|S_0^-\right|^2 & cB_z^-E_y^+-cB_y^-E_z^+ & cB_x^-E_z^+-cB_z^-E_x^+ & cB_y^-E_x^+-cB_x^-E_y^+ \\ cB_z^-E_y^+-cB_y^-E_z^+ & \left|S_0^-\right|^2-\left|E_x^+\right|^2+\left|cB_z^-\right|^2+\left|cB_y^-\right|^2 & -E_x^+E_y^+-cB_x^-cB_y^- & -E_x^+E_z^+-cB_x^-cB_z^- \\ cB_x^-E_z^+-cB_z^-E_x^+ & -E_y^+E_x^+-cB_x^-cB_y^- & \left|cB_z^-\right|^2+\left|cB_x^-\right|^2-\left|E_y^+\right|^2+\left|S_0^-\right|^2 & -E_y^+E_z^+-cB_z^-cB_y^- \\ cB_y^-E_x^+-cB_x^-E_y^+ & -E_z^+E_x^+-cB_z^-cB_x^- & -E_z^+E_y^+-cB_y^-cB_z^- & \left|cB_y^-\right|^2+\left|cB_x^-\right|^2-\left|E_z^+\right|^2+\left|S_0^-\right|^2 \\ \hline \left|S_0^+\right|^2-\left|E^-\right|^2 & cB_y^+E_z^--cB_z^+E_y^- & cB_z^+E_x^--cB_x^+E_z^- & cB_x^+E_y^--cB_y^+E_x^- \\ cB_y^+E_z^--cB_z^+E_y^- & \left|E_x^-\right|^2-\left|S_0^+\right|^2-\left|cB_z^+\right|^2-\left|cB_y^+\right|^2 & E_x^-E_y^-+cB_x^+cB_y^+ & E_x^-E_z^-+cB_x^+cB_z^+ \\ cB_z^+E_x^--cB_x^+E_z^- & E_x^-E_y^-+cB_x^+cB_y^+ & \left|E_y^-\right|^2-\left|S_0^+\right|^2-\left|cB_z^+\right|^2-\left|cB_x^+\right|^2 & E_y^-E_z^-+cB_y^+cB_z^+ \\ cB_x^+E_y^--cB_y^+E_x^- & E_x^-E_z^-+cB_x^+cB_z^+ & E_y^-E_z^-+cB_y^+cB_z^+ & \left|E_z^-\right|^2-\left|S_0^+\right|^2-\left|cB_x^+\right|^2-\left|cB_y^+\right|^2 \end{array}\right)$$

Due to page boundaries constrains, the 8x8 block tensor is presented in a condensed format where the upper four rows correspond to the upper (left) block and the lower 4 rows correspond to the lower (right) block.

The right term in eq.92 can be easily calculated to yield:

$$\frac{1}{4}\eta_{\mp}^{\mu\nu}F_{\lambda\rho}^{\mp}F_{\mp}^{\lambda\rho}=\eta_{\mp}^{\mu\nu}\left(\left|S_0^{\mp}\right|^2-\frac{1}{2}\left|E^{\pm}\right|^2+\frac{1}{2}\left|cB^{\mp}\right|^2\right) \tag{93}$$

such that

$$\pm F_{\mp}^{\mu\lambda}F_{\mp\lambda}^{\nu}\mp\frac{1}{4}\eta_{\mp}^{\mu\nu}F_{\lambda\rho}^{\mp}F_{\mp}^{\lambda\rho}=$$

$$=\left(\begin{array}{cccc} \frac{1}{2}\left(c^2B_-^2+E_+^2\right) & cB_z^-E_y^+-cB_y^-E_z^+ & cB_x^-E_z^+-cB_z^-E_x^+ & cB_y^-E_x^+-cB_x^-E_y^+ \\ cB_z^-E_y^+-cB_y^-E_z^+ & \frac{1}{2}\left(c^2B_-^2+E_+^2\right)-\left|cB_x^-\right|^2-\left|E_x^+\right|^2 & -E_x^+E_y^+-cB_x^-cB_y^- & -E_x^+E_z^+-cB_x^-cB_z^- \\ cB_x^-E_z^+-cB_z^-E_x^+ & -E_y^+E_x^+-cB_x^-cB_y^- & \frac{1}{2}\left(c^2B_-^2+E_+^2\right)-\left|cB_x^-\right|^2-\left|E_y^+\right|^2 & -E_y^+E_z^+-cB_z^-cB_y^- \\ cB_y^-E_x^+-cB_x^-E_y^+ & -E_z^+E_x^+-cB_z^-cB_x^- & -E_z^+E_y^+-cB_y^-cB_z^- & \frac{1}{2}\left(c^2B_-^2+E_+^2\right)-\left|cB_z^-\right|^2-\left|E_z^+\right|^2 \\ \hline \frac{1}{2}\left(c^2B_+^2+E_-^2\right) & cB_z^+E_y^--cB_y^+E_z^- & cB_x^+E_z^--cB_z^+E_x^- & cB_y^+E_x^--cB_x^+E_y^- \\ cB_z^+E_y^--cB_y^+E_z^- & \frac{1}{2}\left(c^2B_+^2+E_-^2\right)-\left|cB_x^+\right|^2-\left|E_x^-\right|^2 & -E_x^-E_y^--cB_x^+cB_y^+ & -E_x^-E_z^--cB_x^+cB_z^+ \\ cB_x^+E_z^--cB_z^+E_x^- & -E_y^-E_x^--cB_x^+cB_y^+ & \frac{1}{2}\left(c^2B_+^2+E_-^2\right)-\left|cB_x^+\right|^2-\left|E_y^-\right|^2 & -E_y^-E_z^--cB_z^+cB_y^+ \\ cB_y^+E_x^--cB_x^+E_y^- & -E_z^-E_x^--cB_z^+cB_x^+ & -E_z^-E_y^--cB_y^+cB_z^+ & \frac{1}{2}\left(c^2B_+^2+E_-^2\right)-\left|cB_z^+\right|^2-\left|E_z^-\right|^2 \end{array}\right)$$

where the square terms inside the brackets are square of the absolute values.

Therefore, the tensor in eq.92 is similar to the 'classic' electromagnetic tensor, specifically, the scalar fields $S_0^{\mp}$ subtracts and do not appear in the final result.

Eq.92 turns to :

$$T_{\mp}^{\mu\nu}=\begin{pmatrix} T_-^{\mu\nu} & 0_{4\times4} \\ 0_{4\times4} & T_+^{\mu\nu} \end{pmatrix}$$

where $T^{\mu\nu}$is the 'standard' electromagnetic stress-energy tensor:

$$T_-^{\mu\nu}=\begin{pmatrix} \frac{1}{2}\left(\left|cB_-\right|^2+\left|E_+\right|^2\right) & S_x^- & S_y^- & S_z^- \\ S_x^- & -\sigma_{xx}^- & -\sigma_{xy}^- & -\sigma_{xz}^- \\ S_y^- & -\sigma_{yx}^- & -\sigma_{yy}^- & -\sigma_{yz}^- \\ S_z^- & -\sigma_{zx}^- & -\sigma_{zy}^- & -\sigma_{zz}^- \end{pmatrix}$$

$$T_{+}^{\mu\nu}=\begin{pmatrix} \frac{1}{2}\left(\left|cB_{+}\right|^{2}+\left|E_{-}\right|^{2}\right) & S_{x}^{+} & S_{y}^{+} & S_{z}^{+}\\ S_{x}^{+} & -\sigma_{xx}^{+} & -\sigma_{xy}^{+} & -\sigma_{xz}^{+}\\ S_{y}^{+} & -\sigma_{yx}^{+} & -\sigma_{yy}^{+} & -\sigma_{yz}^{+}\\ S_{z}^{+} & -\sigma_{zx}^{+} & -\sigma_{zy}^{+} & -\sigma_{zz}^{+} \end{pmatrix}$$

and $\boldsymbol{S}^{\mp}=\boldsymbol{E}^{\mp}\times c\boldsymbol{B}^{\pm}$ corresponds to the Poynting vector. Additionally,

$$\sigma_{ij}^{\mp}=E_{i}^{\pm}E_{j}^{\pm}+c^{2}B_{i}^{\mp}B_{j}^{\mp}-\frac{1}{2}\delta_{ij}\left(\left|\boldsymbol{E}^{\pm}\right|^{2}+c^{2}\left|\boldsymbol{B}^{\mp}\right|^{2}\right)$$

corresponds to the Maxwell stress tensor.

Given that there is only one 4-dimensional spacetime, $T_{-}^{\mu\nu}$ and $T_{+}^{\mu\nu}$ need to be consolidated into a single 4-dimensional stress-energy tensor. There are two options for this consolidation each yielding different result. The first option is to add all the squared field terms linearly such that the phase differences between the fields in $T_{-}^{\mu\nu}$ and $T_{+}^{\mu\nu}$ are ignored. The second option is to add the field terms before squaring them in the energy terms or multiply them in the $S_{i}^{\mp}$and σ_{ij} terms. Since the phase difference between the $'+'$ and $'-'$ field sets plays a role in determine the spin orientation, as described in a previous section, and the spin orientation has no energy contribution for a free particle, the first consolidation option seems more appropriate. Hence the corresponding 4 dimensional stress-energy tensor is-

$$T^{\mu\nu}=T_{-}^{\mu\nu}+T_{+}^{\mu\nu}=\begin{pmatrix} \frac{1}{2}\left(\left|cB_{-}\right|^{2}+\left|E_{+}\right|^{2}+\left|cB_{+}\right|^{2}+\left|E_{-}\right|^{2}\right) & S_{x}^{-}+S_{x}^{+} & S_{y}^{-}+S_{y}^{+} & S_{z}^{-}+S_{z}^{+}\\ S_{x}^{-}+S_{x}^{+} & -\sigma_{xx}^{-}-\sigma_{xx}^{+} & -\sigma_{xy}^{-}-\sigma_{xy}^{+} & -\sigma_{xz}^{-}-\sigma_{xz}^{+}\\ S_{y}^{-}+S_{y}^{+} & -\sigma_{yx}^{-}-\sigma_{yx}^{+} & -\sigma_{yy}^{-}-\sigma_{yy}^{+} & -\sigma_{yz}^{-}-\sigma_{yz}^{+}\\ S_{z}^{-}+S_{z}^{+} & -\sigma_{zx}^{-}-\sigma_{zx}^{+} & -\sigma_{zy}^{-}-\sigma_{zy}^{+} & -\sigma_{zz}^{-}-\sigma_{zz}^{+} \end{pmatrix}$$

By following the previous procedure using the definition from eq.91 for a curved spacetime, it is possible to incorporate the fermionic field stress-energy tensor into the Einstein field equation (up to units conversion)-

$$R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}=\frac{8\pi G}{c^{4}}T_{\mu\nu}\tag{94}$$

E. Local U(1) Symmetry

Consider the Lagrangian in terms of the fields and investigate its behavior under local U(1) transformation [6]. According to eq.93:

$$-\mathcal{L}^{\mp}=F_{\mu\nu}^{\mp}F_{\mp}^{\mu\nu}=4\left(\left|S_{0}^{\mp}\right|^{2}-\frac{1}{2}\left|E^{\pm}\right|^{2}+\frac{1}{2}\left|cB^{\mp}\right|^{2}\right)\tag{95}$$

Since only the absolute value of the fields exist in the Lagrangian, it is indifferent to the field's phases, therefore, it holds U(1) symmetry at the fields level. Hence there is no need for a gauge field.

Next let's check the electromagnetic tensor (potential level) under local U(1) transformation-

$$\acute{F}_{\mp}^{\mu\nu}=\left(\partial^{\mu}\acute{A}_{\mp}^{\nu}-\partial^{\nu}\acute{A}_{\mp}^{\mu}\right)\mp\eta_{-}^{\mu\nu}\left(\partial^{0}\acute{A}_{\mp}^{0}-\partial^{i}\acute{A}_{\mp}^{i}\pm jm^{0}\acute{A}_{\mp}^{0}\right)-jm^{0}\left(\delta^{\mu0}\acute{A}_{\mp}^{\nu}-\delta^{0\nu}\acute{A}_{\mp}^{\mu}\right)\tag{96}$$

grouping terms :

$$\acute{F}_{\mp}^{\mu\nu}=\left[\left(\partial^{\mu}-jm^{0}\delta^{\mu0}\right)\acute{A}_{\mp}^{\nu}-\left(\partial^{\nu}-jm^{0}\delta^{0\nu}\right)\acute{A}_{\mp}^{\mu}\right]\mp\eta_{-}^{\mu\nu}\left[\left(\partial^{0}\pm jm^{0}\right)\acute{A}_{\mp}^{0}-\partial^{i}\acute{A}_{\mp}^{i}\right]\tag{97}$$

under the transformation (same gauge transformation presented in eq.28-29):

$$A^{\mu}\rightarrow\acute{A}^{\mu}=A^{\mu}+\left(\partial^{\mu}-jm\delta^{\mu0}\right)\chi(x)\tag{98}$$

Starting with the first term of eq.97:

$$\begin{aligned}&\left(\partial^{\mu}-jm^{0}\delta^{\mu0}\right)\acute{A}_{\mp}^{\nu}-\left(\partial^{\nu}-jm^{0}\delta^{\nu0}\right)\acute{A}_{\mp}^{\mu}=\\&=\left(\partial^{\mu}-jm^{0}\delta^{\mu0}\right)\left[A_{\mp}^{\nu}+\left(\partial^{\nu}-jm\delta^{0\nu}\right)\chi(x)\right]-\\&\quad-\left(\partial^{\nu}-jm^{0}\delta^{0\nu}\right)\left[A^{\mu}+\left(\partial^{\mu}-jm\delta^{\mu0}\right)\chi(x)\right]\\&=\left(\partial^{\mu}-jm^{0}\delta^{\mu0}\right)A_{\mp}^{\nu}-\left(\partial^{\nu}-jm^{0}\delta^{0\nu}\right)A_{\mp}^{\mu}+\\&\quad+\left(\partial^{\mu}-jm^{0}\delta^{\mu0}\right)\left[\left(\partial^{\nu}-jm\delta^{0\nu}\right)\chi(x)\right]-\\&\quad-\left(\partial^{\nu}-jm^{0}\delta^{0\nu}\right)\left[\left(\partial^{\mu}-jm\delta^{\mu0}\right)\chi(x)\right]\\&=\left(\partial^{\mu}-jm^{0}\delta^{\mu0}\right)A_{\mp}^{\nu}-\left(\partial^{\nu}-jm^{0}\delta^{0\nu}\right)A_{\mp}^{\mu}\end{aligned}\tag{99}$$

where the last transition used to the commutation relation:

$$\left[\left(\partial^\mu - jm^0\delta^{\mu 0}\right), \left(\partial^\nu - jm\delta^{0\nu}\right)\right] = 0 \tag{100}$$

Therefore, according to eq.99 the first term of the electromagnetic tensor in eq.97 is invariant. The second term of eq.97 is to be investigated next-

$$\begin{aligned}
&\left(\partial^0 \pm jm^0\right) \acute{A}^0_\mp - \partial^i \acute{A}^i_\mp = \\
&\quad = \left(\partial^0 \pm jm^0\right)\left[A^0 + \left(\partial^0 - jm\delta^{00}\right)\chi(x)\right] - \\
&\qquad -\partial^i\left[A^i_\mp + \left(\partial^i - jm\delta^{i0}\right)\chi(x)\right] \\
&\quad = \left(\partial^0 \pm jm^0\right)\left[A^0 + \left(\partial^0 - jm\right)\chi(x)\right] - \\
&\qquad -\partial^i\left[A^i_\mp + \partial^i\chi(x)\right] \\
&\quad = \left(\partial^0 \pm jm^0\right) A^0_\mp - \partial^i A^i_\mp + \\
&\qquad + \left(\partial^0 \pm jm^0\right)\left(\partial^0 - jm\right)\chi(x) - \partial^i\partial^i\chi(x)
\end{aligned} \tag{101}$$

using the argument given in eq.70, the term $\pm jm^0$ is equal to $+jm$, hence-

$$\begin{aligned}
&\left(\partial^0 \pm jm^0\right) \acute{A}^0_\mp - \partial^i \acute{A}^i_\mp = \\
&\quad = \left(\partial^0 \pm jm^0\right) A^0_\mp - \partial^i A^i_\mp + \\
&\qquad + \left(\partial^0 + jm\right)\left(\partial^0 - jm\right)\chi(x) - \partial^i\partial^i\chi(x) \\
&\quad = \left(\partial^0 \pm jm^0\right) A^0_\mp - \partial^i A^i_\mp + \\
&\qquad + \left(\partial^0\partial^0 + m^2\right)\chi(x) - \partial^i\partial^i\chi(x)
\end{aligned} \tag{102}$$

Therefore, the electromagnetic tensor is invariant under local U(1) transformation if the last two terms of eq.102 cancel each other. Thus the transformation field $\chi(x)$ needs to satisfy:

$$\begin{aligned}
\left(\partial^0\partial^0 + m^2 - \partial^i\partial^i\right)\chi(x) = 0 \\
\left(\partial_{tt} + m^2 - \nabla^2\right)\chi(x) = 0
\end{aligned} \tag{103}$$

which is the mass shell condition. Hence, given that the transformation field $\chi(x)$ is a massive field with the same mass as the fermionic field, the extended electromagnetic tensor and hence the electromagnetic Lagrangian are both invariant under the transformation described by eq.98. Therefore, unlike the case of the Dirac Lagrangian, no additional gauge field is required to be added to maintain the symmetry, as long as the transformation is of the form of eq.98 and satisfies eq.103. Note that if instead using eq.98, one would use the classical mass-less Lorentz gauge condition, equations of motion would be invariant thought the Lagrangian would not be invariant, as shown in SubSec.II A.

It is important to note here that the field equation of motions can be similarly formulated by Dirac equation framework as described in Sec.III and that the Dirac-like Lagrangian with the 8×8 gamma matrices also describes the same dynamics, but it needs an additional (electromagnetic) gauge field in order to maintain local U(1) symmetry. Thus, symmetry-wise, the Lagrangian suggested in eq.76 and eq.95 is a better framework to work with.

IV. SUPPORTING FORMALISM FOR SPIN 0, 1/2, 1

The compatibility of the above formalism to the Dirac equation was widely discussed in Sec.III and up until now we considered it as description of the fermionic field (spin 1/2). It was also shown in Sec.I that $S_0^\pm$ fields (and every vector field component) satisfy the Klein-Gordon equation (spin 0) just as the components of Dirac's bi-spinors. Next, it will be shown that the Proca equation and Maxwell equations are both degenerate cases of eq.60 (or eq.1).

The Proca equation

Degenerate the potentials $\boldsymbol{A}^\pm$,$\phi^\pm$ and the scalar fields $S_0^\pm$ as follow:

$$\begin{aligned}
S_0^+ = S_0^- \equiv S_0 \\
\phi_0^+ = \phi_0^- \equiv \phi_0 \\
\boldsymbol{A}^+ = \boldsymbol{A}^- \equiv \boldsymbol{A}
\end{aligned} \tag{104}$$

Summing eq.14 with eq.15 while applying the degeneracy described by eq.104 and divide the equation by factor of 2 yields:

$$S_0 = j\tfrac{mc}{\hbar}\phi_0 \tag{105}$$

Subtracting eq.7 and eq.8 while defining yields:

$$\boldsymbol{E}^- - \boldsymbol{E}^+ = 2j\frac{mc^2}{\hbar}\boldsymbol{A} \tag{106}$$

Next, the top and bottom parts of eq.4 are to be summed while using the first row of eq.104 :

$$\begin{aligned}
&\nabla \cdot \left(\boldsymbol{B}^+ + \boldsymbol{B}^-\right) = 0 \\
&\nabla \cdot \left(\boldsymbol{E}^- + \boldsymbol{E}^+\right) = 2j\frac{mc}{\hbar}S_0 \\
&c\nabla \times \left(\boldsymbol{B}^+ + \boldsymbol{B}^-\right) = \frac{1}{c}\partial_t\left(\boldsymbol{E}^- + \boldsymbol{E}^+\right) + j\frac{mc}{\hbar}\left(\boldsymbol{E}^- - \boldsymbol{E}^+\right) \\
&\nabla \times \left(\boldsymbol{E}^- + \boldsymbol{E}^+\right) = j\frac{mc^2}{\hbar}\left(\boldsymbol{B}^+ - \boldsymbol{B}^-\right) - \partial_t\left(\boldsymbol{B}^+ + \boldsymbol{B}^-\right)
\end{aligned}$$

Defining $\boldsymbol{2B} \equiv \boldsymbol{B}^+ + \boldsymbol{B}^-$ and $\boldsymbol{2E} \equiv \boldsymbol{E}^- + \boldsymbol{E}^+$, the above equations can be written as:

$$2\nabla \cdot \boldsymbol{B} = 0 \tag{107}$$

$$2\nabla \cdot \boldsymbol{E} = 2j\frac{mc}{\hbar}S_0 \tag{108}$$

$$2c\nabla \times \boldsymbol{B} = \frac{2}{c}\partial_t \boldsymbol{E} + j\frac{mc}{\hbar}\left(\boldsymbol{E}^- - \boldsymbol{E}^+\right) \tag{109}$$

$$2\nabla \times \boldsymbol{E} = j\frac{mc^2}{\hbar}\left(\boldsymbol{B}^+ - \boldsymbol{B}^-\right) - 2\partial_t \boldsymbol{B} \tag{110}$$

Using eq.104 $\boldsymbol{A}^+ = \boldsymbol{A}^-$ relation in eq.110 cancels the first term on the right hand side.

Additionally, using eq.106 on eq.109 and using eq.105 on eq.108, the above equations yield the following:

$$\nabla \cdot \boldsymbol{B} = 0 \tag{111}$$

$$\nabla \cdot \boldsymbol{E} = -\left(\frac{mc}{\hbar}\right)^2 \phi_0 \tag{112}$$

$$c\nabla \times \boldsymbol{B} = \frac{1}{c}\partial_t \boldsymbol{E} - \left(\frac{mc}{\hbar}\right)^2 c\boldsymbol{A} \tag{113}$$

$$\nabla \times \boldsymbol{E} = -\partial_t \boldsymbol{B} \tag{114}$$

Note that combining the potential relations in eq.104 together with the definitions $\mathbf{2}\boldsymbol{B} \equiv \boldsymbol{B}^+ + \boldsymbol{B}^-$ and $\mathbf{2}\boldsymbol{E} \equiv \boldsymbol{E}^- + \boldsymbol{E}^+$ results that ϕ_0 and $\boldsymbol{A}$ are the potentials of $\boldsymbol{E}$ and $\boldsymbol{B}$ fields. Therefore, using the Maxwell form, the above four equations can be compressed to the Proca equation:

$$\partial_\mu \left(\partial^\mu B^\nu - \partial^\nu B^\mu\right) + \left(\frac{mc}{\hbar}\right)^2 B^\nu \tag{115}$$

where B is the corresponding four potential: $B_\mu = \left(\frac{1}{c}\phi, \boldsymbol{A}\right)$.

An interesting consequences is that by this formalism it can be shown that the massive Proca Lagrangian is local U(1) invariant (under eq.98 transformation) by its 'extended' structure with $A_\mu^\pm$ components. This may suggest that the Higgs mechanism by spontaneous symmetry braking is less needed.

Maxwell equations

To degenerate eq.4 to homogeneous Maxwell equations, the following degeneracy should be taken:

$$S_0^+ = S_0^- \ , \ m = 0 \tag{116}$$

Next, the top and bottom parts of eq.4 are to be summed while using eq.116 and the total field definitions $2\boldsymbol{B} \equiv \boldsymbol{B}^+ + \boldsymbol{B}^-$ and $2\boldsymbol{E} \equiv \boldsymbol{E}^- + \boldsymbol{E}^+$:

$$\nabla \cdot \boldsymbol{B} = 0 \tag{117}$$

$$\nabla \cdot \boldsymbol{E} = 0 \tag{118}$$

$$c\nabla \times \boldsymbol{B} = \frac{1}{c}\partial_t \boldsymbol{E} \tag{119}$$

$$\nabla \times \boldsymbol{E} = -\partial_t \boldsymbol{B} \tag{120}$$

Which are the homogeneous Maxwell equations.

SUMMARY

This study introduce new field equations that extend Maxwell equations, satisfy the Dirac equation, and describe the intrinsic spin momentum phenomenon using a representation of fields instead of bi-spinor wave function. This representation lead to local U(1) invariant Lagrangian with no need for additional gauge field (force carriers). It was also shown that Maxwell equations and Proca equation are both degenerate versions of the original equation set. It was mentioned that the Proca mass term under this new formalism can be shown to be local U(1) symmetric by defining the transformation to include the mass in adjacent with the time derivative, thus, relaxing the need of the Higgs mechanism. Additionally, this study suggests a stress-energy tensor that encapsulates the dynamics of the Dirac equation is presented. This tensor can be integrated into the formalism of Einstein's field equations, serving as a bridge between quantum mechanics and general relativity.

[1] Dirac, P.A.M. (1928). "The Quantum Theory of the Electron". *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences.* 117 (778): 610–624

[2] D.J. Griffiths, (2008). *"Introduction to Elementary Particles"* (2nd ed.). Wiley-VCH. ISBN 978-3-527-40601-2.

[3] D. J. Griffiths (2007). *"Introduction to Electrodynamics"* (3rd ed.). Pearson Education, Dorling Kindersley. ISBN 978-81-7758-293-2.

[4] L.D. Landau and E.M. Lifshitz (1975). "*The Classical Theory of Fields",* (4th revised Eng. ed.) Elsevier. ISBN 978-0-7506-2768-9

[5] J.A. Wheeler, C. Misner, K.S. Thorne, W.H. Freeman & Co (1973). "*Gravitation*". ISBN 0-7167-0344-0

[6] M. Peskin, D. Schroeder (1995). "*An Introduction to Quantum Field Theory*" (Reprint ed.). Westview Press. ISBN 987-0201503975