The Riemann Zeta function and the transcendent Lerch function

Marcello Colozzo

Abstract

Elementary notions of quantum statistical mechanics provide a link between the Riemann Zeta function and the transcendent Lerch function.

1 The Fermi-Dirac integral

Let us consider a perfect gas of N non-relativistic fermions contained in a container D of volume V, and in thermodynamic equilibrium at temperature T. The gas is subjected to a potential energy force field:

$$U(\mathbf{x}) = +\infty, \quad \mathbf{x} \in \mathbb{R}^3 \backslash D$$

while for $\mathbf{x} \in D \backslash \partial D$ it is a regular function. The single fermion Hamiltonian follows:

$$H(\mathbf{x}, \mathbf{p}) = \frac{|\mathbf{p}|^2}{2m} + U(\mathbf{x}) \tag{1}$$

From quantum statistical mechanics:

$$N = \int_{\varepsilon_0}^{+\infty} \frac{g(\varepsilon)\, d\varepsilon}{e^{\frac{\varepsilon - \mu(T)}{k_B T}} + 1} \tag{2}$$

where ε_0 is the minimum energy of a single fermion: $\varepsilon_0 = \min U(\mathbf{x})$;the potential energy is defined up to an inessential additive constant for which we can redefine the energy scale: $\min U(\mathbf{x}) = 0$

$$N = \int_{0}^{+\infty} \frac{g(\varepsilon)\, d\varepsilon}{e^{\frac{\varepsilon - \mu(T)}{k_B T}} + 1} \tag{3}$$

$g(\varepsilon)$ is the density of states i.e. the number of single fermion states between ε and $\varepsilon + d\varepsilon$. If $G(\varepsilon)$ is the number of energy states $\leq \varepsilon$

$$g(\varepsilon) = \frac{d}{d\varepsilon} G(\varepsilon)$$

Classically

$$G_{cl}(\varepsilon) = \int_{\Lambda(\varepsilon)} d^3x d^3p, \quad (d^3x = dxdydz, \ d^3p = dp_x dp_y dp_z)$$

where

$$\Lambda(\varepsilon) = \left\{ (\mathbf{x}, \mathbf{p}) \in \mathbb{R}^6 \mid \frac{|\mathbf{p}|^2}{2m} + U(\mathbf{x}) \leq \varepsilon \right\}$$

According to quantum mechanics

$$G(\varepsilon) = \frac{g_s}{h^3} G_{cl}(\varepsilon)$$

where h is Planck's constant, while $g_s = 2s + 1$ is the statistical weight due to the spin s of a single fermion. So

$$G(\varepsilon) = \frac{g_s}{h^3} \int_{\Lambda(\varepsilon)} d^3x d^3p \tag{4}$$

The integral can be calculated only in the simplest cases. For example, for free fermions:

$$G(\varepsilon) = \frac{g_s}{h^3} V \int_{p^2 \leq 2m\varepsilon} d^3 p \tag{5}$$

and it is immediate to move to spherical coordinates in the pulse space. The interesting aspect is that the volume of the container appears. In the general case, we expect a dependence on ε of the power law type:

$$G(\varepsilon) \propto \varepsilon^{l+1}$$

so

$$g(\varepsilon) = \frac{d}{d\varepsilon} G(\varepsilon) = AV\varepsilon^l$$

wgere $A > 0$ is a constant while V is the volume. In the integral (3) we pass to dimensionless variables:

$$t = \frac{\varepsilon}{k_B T}, \quad x = \frac{\mu(T)}{k_B T}$$

I follows

$$N = n_c(T) V \int_0^{+\infty} \frac{t^l dt}{e^{t-x} + 1} \tag{6}$$

where

$$n_c(T) = A (k_B T)^{l+1} \tag{7}$$

So if $n = N/V$ is the concentration of fermions:

$$\frac{n}{n_c} = F_l(x) \tag{8}$$

where

$$F_l(x) = \int_0^{+\infty} \frac{t^l dt}{e^{t-x} + 1}$$

is the Fermi-Dirac integral of order l. This integral converges for $l > -1$. The (7) is the quantum concentration of fermions. If $n > n_c$ the gas is not rarefied: the fermion wave functions tend to overlap giving rise to a deviation from classical behavior. Vice versa for $n < n_c$. It follows that the deviation from classical behavior is measured by $F_l(x)$

$$F_l(x) = \int_0^{+\infty} \frac{t^l dt}{e^{t-x} + 1} = e^x \int_0^{+\infty} \frac{t^l dt}{e^t + e^x}$$

But

$$\int_0^{+\infty} \frac{t^l dt}{e^t + e^x} = \Gamma(l+1) \Phi(-e^x, l+1, 1) \tag{9}$$

where Γ is the gamma function, and Φ is the *transcendent Lerch function*, a function of the complex variable w and which depends on two parameters $s \in \mathbb{C},\ b \in \mathbb{N} \setminus \{0\}$. A representation in $|w| < 1$ is

$$\Phi(w, s, b) = \sum_{k=0}^{+\infty} \frac{w^k}{(k+b)^s} \tag{10}$$

So

$$F_l(x) = e^x \Gamma(l+1) \Phi(-e^x, l+1, 1) \tag{11}$$

If the chemical potential is zero:

$$F_l(0) = \Gamma(l+1)\,\Phi(-1, l+1, 1) \tag{12}$$

$\lambda := l + 1 > 0 \ (l > -1)$

$$F_{\lambda-1}(0) = \Gamma(\lambda)\,\Phi(-1, \lambda, 1) \tag{13}$$

On the other hand:

$$F_{\lambda-1}(0) = \int_0^{+\infty} \frac{t^{\lambda-1}dt}{e^t+1}$$

But

$$\int_0^{+\infty} \frac{t^{\lambda-1}dt}{e^t+1} = \left(1 - 2^{1-\lambda}\right)\Gamma(\lambda)\,\zeta(\lambda), \quad \forall\lambda > 0$$

where $\zeta(\lambda)$ is the Riemann zeta function. It follows

$$F_{\lambda-1}(0) = \left(1 - 2^{1-\lambda}\right)\Gamma(\lambda)\,\zeta(\lambda)$$

which compared with the (13):

$$\Gamma(\lambda)\,\Phi(-1, \lambda, 1) = \left(1 - 2^{1-\lambda}\right)\Gamma(\lambda)\,\zeta(\lambda)$$

Γ has no zeros:

$$\Phi(-1, \lambda, 1) = \left(1 - 2^{1-\lambda}\right)\zeta(\lambda), \quad \forall\lambda > 0$$

which extends immediately to the complex field: $s = \lambda + i\omega$

$$\Phi(-1, s, 1) = \left(1 - 2^{1-s}\right)\zeta(s), \quad \forall\,\mathrm{Re}\,s > 0$$

As is known, the non-trivial zeros of $\zeta(s)$ fall into

$$S_{crit} = \{s \in C \mid 0 < \mathrm{Re}\,s < 1,\ -\infty < \mathrm{Im}\,s < +\infty\}$$

$1 - 2^{1-s} \neq 0,\ \forall s \in S_{crit} \Longrightarrow$

$$\Phi(-1, s, 1) = 0 \Longleftrightarrow \zeta(s) = 0 \tag{14}$$