Estimating the number of primes within a limited boundary

Junho Eom*

Department of Zoology, University of British Columbia, Vancouver, BC, V6T 1Z4, Canada

* Corresponding Author

Junho Eom

Dept. of Zoology, University of British Columbia, Vancouver, B.C., Canada V6T 1Z4

Email: zuno3302@gmail.com

ORCID ID:

Junho Eom 0000-0003-0278-0082

Abstract

Within n^2, n boundaries were generated from the 1st to the nth, each containing n numbers. Primes less than $\frac{n^2}{2}$ were multiplied, intersected, and formed composites. At least one prime less than n or in the 1st boundary was used as a factor for the composites between n and n^2, or 2nd and nth boundaries, limiting the number of composites to $\frac{2n^2}{\lambda}$, where λ represented the wavelength of primes in the 1st boundary. Under these conditions, passively remaining numbers that were not connected to the wave of primes in the 1st boundary were all new primes between the 2nd and nth boundaries. Considering the cause-and-effect relationship among the primes less than n and the composites and new primes between 2nd and nth boundaries, the characteristics of composites could represent the characteristics of primes, and both were defined within a limited n^2 boundary. In this paper, these boundary characteristics were utilized to obtain the average number count per boundary, which led to obtaining the average number of primes per boundary. The average number of primes was multiplied by n boundaries with a coefficient of either β_1 or $\beta_{\sqrt{2}}$, denoting the ratio of the number of primes. Using either β_1 or $\beta_{\sqrt{2}}$, the number of primes was estimated between 10^6 and 10^{28} and compared to the actual number of primes. Considering the relative error between β_1 (Average 1.42%: maximum 2.92%, minimum 0.16%) or $\beta_{\sqrt{2}}$ (Ave. 0.37%: max. 0.96,

min. 0.04%), it was concluded that the number of primes could be estimated with $\beta_{\sqrt{2}}$, allowing for a relative average error of 0.37%, in an equation of

$$\pi(n^2) = \frac{\pi(n)}{\beta_{\sqrt{2}}} \cdot n$$

, where $10^3 \leq n \leq 10^{14}$, $\pi(n)$ was the known number of primes within n, and $\beta_{\sqrt{2}} = \frac{ln\,(2\sqrt{2}\cdot n)}{ln\,(n)} + 1$.

1. Introduction

Historically, Gauss conjectured the distribution of primes after counting all the primes up to 3 million, forming an equation involving $\pi(x) \approx \frac{x}{ln(x)}$, where $\pi(x)$ and $\frac{x}{ln(x)}$ represent the actual and estimated number of primes less than integer x, respectively (Zagier 1977). In an ideal circumstance, the ratio of the actual to the estimated number of primes converges to 1, aiming to accurately estimate the distribution of primes (Stein 2000). Although Gauss's prime-counting function behaves similar to $\pi(x)$, the estimated number of primes allows for a consistent margin of error, resulting in the ratio deviating from 1. As expected, increasing of x affects the increasing the gap difference between the actual and estimated number of primes. As the number gap difference increases, the actual number of primes also increases correspondingly, resulting in a decrease of relative error. Including Riemann zeta function, it has been studied to minimize the gap different between the actual and estimated number of primes (Stein 2000). Therefore, it is more reasonable to approach prime-counting problems by focusing on improving accuracy rather than questioning the feasibility of prime-counting estimation.

In this paper, Gauss's prime-counting function was used to provide a ratio between two different prime boundaries, but the fundamental estimation of prime distribution heavily relied on the prime wave analysis described by Eom (2024a). Prior to estimating the number of primes, the boundary was initially characterized and it was used to derive the average number count per boundary. Similarly, the average number of primes per boundary was characterized using the characteristics of average number count per boundary. The average number of primes per boundary was multiplied by the total number of boundaries with a formulated coefficient β, the prime-counting equation was formulated, and it was tested against a known number of primes up to 10^{28}.

2. Materials and methods

The general calculations and trends analysis were performed using Excel (version 2016, Microsoft, Redmond, WA, USA). The calculations of large numbers were performed in the online calculators: MathisFun (http://www.mathsisfun.com) and dCode (http://www.dcode.fr/en). The list of known primes between 10 and 10^{29} was obtained from Wikipedia (http://en.wikipedia.org/wiki/Prime-counting_function) and cited references in the main text were also referenced in this paper.

A key point in this section was to use the similar method to obtain the average number of primes after obtaining the average number count within a characterized boundary. By multiplication of total boundary numbers to the average number of primes per boundary, the total number of primes was estimated. Meanwhile, a coefficient β was considered to improve the accuracy of estimation.

2.1. Characterize the boundary within n^2

The primes less than integer n in the 1^{st} boundary were used to determine the new primes from the 2^{nd} to n^{th} boundaries (Eom 2024a and c). Within n^2, therefore, it was possible to generate n boundaries, each containing n numbers (Figure 1).

$$1^{st}, 2^{nd}, 3^{rd}, \ldots, (n-2)^{th}, (n-1)^{th}, n^{th}$$

Using the midpoint of the n boundaries, $\frac{1^{st}+n^{th}}{2}$, the array could be written as

$$\left(\frac{1^{st}+n^{th}}{2}-m\alpha\right),\left(\frac{1^{st}+n^{th}}{2}-(m-1)\alpha\right),\left(\frac{1^{st}+n^{th}}{2}-(m-2)\alpha\right),\ldots,\boldsymbol{\frac{1^{st}+n^{th}}{2}},\ldots,$$

$$\left(\frac{1^{st}+n^{th}}{2}+(m-2)\alpha\right),\left(\frac{1^{st}+n^{th}}{2}+(m-1)\alpha\right),\left(\frac{1^{st}+n^{th}}{2}+m\alpha\right)$$

, where m and α were integers. The paired boundaries, which had the same distance from $\frac{1^{st}+n^{th}}{2}$, could be replaced with boundaries between the 1^{st} and n^{th}. For example, the minimum and maximum boundaries shown by $\left(\frac{1^{st}+n^{th}}{2}-m\alpha\right)$ and $\left(\frac{1^{st}+n^{th}}{2}+m\alpha\right)$ could be replaced with the 1^{st} and n^{th} boundaries, respectively. The average value of the 1^{st} and n^{th} boundaries was

$$x_{ave}^{th}=\frac{\left(\frac{1^{st}+n^{th}}{2}-m\alpha\right)+\left(\frac{1^{st}+n^{th}}{2}+m\alpha\right)}{2}$$

$$=\frac{1^{st}+n^{th}}{2}$$

, where x_{ave}^{th} represented the average of the 1st and n^{th} boundaries. Considering the same distance from the midpoint, x_{ave}^{th} could represent the average value of each paired boundaries, including the paired 1st and n^{th}. Therefore, x_{ave}^{th} could be written as

$$x^{th} = \frac{1^{st} + n^{th}}{2}$$

, where $1 \leq x \leq n$.

2.2. Characterize the number count per boundary using the boundary characteristics within n^2

Supposing that the number count in each boundary was $C(x^{th})$, where C represented the number count. The average number count in the paired left and right boundaries from $\frac{1^{st}+n^{th}}{2}$ could be written as

$$C(x^{th}) = \frac{C(1^{st}) + C(n^{th})}{2}$$

As each boundary contained n numbers, $C(n^{th})$ could be replaced with $C(1^{st})$.

$$C(x^{th}) = \frac{C(1^{st}) + C(1^{st})}{2}$$

$$= C(1^{st})$$

The limited n^2 was evenly divided into n boundaries, thus the total number count could be calculated by multiplication of n boundaries to $C(x^{th})$ as

$$C(x^{th}) \cdot n \cdot B = C(1^{st}) \cdot n \cdot B$$

, where B represented the unit of boundary. Each boundary contained n numbers, so the value of $C(x^{th})$ was equal to $C(n)/B$ and the above equation could be written as follows.

$$C(n)/B \cdot n \cdot B = C(1^{st}) \cdot n \cdot B$$

$$C(n) \cdot n = C(1^{st}) \cdot n \cdot B$$

Also, there were n numbers in the 1st boundary, so $C(1^{st})$ could be replaced with n/B.

$$C(n) \cdot n = n/B \cdot n \cdot B$$

$$C(n) \cdot n = n \cdot n$$

$C(n) \cdot n$ represented the total number count within n^2, so it could be replaced with $C(n^2)$.

$$C(n^2) = n^2$$

$$n^2 = n^2$$

Using the characteristics of the boundary in section 2.1, it was explained that the total number count within n^2 could be calculated using the number count in the 1st boundary.

2.3. Characterize the number of primes per boundary using the characteristics of number count per boundary within n^2

Following the characteristics of number count per boundary in section 2.2,

$$C(x^{th}) = \frac{C(1^{st}) + C(n^{th})}{2}$$

, the average number of primes per boundary could be derived as

$$\pi(x^{th}) = \frac{\pi(1^{st}) + \pi(n^{th})}{2}$$

, where $\pi(x^{th})$ represented the number of primes in each boundary ($1 \leq x \leq n$). Unlike $C(1^{st})$ and $C(n^{th})$, $\pi(1^{st})$ and $\pi(n^{th})$ were not the same. As primes in the 1st boundary were used to generate the new primes from the 2nd to nth boundaries (Eom 2024c), $\pi(1^{st})$ should be higher than others. Also, the number count in each boundary was limited by n, therefore, $\pi(x^{th})$ could not exceed n. Thus, the relationship between $\pi(1^{st})$ and $\pi(n^{th})$ could be expressed as

$$\pi(1^{st}) = \frac{1}{\gamma} \cdot \pi(n^{th})$$

, where $\frac{1}{\gamma}$ was a ratio of $\pi(1^{st})$ over $\pi(n^{th})$. After replacing $\pi(n^{th})$ with $\gamma \cdot \pi(1^{st})$, $\pi(x^{th})$ could be written as follows.

$$\pi(x^{th}) = \frac{\pi(1^{st}) + \pi(n^{th})}{2}$$

$$= \frac{\pi(1^{st}) + \gamma \cdot \pi(1^{st})}{2}$$

$$= \frac{(1+\gamma)}{2} \cdot \pi(1^{st})$$

Similar to section 2.2, each boundary contained n numbers and assumed to have the same number of primes within n^2. Thus, $\pi(x^{th})$ could be replaced with $\pi(n)$/B.

$$\pi(n)/B = \frac{(1+\gamma)}{2} \cdot \pi(1^{st})$$

By multiplication of n boundaries, the total number of primes could be estimated within n^2 and could be written as follows.

$$\pi(n)/B \cdot n \cdot B = \frac{(1+\gamma)}{2} \cdot \pi(1^{st}) \cdot n \cdot B$$

$$\pi(n) \cdot n = \frac{(1+\gamma)}{2} \cdot \pi(1^{st}) \cdot n \cdot B$$

The left side represented the total number of primes within n^2, so it could be replaced with $\pi(n^2)$. Also, $\pi(1^{st})$ was limited by n in the 1st boundary, so it could be written as $\pi(n)$/B.

$$\pi(n^2) = \frac{(1+\gamma)}{2} \cdot \pi(n)/B \cdot n \cdot B$$

$$\pi(n^2) = \frac{(1+\gamma)}{2} \cdot \pi(n) \cdot n$$

Overall, it could explain that the total number of primes within n^2 could be estimated using the known number of primes in the 1st boundary. Meanwhile, the accuracy of estimation relied on the equation $\frac{1+\gamma}{2}$, which represented the ratio between $\pi(n) \cdot n$ and $\pi(n^2)$. For the convenience of calculation, β was used to represent $\frac{2}{1+\gamma}$, which denoted the ratio of $\pi(n) \cdot n$ over $\pi(n^2)$.

$$\frac{2}{1+\gamma} = \frac{\pi(n) \cdot n}{\pi(n^2)}$$

$$= \beta$$

2.4. Practical approach to formulate β

As n increased, β also changed. Thus, β should be formulated with a variable n. Prior to formulating β, the actual β (β_{actual}) was calculated and visualized using the primes between 2 (1st prime) and 1009 (168th prime) to understand the behavioral pattern of β_{actual} (Figure 2). In general, β_{actual} initiated from 1 (1st prime or 2), reached a maximum of 2.2257 (30th prime or 113), and decreased with fluctuation. Theoretically, β converged to 2, indicating that β_{actual} would also converge to 2. For example, β could be calculated using the prime-counting function (Stein 2000) as

$$\beta = \frac{\pi(n) \cdot n}{\pi(n^2)}$$

$$= \frac{\frac{n}{ln(n)} \cdot n}{\frac{n^2}{ln(n^2)}}$$

$$= \frac{2n^2 \cdot ln(n)}{n^2 \cdot ln(n)}$$

$$= 2$$

, where $\pi(n)$ and $\pi(n^2)$ were $\frac{n}{ln(n)}$ and $\frac{n^2}{ln(n^2)}$, respectively. Based on the results, it was reasonable to hypothesize that β_{actual} would converge to 2 as n increased infinitely. Overall, it was thought that β_{actual} behaved in two different regions, before and after a maximum of 2.2257: $1 \leq \beta_{\text{actual}} \leq 2.2257$ or $2 < \beta_{\text{actual}} \leq 2.2257$. Considering the asymmetrical decreasing patterns from maximum 2.2257, the most practical approach for β would follow β_{actual} after 2.2257 by covering the increasing n, while allowing an error in a range of $1 \leq \beta_{\text{actual}} \leq 2.2257$ by disregarding the decreasing n (shaded area in Figure 2).

Logically, β, which followed β_{actual} and ranged between 2.2257 and 2, should be formulated from $\frac{2}{1+\gamma}$.

$$\beta = \frac{2}{1+\gamma}$$

Although β was assumed to converge closely to 2 as n increased, formulating β to converge to 1 was preferred because convergence to 1 was easier than to 2. Subsequently, 1 was added to achieve convergence to 2. Prior to formulating β, both the numerator and denominator were multiplied by $\sqrt{2}$, which was known as a balance constant in the increasing boundaries from n^x to n^{2x} (Eom 2024a), without changing the value of β.

$$\beta = \frac{2\sqrt{2}}{(1+\gamma) \cdot \sqrt{2}}$$

Assuming that n increased infinitely and β converged closely to 2, then γ should converge to 0.

$$\beta = \lim_{\gamma \to 0} \frac{2\sqrt{2}}{(1+\gamma) \cdot \sqrt{2}}$$

$$= \frac{2\sqrt{2}}{(1) \cdot \sqrt{2}}$$

By applying the logarithmic function with variable x to each numerator and denominator, β could be formulated in a function that converges to 1. Considering the original value of β, the left side should be subtracted by 1 to balance the equation.

$$\beta - 1 = \frac{ln(2\sqrt{2} \cdot x)}{ln(\sqrt{2} \cdot x)}$$

The variable x could be replaced with n, then β was formulated as

$$\beta_1 = \frac{ln(2\sqrt{2} \cdot n)}{ln(\sqrt{2} \cdot n)} + 1$$

, where β_1 was the formulated β to follow β_{actual}.

2.5. Formulate $\beta_{\sqrt{2}}$ as an alternative to β_1

β was formulated through the balance of the numerator and denominator, and formed β_1. If β was formulated while maintaining the balance between left and right sides instead of the numerator and denominator, it could be possible to formulate another β as an alternative to β_1.

$$\beta = \frac{2}{1+\gamma}$$

For example, a balance constant $\sqrt{2}$ could be multiplied on both the left and right sides.

$$\sqrt{2} \cdot \beta = \sqrt{2} \cdot \left(\frac{2}{1+\gamma}\right)$$

The method used to formulate β to β_1 could also be applied as follows.

$$\sqrt{2} \cdot \beta = \lim_{\gamma \to 0} \sqrt{2} \cdot \left(\frac{2}{1+\gamma}\right)$$

$$\sqrt{2} \cdot \beta = \frac{2\sqrt{2}}{1}$$

$$\sqrt{2} \cdot \beta = \frac{2\sqrt{2} \cdot x}{x}$$

$$\sqrt{2} \cdot \beta - 1 = \frac{ln(2\sqrt{2} \cdot x)}{ln(x)}$$

$$\sqrt{2} \cdot \beta = \frac{ln\left(2\sqrt{2} \cdot n\right)}{ln(n)} + 1$$

$$\beta_{\sqrt{2}} = \frac{ln\left(2\sqrt{2} \cdot n\right)}{ln(n)} + 1$$

, where $\beta_{\sqrt{2}}$ represented $\sqrt{2} \cdot \beta$.

Overall, it was proposed that there could be two different formulated β, which followed β_{actual} and ranged between 2.2257 and 2, for estimating $\pi(n^2)$ using the known $\pi(n)$ within a limited n^2:

$$\pi(n^2) = \frac{\pi(n)}{\beta_1} \cdot n$$

, where $\beta_1 = \frac{ln\ (2\sqrt{2} \cdot n)}{ln\ (\sqrt{2} \cdot n)} + 1$,

or

$$\pi(n^2) = \frac{\pi(n)}{\beta_{\sqrt{2}}} \cdot n$$

, where $\beta_{\sqrt{2}} = \frac{ln\ (2\sqrt{2} \cdot n)}{ln\ (n)} + 1$.

3. Results

Using the known number of primes between 10^3 and 10^{14}, the number of primes between 10^6 and 10^{28} was estimated with β_1 (Table 1) and $\beta_{\sqrt{2}}$ (Table 2), respectively. Compared to the actual number of primes between 10^6 and 10^{28}, the estimated number of primes generated by $\beta_{\sqrt{2}}$ (0.37 ± 0.29% relative average error ± S.D.: maximum 0.95% ~ minimum 0.07%) was 3.8 times more accurate than β_1 (1.42 ± 0.75%: 2.92% ~ 0.61%). It was tested between $\pi(2^{17})$ and $\pi(2^{43})$ with $\beta_{\sqrt{2}}$, and the results were consistent

(0.31 ± 0.20%: 0.78% ~ 0.09%) (*see* Appendix A). These results indicated that $\beta_{\sqrt{2}}$ was more accurate for estimating the number of primes than β_1 within a limited boundary between 10^6 and 10^{28}.

Using the known number of primes between 10^{15} and 10^{29}, the number of primes between 10^{30} and 10^{58} was estimated with β_1 (Table 1) and $\beta_{\sqrt{2}}$ (Table 2), respectively. Also, the known number of primes less than 2^{86} or 2^{80} was used to estimate the number of primes either from 2^{43} to $2^{10.75}$ or 2^{160} to 2^{81920} (*see* Appendix B and C). If the actual number of primes within the above range could be found using an appropriate computational technology, the functionality of $\beta_{\sqrt{2}}$ could be confirmed over 10^{30}.

4. Discussions

In the previous sections, the average number of primes per boundary was logically derived based on the characteristics of the boundary and used for estimating the total number of primes with a coefficient β_1 or $\beta_{\sqrt{2}}$. Using either β_1 or $\beta_{\sqrt{2}}$, the number of primes was estimated, compared with the actual number of primes, and concluded that $\beta_{\sqrt{2}}$ was more accurate than β_1 for estimating the number of primes within 10^6 and 10^{28}.

The number of primes showed dual characteristics: irregularity per boundary and regularity within n^2. These characteristics were thought to derive from the dual characteristics of composites within a limited n^2 boundary, which was divided by n, with each boundary containing n numbers from the 1st to nth. The dual characteristics could be explained using prime wave analysis: the wave of primes less than integer n, or the 1st boundary, oscillated and connected (or eliminated) the composites; passively remaining numbers, without being connected (or eliminated), became new primes between n and n^2, or the 2nd and nth boundaries (Eom 2024a). Here, two types of composites were considered: composites consisting either of multiples of a single prime (Type I) or of two or more primes (Type II) as factors. Type I oscillated independently, intersected with other Type I, and formed hypothetical oscillation, representing Type II. In general, the role of Type I was to evenly distribute the number of composites, while Type II regulated the increasing number of composites to 1, regardless of number of Type I intersections. Due to the different roles of Type I and II, only selected boundaries were regulated based on the wavelength (λ) of primes, causing irregularities in the number of composites per boundary. When extending the boundary to n^2, it was observed that prime factors of Type I and II originated from the primes less than $\frac{n^2}{2}$, and at least one primes in the 1st boundary was used as a factor for both Type I and II (Eom 2024c). Thus, total number of Type I and II was constrained by $\frac{2n^2}{\lambda}$, where λ represented the wavelength of primes in the 1st boundary, limiting the total number of composites within n^2. As a result, the constrained number of composites ranged within certain minimum and maximum limits, showing regularity within n^2. Considering the cause-and-effect relationship among the primes in the 1st boundary and the composites and passively remaining new primes between the 2nd and nth boundaries (Eom 2024b), the dual characteristics of composites could represent the characteristics of new primes between the 2nd and nth boundaries, with changes in numbers increasing or decreasing.

Instead of following in the same direction, $\beta_{\sqrt{2}}$ followed β_{actual} in the opposite direction (Figure 2). This might have two advantages: minimizing errors by reducing of the crossing phenomenon and enabling the estimation of a large number of primes. If $\beta_{\sqrt{2}}$ followed β_{actual} from the same direction, $\beta_{\sqrt{2}}$ would converge to 2 from below 2, whereas β_{actual} converged to 2 from above 2. The relative error between $\beta_{\sqrt{2}}$ and β_{actual} spread around 2 and did not converge absolutely to 0, indicating that the estimated number could not equal the actual number. If $\beta_{\sqrt{2}}$ exceeded 2 and followed β_{actual}, it would eventually reach a relative error of 0. Thereafter, the crossing phenomenon would occur, indicated by the ± sign of the relative error value switching, resulting in an increase in error. Therefore, the ideal approach would follow the hypothetical trend line through the center of the minimum and maximum ranges in β_{actual} within a limited n^2 boundary. If both $\beta_{\sqrt{2}}$ and β_{actual} could maintain parallel convergence towards 2, the relative error would decrease as the estimation number increased, eventually converging to 0 and preventing the crossing phenomenon. This advantage might provide stability in relative errors as the estimation number of primes increased, enabling the estimation of a large number of primes. Also, $\beta_{\sqrt{2}}$ was expected to self-adjust as the boundary increased from n^x and n^{2x}, because the boundary increasing rate was adjusted by $\sqrt{2}$ in advance. The functionality of the self-adjustment could be confirmed later by comparing the estimated large number of primes (Table 1 and 2, also *see* Appendix B and C) to the actual numbers discovered later.

5. Conclusions

Considering the cause-and-effect relationship among the primes in the 1st boundary and the composites and new primes between the 2nd and nth boundaries, the characteristics of composites could apply to the characteristics of primes, with only the increase or decrease in numbers changing; both were defined within the limited characteristics of the n^2 boundary. Therefore, the number of primes could be estimated using the characteristics of boundary, allowing for a relative average error of 0.37%, in an equation of

$$\pi(n^2) = \frac{\pi(n)}{\beta_{\sqrt{2}}} \cdot n$$

, where $10^3 \leq n \leq 10^{14}$, $\pi(n)$ was the known number of primes within n, and $\beta_{\sqrt{2}} = \frac{ln\,(2\sqrt{2} \cdot n)}{ln\,(n)} + 1$.

5. References

Baugh, D. (2020) New prime counting function record, $\pi(10^{28})$. https://www.mersenneforum.org/index.php

Buethe, J. (2015) Analytic computation of the prime-counting function. http://www.math.uni-bonn.de/people/jbuethe/topics/AnalyticPiX.html

Caldwell, C., Buethe, J., Franke, J., Jost, A., and Kleinjung, T. (2010) Conditional calculation of $\pi(10^{24})$. http://t5k.org/notes/pi(10to24).html

Eom, J. (2024a) Characteristics of primes within a limited number boundary. http://vixra.org/abs/2406.0046

Eom, J. (2024b) Approaching Goldbach's conjecture using the asymmetric relationship between primes and composites within a limited even boundary. http://vixra.org/abs/2406.0072

Eom, J. (2024c) A technical approach to finding primes within a limited boundary. http://vixra.org/abs/2406.0157

Platt, D. J. (2012) Computing $\pi(x)$ analytically. http://arxiv.org/abs/1203.5712v3. Doi.org/10.48550/arXiv.1203.5712

Staple, D. B. (2015) The combinational algorithm for computing $\pi(x)$. *Master thesis,* University of Dalhousie, Halifax, Nova Scotia.

Stein, W. (2000) Prime numbers. In *Elementary number theory: Primes, congruences, and secrets* (ed. S. Axler and K. A. Ribet). Springer, New York, pp 1-20. Doi 10.1007/978-0-387-85525-7

Walisch, K. (2015) New confirmed $\pi(10^{27})$ prime counting function record. https://www.mersenneforum.org/index.php

Walisch, K. (2022) New prime counting function record: $\pi(10^{29})$. https://www.mersenneforum.org/index.php

Zagier, D. (1977) The first 50 million prime numbers. *The Mathematical Intelligencer* 1: 7-19. Doi 10.1007/BF03039306

Table 1. List of the estimated number of primes with β_1. Using the known number of primes between 10^3 and 10^{14}, $\pi(n^2)$ was estimated ($\pi(n^2)_{\text{estimate}}$) between 10^6 and 10^{28} and compared with actual $\pi(n^2)$. The relative error was calculated in an equation of $\frac{\pi(n^2_{estimate})-\pi(n^2)}{\pi(n^2)} \cdot 100(\%)$. The known number of primes higher than 10^{24} was supported by Baugh (2020), Buethe (2015), Caldwell et al. (2010), David (2020), Platt (2012), and Walisch (2015 and 2022).

n	$\pi(n)$	β_1	$\pi(n^2)_{\text{estimate}}$	$\pi(n^2)$	% error
10^3	168	2.0955	80169	78498	2.12
10^4	1229	2.0725	5929955	5761455	2.92
10^5	9592	2.0584	465982459	455052511	2.40
10^6	78498	2.0489	38311444845	37607912018	1.87
10^7	664579	2.0420	3254391578699	3204941750802	1.54
10^8	5761455	2.0369	282849389842964	279238341033925	1.29
10^9	50847534	2.0328	25012343938611103	24739954287740860	1.10
10^{10}	455052511	2.0296	2242017225292467184	2220819602560918840	0.95
10^{11}	4118054813	2.0269	203160384371463744394	201467286689315906290	0.84
10^{12}	37607912018	2.0247	18573871422674697755935	18435599767349200867866	0.75
10^{13}	346065536839	2.0228	17107472255598307707 52900	16992467508724371413 27603	0.67
10^{14}	3204941750802	2.0212	158560524401571241687412154	157589269275973410412739598	0.61
10^{15}	29844570422669	2.0198	1477549589539506070353499 3314	n.a.	n.a.
10^{16}	279238341033925	2.0186	1383300017249748282775997792619	n.a.	n.a.
10^{17}	2623557157654233	2.0175	130036640567917683718134953482365	n.a.	n.a.
10^{18}	24739954287740860	2.0165	12268241499295640565517671765352214	n.a.	n.a.
10^{19}	234057667276344607	2.0157	1161162090453248207501705577266251562	n.a.	n.a.
10^{20}	2220819602560918840	2.0149	110217705009157654933354125900305243024	n.a.	n.a.
10^{21}	21127269486018731928	2.0142	10488991147804549515957564917876998181801	n.a.	n.a.
10^{22}	201467286689315906290	2.0135	1000537678361017268675994291326831203550511	n.a.	n.a.
10^{23}	1925320391606803968923	2.0130	95644180986461491229147857585239669380936587	n.a.	n.a.
10^{24}	18435599767349200867866	2.0124	9160706948529276893838643835669906918472104855	n.a.	n.a.
10^{25}	17684630939914376941 1680	2.0119	878971282516294613334058105026384415602146356013	n.a.	n.a.
10^{26}	1699246750872437141327603	2.0115	84476116811837308727065306252868606107373285886916	n.a.	n.a.
10^{27}	16352460426841680446427399	2.0110	81311533326935435611513340283627886803008823281799 74	n.a.	n.a.
10^{28}	157589269275973410412739598	2.0106	783755765943916097720984879062462404658810637453580325	n.a.	n.a.
10^{29}	1520698109714272166094258063	2.0103	7564432557654210970613725353987814749268121244192205 6971	n.a.	n.a.

Not available (n.a.)

Table 2. List of the estimated number of primes with $\beta_{\sqrt{2}}$. Using the known number of primes between 10^3 and 10^{14}, $\pi(n^2)$ was estimated ($\pi(n^2)_{\text{estimate}}$) between 10^6 and 10^{28} and compared with actual $\pi(n^2)$. The relative error was calculated in an equation of $\frac{\pi(n^2_{estimate})-\pi(n^2)}{\pi(n^2)} \cdot 100(\%)$. The known number of primes higher than 10^{24} was supported by Baugh (2020), Buethe (2015), Caldwell et al. (2010), David (2020), Platt (2012), and Walisch (2015 and 2022).

n	$\pi(n)$	$\beta_{\sqrt{2}}$	$\pi(n^2)_{\text{estimate}}$	$\pi(n^2)$	% error
10^3	168	2.1505	78121	78498	-0.4803
10^4	1229	2.1128	5816688	5761455	0.9587
10^5	9592	2.0903	458879524	455052511	0.8410
10^6	78498	2.0752	37825667436	37607912018	0.5790
10^7	664579	2.0645	3219069658014	3204941750802	0.4408
10^8	5761455	2.0564	280166027070540	279238341033925	0.3322
10^9	50847534	2.0501	24801598248889743	24739954287740860	0.2492
10^{10}	455052511	2.0451	2225027552415802422	2220819602560918840	0.1895
10^{11}	4118054813	2.0410	201761629126737751320	201467286689315906290	0.1461
10^{12}	37607912018	2.0376	18456704651425622702551	18435599767349200867866	0.1145
10^{13}	346065536839	2.0347	1700789870691182132970321	1699246750872437141327603	0.0908
10^{14}	3204941750802	2.0322	157703859391414784324113659	157589269275973410412739598	0.0727
10^{15}	29844570422669	2.0301	14701012918577720857727846973	n.a.	n.a.
10^{16}	279238341033925	2.0282	1376764482997112002897118409294	n.a.	n.a.
10^{17}	2623557157654233	2.0265	129458553131773863297848882519258	n.a.	n.a.
10^{18}	24739954287740860	2.0250	12216743549794488125201284107412579	n.a.	n.a.
10^{19}	234057667276344607	2.0237	1156545381727253849374247927110845990	n.a.	n.a.
10^{20}	2220819602560918840	2.0225	109801472498417292944423712477729151275	n.a.	n.a.
10^{21}	21127269486018731928	2.0215	10451272368870643948908084453828636907897	n.a.	n.a.
10^{22}	201467286689315906290	2.0205	997103769467061053241201459836506483759489	n.a.	n.a.
10^{23}	1925320391606803968923	2.0196	95330239313152656042179148845855536148357950	n.a.	n.a.
10^{24}	18435599767349200867866	2.0188	9131894441668989499971235376473126472359391959	n.a.	n.a.
10^{25}	176846309399143769411680	2.0180	876317610401931929129667009074585692393652703991	n.a.	n.a.
10^{26}	1699246750872437141327603	2.0173	84230913556810896707461236697237348186449585289791	n.a.	n.a.
10^{27}	16352460426841680446427399	2.0167	8108427990013190928126437283621789064063252499458929	n.a.	n.a.
10^{28}	157589269275973410412739598	2.0161	781643715904611427229763065761451110805475356748390394	n.a.	n.a.
10^{29}	1520698109714272166094258063	2.0155	75447526984309860848694897988399235260260215739670216543	n.a.	n.a.

Not available (n.a.)

Figure 1. The boundary structure for estimating the number of primes within n^2 involved generating a total of n boundaries from the 1^{st} and n^{th}, with each containing n numbers. From the midpoint, $\frac{1^{st}+n^{th}}{2}$, the boundaries were paired between 1^{st} and n^{th} for further analysis.

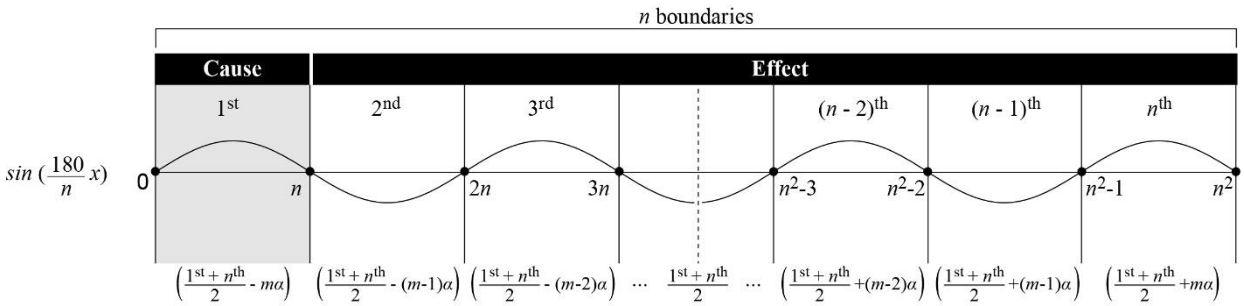
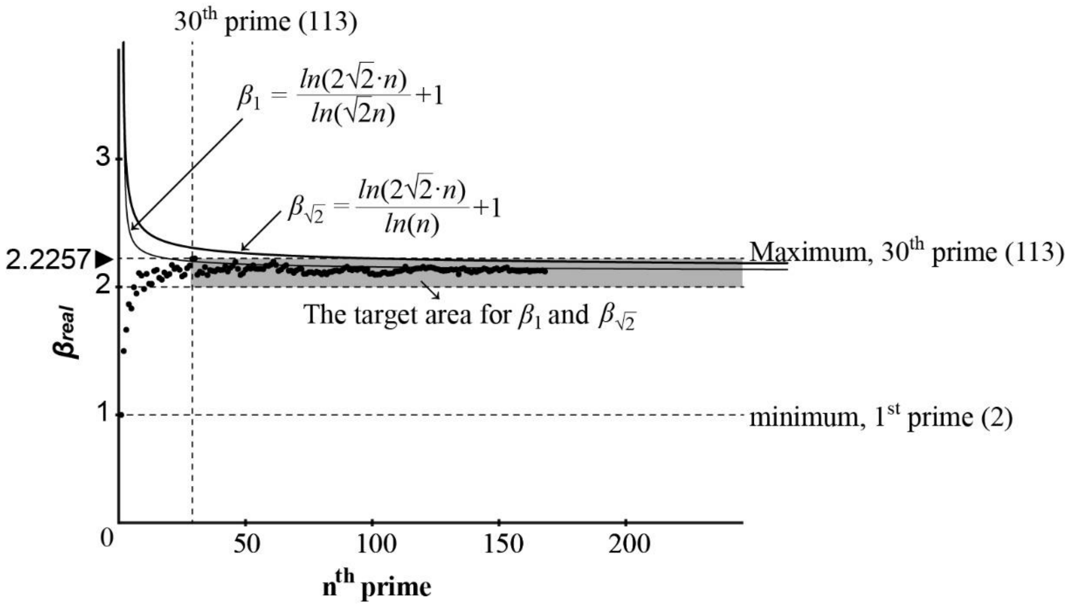


Figure 2. The actual β (β_{real}), which denotes the ratio of actual $\pi(n)\cdot n$ over actual $\pi(n^2)$, was visualized between 2 (1st prime) and 1009 (168th prime). β_{actual} initiated from 1 (1st prime or 2), reached a maximum of 2.2257 (30th prime or 113), and decreased and converged closely to 2 with oscillation. Two different formulated β were suggested: β_1 and $\beta_{\sqrt{2}}$ that followed β_{real} after a maximum of 2.2257.



Appendix A. List of the estimated number of primes with $\beta_{\sqrt{2}}$. Using the known number of primes between 2^{17} and 2^{43} (Staple 2015), the $\pi(n^2)$ was estimated ($\pi(n^2)_{\text{estimate}}$) between 2^{34} and 2^{86} and compared with actual $\pi(n^2)$. The relative error (%) was calculated in an equation of $\frac{\pi(n^2_{estimate})-\pi(n^2)}{\pi(n^2)} \cdot 100(\%)$.

n	$\pi(n)$	$\beta_{\sqrt{2}}$	$\pi(n^2)_{\text{estimate}}$	$\pi(n^2)$	% error
2^{17}	12251	2.0882	768956964	762939111	0.78
2^{18}	23000	2.0833	2894069760	2874398515	0.68
2^{19}	43390	2.0789	10942487850	10866266172	0.70
2^{20}	82025	2.0750	41450335614	41203088796	0.60
2^{21}	155611	2.0714	157543409593	156661034233	0.56
2^{22}	295947	2.0681	600184990978	597116381732	0.51
2^{23}	564163	2.0652	2291546776155	2280998753949	0.46
2^{24}	1077871	2.0625	8767842224065	8731188863470	0.41
2^{25}	2063689	2.0600	33614520494974	33483379603407	0.39
2^{26}	3957809	2.0576	129078611474642	128625503610475	0.35
2^{27}	7603553	2.0555	496474836512878	494890204904784	0.32
2^{28}	14630843	2.0535	1912491066892929	1906879381028850	0.29
2^{29}	28192750	2.0517	7377145457552134	7357400267843990	0.26
2^{30}	54400028	2.0500	28493456239205400	28423094496953330	0.24
2^{31}	105097565	2.0483	110181958594419381	109932807585469973	0.22
2^{32}	203280221	2.0468	426543829554150798	425656284035217743	0.20
2^{33}	393615806	2.0454	1652998858092932118	1649819700464785589	0.19
2^{34}	762939111	2.0441	6412152520280096178	6400771597544937806	0.17
2^{35}	1480206279	2.0428	24896258974809488720	24855455363362685793	0.16
2^{36}	2874398515	2.0416	96747997656262259774	96601075195075186855	0.15
2^{37}	5586502348	2.0405	376274335659412686536	375744164937699609596	0.14
2^{38}	10866266172	2.0394	1464542800811852151457	1462626667154509638735	0.13
2^{39}	21151907950	2.0384	5704490445825358449992	5697549648954257752872	0.12
2^{40}	41203088796	2.0375	22234736309933265274942	22209558889635384205844	0.11
2^{41}	80316571436	2.0365	86722614900031941989455	86631124695994360074872	0.10
2^{42}	156661034233	2.0357	338457375806366997218429	338124238545210097236684	0.09
2^{43}	305761713237	2.0348	1321701306368920962838785	1320486952377516565496055	0.09

Appendix B. Using the known primes within either 2^{86} or 2^{80} (Staple 2015), the number of primes was sequentially estimated either from 2^{43} to $2^{10.75}$ or from 2^{160} to 2^{81920}. As the number of primes was estimated, the relative errors were expected to increase due to accumulated errors. Using the trends of accumulated errors from $\pi(2^{20})$ to $\pi(2^{80})$ ($y = 0.4246 \cdot ln(x) + 0.1412$, $R^2 = 0.9827$), the accumulated error was estimated.

n	$\pi(n)_{\text{estimate}}$	$\beta_{\sqrt{2}}$	$\pi(n^2)$	$\pi(n)$	**Estimated % error**
2^{86}				1320486952377516565496055	n.a.
2^{43}	305480785197	2.0348		305761713237	0.09
$2^{21.5}$	213186	2.0697		214516	0.62
$2^{10.75}$	264.85	2.1395		268	1.17

n	$\pi(n)$	$\beta_{\sqrt{2}}$	$\pi(n^2)_{\text{estimate}}$	$\pi(n^2)$	**Estimated % error**
2^{80}	22209558889635384205844	2.0187	$1.33001655400281512 \times 10^{46}$	n.a.	0.12
2^{160}		2.0093	$9.67376110159785786 \times 10^{93}$	n.a.	0.47
2^{320}		2.0046	$1.03073562845126765 \times 10^{190}$	n.a.	0.58
2^{640}		2.0023	$2.34858280316574171 \times 10^{382}$	n.a.	0.72
2^{1280}		2.0011	$2.44295763640031015 \times 10^{767}$	n.a.	0.82
2^{2560}		2.0005	$5.29112015486709132 \times 10^{1537}$	n.a.	0.90
2^{5120}		2.0002	$4.96628701706807671 \times 10^{3078}$	n.a.	0.96
2^{10240}		2.0001	$8.75236883710555649 \times 10^{6160}$	n.a.	1.02
2^{20480}		2.00007	$5.43739532892201376 \times 10^{12325}$	n.a.	1.07
2^{40960}		2.00003	$4.19735168078446281 \times 10^{24655}$	n.a.	1.11

Not available (n.a.)

Appendix C. Using the known number of primes within 2^{80}, the number of primes within 2^{81920} was estimated, and 24656 digits of numbers were listed. The accumulated error from $\pi(2^{160})$ was assumed to be 1.1188%.

4197351680784462819889933924219033387862376460470259482826015308236633786840576069544479914929208885863667140145955895703599528979629022650166065828328878 61
62129506110080224772380574357933800118895754542587842224858471611607660474189210403133453396960924591409655176754255134700516614078538789989523105660776583 7
89388800183087706636143739649294474100083760601940243209721900305084719619618331262956239338929049371839648685137260167446446938473761369625523149905053225 9
23848748889173285808636858666670039131977700014566088417058046626564815488386814920627370595183781164231763397343808216729638455175178545788437844709964444 7
45928685104326764836466690524003859958601984753395610421194589705574321638934358675503112817219387824339932366131914858977294053159050906962527727423106239
38708351729509196586403863326230400068858505486054040100229660742241721737691374425860518476410564688975494238447189472365621298785972079107656460262623509 1
69596083424888963839919900674282261470717609481644842606011679232319497756164286947380376114796655911042554925418827709741059318156833295222926356464611812 1
25736207135641845123869484663759227724104070698845628147186262450973199861883441623539621389731589635185780369663738120796203986773150003993416384759699760 9
52440756424018645814314495923142807546417848300766982452170510717191970377602018364393613150146477609358511699485586451106709513063303469 86
30679786088458868841460583739564630363510315025528072905264298614995515190925844138807930864976498132059046429070365652180827991362917020845200821804615788 5
73021402607434956436475835742224258111676715081057327611649025364156204472772392092609002713795695171937700066236690519694115024211106365161364800590289612 5
52419908578207418732235739743062234457041930770905224812565784583148094497296320860788531417086134658765382707354615483244154472700684359130366457293068279 7
65211546732501737303846964668903388458666674387312157436319817248966429369670246817138669590023799742412136383014950100926649023002682191424081415474139742 4
60159332358922075118226666768495400119847383900058501325324894352411205331617606633853498415290560798284868315506416839861178818904289869281846141209811428 1
88180235844257054935206031328326843581424431376928810392910070496011533167763692729203696686141451510189317340604471238540591044095258354552081530778680259 1
20568955071398955353574852345734076556351332144238804301177758382040709881946750289961595581050806069035436600803365355464747711745477164625779223600878887 2
10487300109395347910476618969578204616700120690252038040636112393142902745218997260509345057771404703882986687398683908954357201344727108784433584396108146 5
66992910864614463651514962719993394784684445850727582141896213458638387344688474683223000485530927367663793245101690265634958969772883218163704665230467455 5
64129794475661156678053693336382291194151091885497305062143356053404485474757625380682952739368833147226060856696563733669353561318782426727700067921945003 2
89647329323101973583582415455719712591358257867252328472671904151613121855511841044800630484673697603604054503389133297727386606735120391849223625489364556 5
77690025549827781001315131662019936928267671784985977364843638146648909727850448202932322858208000645808045589778331398357626463597165352220425002557234598 5
66792030913677628279590383715347351541919307634630873243307484302070576370164637267579168649135666204963517398969583570987694042926578263581349754785146048 5
72867581841729378194913998806192092388916988117117838115561592675200641870437382064076854156760240589155802888594194502802033035282829208877494700264329271 0
93482387116588132526924605175401577384220256698024477309866700856274162290051330422005104199356547768314242794997292779128779124741649193293662690310610300 1
32718708178478691991860933248448756392337216385078605134137356217105388275752299587696044571142750900352487807610524326392697872255547647149550510882100269
34529583487244479009415646860165124470722412756874149801475954867296878965870383726327156250641195116090931382756099695709468317790602737093117386813353477 5
63239866034552976623689911022603559806655773374992460796567313596551001352733701168183234502693002336901994712239200702319249058293150719583079656087412091 6
75530115885272587345666050332609621735468080990821191227681328626632143057925424037021824871727733550486360693367564430718283548423701944299709558443872005 9
58973107644745861275478650017340638679316641117463416549259481732178892622363531812576661917894448183216348815196700205828757416395026670845472310842219470 0
96644556028920795055459267528837008310133488146137027235280204114536995675147169390379769172573828429805852957906180063442631017949757267609360903535602979
02544806475776153000447641582009336079368879754845391083892017709100011598810320634760943434973662624459844749466579840324637923464415056439725074407885922 7
40932514117808749674009693282564378602178884928156709362722499250053750990115636087790178031144901984129273111295416524050386836502243128695395216904813826 8
88185872277748168015184979805563408737396928182760714300797226707442063473300902887282093377466523548616087644553414368720690005845241388933679267178415406 1
94050006055831631292327513870541764261529396981737795567651752558778162063778047401497314764644201454630473749225476050380725776961379156407707873163639337 3
18911287040256921950079837751589542962297390807044858550048565960959117607756434371362145315098112304617532704081740744402888446750450149700941986344491748 0
97470227186629989815831295915138777447939762133892361878413030393655321330158108548898326001883927325686491889144961445098267338786541812400391202928032880 6
98842632506052697954265192970664275122200475392602613034143158787278546784045857740011683967921203072684210379044162473036471981901543790972939999544258122 3
31088377801118048606361262783643707949769744024490068713566386111077093413403487332771283180707050844178390773979126081271492255738938418570387622232127664 7
32487507186144205184716431320681799358623526593859666324289128836632465593849714486808327129659771098145660703925780344341536754855849838357174432640938893 1
64154783923231763673355981657754010163330464239675511214809944374435988344466566442831151357911063241484527010424408231554788602363670297397053461444406627 2
07342772604728257486094699454267501250451504297823170029903995285950036939246413619923507859550755367261791773320022238649367121470787355687241784887302019
04007996510758703645730551366524290078416991601446412571560829900390667782722606165165028871163068083923584716125784928688872909409411982201855701932259675 1
92276491398128525123527240666450505141709755357803728167712163759799666007139434709519028255483295648067225202428485119325014764408379763297736780290826020 9
56203099451515579744239007363678696256892229488728341729821519806758994347969822844199895872187343247708254822011251371514469483173563971846723374986250332 9
05918338629750654425128494046947916233563538390430374010235574317751978412674171189223982466413296619015276855682902694880540701901958406384843098761524489 2
87086086199220994777378618916719621392229987186483543343343426479968803894272410816904004730518577311533389364521545943192534884859826853802518405598499352 6
84183900637950556648916890541642146793515959169631828803315275814683571282403309419626412405428566949849413236279798339232861880236843172615836352672110679 5
66971023411565920607092410877057919837692381459835906926961710249833843825616804708573962174500521370143960759026927367100014238309267929739234743515922131 7
03969311164531464846897284906969291614103365468656506613735184614358172482970265193338769526841132294637057633490817354591180286981091484943404853800132480 0
64039585850616831036083975125715878007931845661557195042700377131515879724461790303180957503530221805556949600481652523340643228383046625782776979834717974 2
43206042273455815222910970064526120791979379121487741569495516013152044044829827860978718577392578526864136023551805724405282372517294167041855331075893065 6
21725656073982780513298815080689460685808245415873009967252807937912917457863067046614392847421025950163259336446422471846450666381138897988170484897495198 7
71074077108310526312371801477109332955660415721670300856738046307882511188272008176169180738980447027286867155355986199505293208621058269100986433687527875 7
80456871432050243898296055250552643137766805885012110192783126263298821890896456199883953332290957656242765747181528149536545953420376956178270977723909389 7
56540030403228909670211453286600603532795807757631620326207743352421909501719541699983833954074794056160800071942647251397666278351977631094918218971663895 5
58484941502896092153500950458155819349493770793484058595402363931271603222182877637156802816095876089295714111608584472743404292568002259238226596310049091 54
45199190107098193936705854034728715079542780830845848655078068328622768748030540422020162672832407136784645513122104074194766915133847498834305643851349184 2
70185767538265139523915139884207609595330458920679168965040620784396006610938704374703179551541828458723593694502820577394133963957541073922403105886355543 1
35688734049543106106879657801182554230206007702954762708525263775441907814331021682157370737747413383674961562713852002510402731525195685775435608553437779 5
84945494394602786190806516982468191489387180871936424620114924423138691207787724028042315021539080018458691464438141778528790561080730502491152723607138041 4
40035643393859736259018780223821309414155634278463333029343208190191537607739383888417926762046513905124464906816548219652823998206163702406936403375397800 2
83039677099026800535021226803878152186634977898549676605518330955711855081483732451949152333169462804972244409548440623028116633866740241961037457696689057 3
51183699977052825051135824769153373705759675150266325268070405890136794537743202661316907292722980862604742713932895415953997555689794587619919872228653421 1
93973908933565981766228329475418966951055532823833064309456116693596477536650430696107272962716229504554175087685170920116488889645075274565912076603064475 6
73359064162424648083153889582245656246799101345617794098410477892922811045175759571669491786328535877911554474149045970601745777362760186786641496373147933 6
28696646596908823245080240777382331046162759011074133264503486016245083721194973855035143276237492620405920473124046830974255217201851700148888056167828240 8
91498861596833766020686862986708044466699666912036870647079582720734283409566867590273406433753077039073371255838335135236692310801593130826678516313313548 3
00385585365780460668455731401423655453903071891255622268828472986383953307715161349270041490299173818140208665025642777615117755280804990066972101516735936 2
05385029415831714576118630602459380413502110658856202105290001745799778686733437804607018444732329024344032049324445053645252795097462970003339172154794227 1
23658294084157301896407878727975299335724341593280755225452389750382415672363189021101867246825653566335992869409988691821750622774004466844683063365487868 4
71221196908679269877535433884927375740138635819140209879134508863811311411695223740041326120067764024345323801165941208742544860807881375477919885346731764 2
40781176066705994068166479597408862104419104914319660418940010043679509698447615228817916623425327940098566406604296136156038041804701631499259903191293051 4
04263275629624836924267363918311385556708041945853429264845736027267459815839282078062224314057074651276135736113225779889389165341961689785189507236277679 5
00331389392289717706722284305436272744589576744452189025395132950694948477769641167611460453365098892807208197289231869867731196138282965482152811789110017 9
03322914658066536829079975488047999039168703992701248813516364060181881542952691385270244449388959592038109292472340072474194748257486600230325109611320483 5
51489318414964610252718235597801244879203922450368172597260408903539412252762945605147498538685302210962111080551306388015040823645117525188897810243982839 2
73931667981103924843985136240487870339058093807230981885464460702748136131381728515948366342452229049697104942512999049976180098349659659808581539550495319 7
93872053587057361285461491769747388252048578788673506775015077650156315042751263284130448923848537423791041102087718249938325917804090276456591671228934714 8
56553114199936554377680656091936259870285430018243157737116490453200233928531301357911503837520329940208463163054482552795551348367508194723760043485170829 1
21328160817785297595968911895125720240019342746326460191215164440452253575936740551241134175180790353065420969634272768106061105915304451018449468320993878 5
95789564328441887159239956314255427324578120473358956826996646144260899939807267058948901395498049469470053393832363184914752702980620323275845752691289162 4
66735440693884082854520586453913890631932657496954991341347891212168835471543541766663897662117992305594747254450322606040734110059454760013553720066846107 2
88952993828958519508953007899843831819981818484458515683073498894230276437185895341745994869291425343027154135325306380423540822347854118194740193516759946 2
32343865159756132260970782806198579303347142783819975871240652493259926936372122916249513810957747128805624281659352023696994898779811709706236921344318271 8

964409660619066863742080992610803447322113230374665193775276881203075457922320843993004476585308912796885705343154833661460871796405193301302371130918741910
432902298083175617822066582371811927208676405760611741049831326960784443738605259360057242916386979326804615994252282602027234561730549378772667370247638997
741684728667002546497832517288593316507750785236802511365418729806824000726907660124633496520717782322298435467200932121841534443141157278986040900425728446
149369684420569030691163844772915751707167384856611830635392611516585700278159011188133538906431359467567899562065166592600205966135407928755176217548981739
336421013706617461849169817252091048026516482595884476642828095106269962071646517591926720815325711517913700760360534973121072346556731272065770856980422440
340504368467825331411034779361525832736606221400681917978124800065946368055482881263447143096211121960778885325575363047717194085260159827397214905357050219
342490213244014423699084640533212159116684598134693515561852279148598181868172022791212269937278480692920672540021366584564399061709321393152718916889207324
304035723149884389672887932483113586375053469525804492847556265781530182401944076784531225603188835165851958522582081582797258927626068154180933275620600069
859581146246937314056027556064289503040433759130447103894144378383870715412098020819187085366039912169808750171213624773846518798802252641368355869398446191
981242655119098853452903191801271335127996942730464098960657993981235568836264770116822796830295012061788218139518028501026422042239139169110182386675229946
849629218246809150390847059910327821963677794189754448476085614395830188754457857051717880647884643434854761125760618012367573924646923256462896235522875346
676483218637462396687528249435966326406230137055390110336253120535882722168719229392577332228895648394558201645988952497102814286461341242101899795492102806
439649533720115148383407876355270028825086014568243186751417920223652023811746853946645746139385452590765336045427715416567552865424235796045318434150820669
614624263478329446913068205114623191765588557301069595257089516835074218925582141491027534421792010568891343081615648963783895057124626052130269587045934612
622618545838456698332400002553382152025984043935156967740967440708604039156654756292168802511188048924968998988677300558608734024918560859691077758068974028
272069737423807243672010900233377354404014074427989249080667836691701463656168507989755405992013037470886653271496390271096710138000222853498600571348485278
391299820707655839278330828086314938915684551028974276037132503459927579358396213762446318343909281207776002118995671978123248496735820636608536598625172842
015464013036758024977055770327373371899393708074438732276463012944461358756429591363300635622940756300807521753554101288333789273321608387132082379371164349
326455443222908401420854846391573736525283012224048635121951649615747490111974708220526983994148037917378983467923372200848704521188584337829783967784427626
611513747525589603259687176072794810801113093983622263976041121064649381297648503260350138813818432641295468626749470205410398475305506058728802344864193803
699812521783481860979071181252942580179600638844554691685368520842424970332143251060830597172526551761334271126439858653114387402870042868491203857894398843
304513863778042496055634253991593837034470000217382191502008814588997732116189146365057760760207331205591180345165609386482081710536861969586238049077866557
172824586036086647287312671318153962790639117807272504531728851213899000673345510213828174584493657011291897821895744565449186471632256884038262062526406024
332711495083469050890268134500292512346057226893667042541437087822268786084391066010768197767915552885518609330502090513541439278029957715191076404504083301
108880013781404435278164820111401390351811599834070456573415741157635330703662988452319535296891394223996142224511422328730404045350677331332779868133194482
484002242327680526898942096257101380095477537823608310300196471298571546151517612296495958540531164608724914565062063327156527990117024078502840455988748232
263645598539527206618456872818916639742417175018712984764505606234099993123345547456168413332118568058214660787255433955121771185837103239843657698954464517
088517906252946101702483761216767357204116949623224517882300616979060094404079975248771345405871393524245225037070793156855161954117990414564318084638417468
988640670238110754123836017915089263744855093550766149289160839066489472338341371723841881156238418641493595899517909276892557527343768639952566041785457456
195482622168025030257750955379236287312858358439453182949334399717712898999238888297310008962177737303514827746819582162947328967644848086162835756531904553
524909022152019299549001602248513045793778990961405754530551942982144892373658128228048106091057670253627881163032833592777246012883179026614538695868949523
117283386930656968369769311611243228189087623651478876040927283966128254404619593393823590093796818493186296859547543563527505645760412225258717399769815108
127871620523687756725100315279841165184216506941720010480874935170290269262514759790310423070687508678412050524319008546827754105345348910260763234122339291
320431703376300130736360606653234316675645827964488818849669333095560160621605861209018066998961594171466188642739719732426663857844853503127051875381756648
487311953338687901868980639609763800665861141479366053060062895318379370481831415003153084947729993087252880157073469681559523815065157501495289876105463017
269145851177446618291432682570708808206517284177153458255323457019044137151849340427359489498278684886289770022597911978913586372137423838092828540222160129
842795179724653633341073520644010893725013832901127774235446974761997384156173743519485650756178619616183386002955771373443769454889717912875156708549731549
078376911547176776178340815453591692055678972347292189396171437138160983013972392652728880040902197821136680512160880655465894158641498317490091385539325886
764329892113873143410374266662751210056352652359126761012570156546136173800877167349408348217809381252002979227928106200573438611388902955393742492239145669
033203486114900770795996254413261200181205069141477060087426963692580214769043578388589475846776339647969517274212193219699413414907042877064866915707709443
902629686175663087925145159931593797909091586496453123887290789184937069416417316383214267149139740718333681795273832918271498261014173728251982654095897752
175258890451237257021994788014198863451383858970654475390626044783708402838818807063201977136955576970522353163770078581614142036294944013931049883874578858
942909633024300475457899024166172748927118049304227699170087517387483517080367122406577207601404403930068035343044961060835726654659125005469136164176435706
110054259904263953282782813944532219656841222381192067936687916721218323921825995761627944770151582287665311422743420621573582801576510197260772174783850863
472406700876402043977498536528186701647533965884858779123595547235048400880500768108925824461333775267657411057975761433216627127911367910620735826245483090
195909071968298758407830869843423935069482081236713528620900888826379718402910712623991934673985669269978355729303895836838814956446848226685744668411304895
812389548876574091283381823132530899127273783017961752799565359638491706998102720518533145898126034944029526774745492456782814304117885840382963519662416915
781801030605933234386083553660201206894278710075617284050418190092200286673656269315575008386913209010303266912473927486481635749241524754901191057387164613
329723726401321391125425383862445438933076847413728623617259583973397567921631996354606907921471697810497295022819288281087233921929895489511143449021205385
056656348209442916440910740827390631781170912911610112978479104828722339259261768888950966330387273070209629016069253684004557123138701155917319248834870041
521884343792303904438419879012366895669332702464684957362733803980826249586219607215771731416714291973839285959655992710254142944445720890002031823752051919
679663721796160469299525631419878235116340671774147616881937177505609970802779969318776459134552287254364443970793334695789527431538515151099905007914897987
575528896185733812983638154468811553463746158091767196846000875388748476833547981489802454797298184944492165976310514214917519654244898004983033707132182983
554719431071659706217848032935934134711478811032667960821078126806695756295103454071008532808721198532632222404239069392273337027107750986584472728633170224
699682954553133523872288694452733736344812276008608110398795067820311514541474400839437304516100954246806703041109141679008739220638161967561852330167919703
403046735012574428548946113918275236025700790982611779870479553143165462069556421052014211685568480684558468067928871031223962574299839984717820208548590562
690126401450991095301716295604884378097334364768989981345637383303329188082226614096210284020581551367396894088143934864435862242379287007575994371529588371
620483825736020200353752024597411997122105046563158270241429776968781024014291678693898162116631247747212609194040125617295404985737832713354543797853344723
808425485504306156321766842291035868947461639228045580146356891951834532262284917963634093955863754398874277494894829060220017111864175544349443098494496963
678742925126974769106525330107566473897196952292880620951612273707324795452565696875218714757116585914974078775731424918206904467766379677545924557944237520
127821488043181012510329669982172531651515881939766084510017644276488139342623618360298161826195372252563746655382275125784222232197945088916631999072109205
000381778700907737393132238416960544516989351692221116281860259328020231621244501235713603055020953079308211379400911394833209912259316277542899735906678193
523534606932391965985678903005218051068146650322976134616130509398967259534094360832522872678745243835864393122995836521761450642203174288485361667424714418
742679529888675713675474694488915445976381070852926785995418787209316071348086100057761749246710483102086534083529281222288770141135065113570082823753950340
813659985661358207817419260049944360939037653959977055286262025112358443764441845231944028705190258722712223006372728943628274276000319007247475091042510351
666045550579888510409608114385396096812447285543195934836198453084728116971246309678523378967705840064147910742922569460849268143135289499417839820376055044
726247501461364880301267382737632060966873043516179237042389584609845056583054233250409008941748988161936555559062784302575066673258644797047785687635627944
803438607609462345198811546054227778602857825235954235065528769918980595857246166034249990839042313524545197616527095557642675978519730520221285578829116865
120030578108715071441884711425176265604920510533343379942358010023469786366561379014785615261743589280662016081668713836196343708073340399952859139805990312
329041165973369507673720150118431259831230306585493643189172904542969227262664998082031542409196848417001118499390262559563390654186039246001825169016483096
320484007297560177076664560617650593816946752912072042405989873909472097167204003658464498291386021485692339748114689026550938669882959845306202077922120533
392895192148367665259417428470010669734970306585645620655751570553804015848279603004633290104408470424191319585551554518552568315928827858502486744611958503
957474389785552818428104991749311675042303707070340089689122731115863200808965767674711510539202007078033385566527749287320681599973182826449491193743150033
958754048616498679922804688562572443297351718258250676073242050759937938875109981900242349940279720206706557420837051714671218359137385253558261966653930384
55433025