A proof of the Collatz (3x+1) conjecture

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Abstract

In this paper, we had given a proof of the Collatz conjecture in elementary algebra. Since any given positive integer is conjectured to return to odd 1 in operations, we analyze continuous inverse operations starting with odd 1, it had proved that all of the inverse path numbers of a given non-triple is obtainable and any inverse operation path tends to infinity, we can get any odd and even, to do continuous forward operations for a positive integer obtained it will return to the odd 1 along the inverse operation paths.

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The Collatz conjecture is the 3x+1 conjecture, also known as Kakutani's problem, and it has not been proved since it was proposed [1]. The Collatz operation is described by the following function

$$f(x) = \begin{cases} 3x+1 & \text{if } x \text{ is odd} \\ \frac{x}{2} & \text{if } x \text{ is even} \end{cases},$$

where $\forall x \in N^+$, for this function there exists $s \in N^+$such that

$$f^{(s)}(x) = 1.$$

As a given even number will be firstly converted to an odd number from the function above and then we again get another odd number, it is clear that we can take odds directly to analyze, and then we have the operation formula as bellow

$$p = \frac{3n+1}{2^k}, \tag{1.1}$$

or

$$2^k p = 3n + 1, \tag{1.2}$$

where n and p are both odds, $k \in N^+$. It is not difficult to see from (1.2) that odd p is not a triple, but a non-triple. To do an inverse operation for (1.1) we then have

$$n = \frac{2^k p - 1}{3}. \tag{1.3}$$

Definition 1.1. *The operation process from n to p using (1.1) is called one time of forward operation and times of operations is called continuous forward operations, and for n, p is called its forward path number. The operation process from p to n using (1.3) is called one time of inverse operation and times of operations is called continuous inverse operations, and for p, n is called its inverse path number.*

In the odd set, each side of an odd triple has a non-triple such as 1, $\underline{3}$, 5, or there are two non-triples between two adjacent triples such as 5 and 7 between 3 and 9. We called the left one as left non-triple and the right one as right non-triple.

Lemma 1.2. *In the odd set, a triple has not any inverse path number, while a non-triple has an infinite number of inverse path numbers and they are obtainable.*

Proof. Let a triple in the odd set be $3(2t-1) = 6t-3$, and then we can get the left non-triple $6t-3-2 = 6t-5$ and right non-triple $6t-3+2 = 6t-1$ respectively, where $t \in N^+$ and $2t-1$ is an odd, for example, when $t = 1$, the triple is 3, its left non-triple is 1 and its right non-triple is 5. Let the inverse path numbers of $6t-3$, $6t-5$ and $6t-1$ be n_0, n_1 and n_2 respectively. Next, we analyze these three inverse path numbers.

a) The triple $6t-3$. By (1.3) we have

$$n_0 = \frac{2^k p - 1}{3} = \frac{2^k(6t-3) - 1}{3} = 2^{k+1}t - 2^k - \frac{1}{3}, \tag{1.4}$$

obviously, the right-hand side of (1.4) is not an integer, thus a triple has not any inverse path number.

b) The left non-triple $6t-5$. By (1.3) we have

$$\begin{aligned} n_1 &= \frac{2^k p - 1}{3} = \frac{2^k(6t-5) - 1}{3} = \frac{2^k(6t) - 5 \cdot 2^k - 1}{3} = 2^{k+1}t - \frac{6 \cdot 2^k - 2^k + 1}{3} \\ &= 2^{k+1}t - 2^{k+1} + \frac{2^k - 1}{3} = 2^{k+1}(t-1) + \frac{2^k - 1}{3}. \end{aligned} \tag{1.5}$$

In three consecutive positive integers $2^k - 1$, $2^k$and $2^k + 1$, there must be a triple. If k is even, $2^k - 1$ is a triple, if k is odd, $2^k + 1$ is a triple. Let $k = 2m$, where $m \in N^+$, by (1.5), we have

$$n_1 = 2^{2m+1}(t-1) + \frac{2^{2m} - 1}{3}. \tag{1.6}$$

It is not difficult to see from (1.6) that the non-triple $6t-5$ has an infinite number of inverse path numbers and they form an inverse path number sequence, to do one forward operation for each of them they all return to $6t-5$.
In (1.6), let $m=1$, then we have

$$n_1 = 8t - 7, \tag{1.7}$$

it is the first inverse path number of a left non-triple and also the smallest one.
In (1.6), let $t=1$ ($6t-5=1$, the smallest left non-triple), we then have

$$n_1 = \frac{4^m - 1}{3}. \tag{1.8}$$

As $m$increases in sequence, we obtain the following infinite increasing sequence

$$1,\ 5,\ \underline{21},\ 85,\ 341,\ \underline{1365},\ 5461\ldots$$

where the triples is underlined, and they are all the inverse path numbers of the first left non-triple 1. We called this sequence as basic sequence.
Next, we analyze the relationship between two adjacent inverse path numbers in a sequence. Let $n_{1,m}$ be an inverse path number from (1.6), $n_{1,m+1}$ be its next, thus from (1.6), we have

$$n_{1,m} = 2^{2m+1}(t-1) + \frac{2^{2m}-1}{3}. \tag{1.9}$$

We replace m in (1.9) with $m+1$ and to find $n_{1,m+1}$, then from (1.6) we have

$$\begin{aligned} n_{1,m+1} &= 2^{2(m+1)+1}(t-1) + \frac{2^{2(m+1)}-1}{3} = 2^{2m+1+2}(t-1) + \frac{2^{2m+2}-4+3}{3} \\ &= 4(2^{2m+1}(t-1)) + 4\left(\frac{2^{2m}-1}{3}\right) + 1 \\ &= 4\left(2^{2m+1}(t-1) + \frac{2^{2m}-1}{3}\right) + 1 = 4n_{1,m} + 1, \end{aligned} \tag{1.10}$$

that is

$$n_{1,m+1} = 4n_{1,m} + 1. \tag{1.11}$$

In all the sequences which are obtained from the left non-triples, two adjacent inverse path numbers satisfy (1.11).

c) The right non-triple $6t-1$. By (1.3) we have

$$n_2 = \frac{2^k p - 1}{3} = \frac{2^k(6t-1)-1}{3} = \frac{2^k(6t) - 2^k - 1}{3} = 2^{k+1}t - \frac{2^k+1}{3}. \tag{1.12}$$

Let $k = 2m-1$, from (1.12) we then have

$$n_2 = 2^{2m}t - \frac{2^{2m-1}+1}{3}. \tag{1.13}$$

It is not difficult to see from (1.13) that the right non-triple $6t-1$ has an infinite number of inverse path numbers and they also form an inverse path number sequence, and to do one forward operation for each of them they all return to $6t-1$. In (1.13), let $m=1$, we then have

$$n_2 = 4t - 1. \tag{1.14}$$

It is the first inverse path number of a right non-triple, and also the smallest one. In (1.13), let $t=1$ ($6t-1=5$, the smallest right non-triple), we then have

$$n_2 = 4^m - \frac{2^{2m-1}+1}{3}. \tag{1.15}$$

As m increases in sequence, we obtain the following infinite increasing sequence which is composed of all the inverse path numbers of the right non-triple 5

$$\underline{3},\ 13,\ 53,\ \underline{213},\ 853\ldots$$

As the same, to do one forward operation for each of them they all return to the second odd 5 in the basic sequence.

In the same way, from (1.13) we can also derive a formula of the relationship between two adjacent inverse path numbers in a sequence obtained from the right non-triples as below

$$n_{2,m+1} = 4n_{2,m} + 1. \tag{1.16}$$

In all sequences which are obtained from the right non-triples, two adjacent inverse path numbers satisfy (1.16). By (1.11) and (1.16), we have

$$n_{i,m+1} = 4n_{i,m} + 1, \tag{1.17}$$

where $i = 1, 2$. From (1.17), we can see that any inverse path number sequence is an infinite increasing sequence. □

Corollary 1.3. *In a sequence, there is a triple in every three consecutive inverse path numbers.*

Proof. Let a triple be $3t$, where t takes odds. Thus using (1.17) we can get one by one three consecutive inverse path numbers after $3t$ as follows

- the first $4(3t)+1 = 12t+1 = 3(4t+1)-2$ it is a left non-triple,
- the second $4(12t+1)+1 = 48t+5 = 3(16t+1)+2$ it is a right non-triple,
- the third $4(48t+5)+1 = 192t+21 = 3(64t+7)$ it is a triple.

This is the order of every three consecutive inverse path numbers in any sequence. Using this property, we can easily determine whether a non-triple is a left one or right one. □

Remark. *Expanding n_1 in (1.6) and n_2 in (1.13) out for any $t \geq 1$ and $m \geq 1$, the odds of them just be all of the odds in the odd set, but we still needs a proof because this argument is the key to proving this conjecture. Here, we further analyze this issue.*

From lemma (1.2) we know that the inverse path numbers of any non-triple form a sequence in which every two adjacent inverse path numbers satisfies (1.17), and we have two first inverse path numbers $8t-7$ from (1.7) and $4t-1$ from (1.14). The odds from $8t-7$ and $4t-1$ for any $t \geq 1$ is a part of the all odds in the odd set. It is clear that the rests is $8t-3$ for any $t \geq 1$. To verify. We let the odds be represented as $2t-1$ for any $t \geq 1$, where $t \in N^+$, and let the inverse path numbers in any sequence except the first be $T(t)$, then by (1.17) we have

$$T(t) = 4(2t-1) + 1 = 8t - 3.$$

Based on the starting odds 1 of $8t-7$ and 5 of $8t-3$, and the same gap of 8, it is not difficult to see that they can be represented as $4t-3$. As $4t-1-(4t-3) = 2$, thus we obtain the continuous odd series, that is, we get any odd in the odd set. For this issue, we will still analyze in Theorem 1.5.

Now we analyze the continuous inverse operations starting with the odd 1. By odd 1 we get the basic sequence, for all the non-triples in the basic sequence except the odd 1 we can get countless sequences (Lemma 1.2). To take it repeatedly we can get continuous inverse operation paths. We skip the triples in any sequence. If there has not any cycle in the continuous inverse operation paths, then all the paths will tend to infinity because the quantity of the odds smaller than a given odd is finite.

Lemma 1.4. *Any two inverse path numbers from the same or different sequences are not equal to each other.*

Proof. From (1.17) we can see that there has not any two equal inverse path numbers in a sequence. Suppose $n_1(t_1, m_1) = n_2(t_2, m_2)$, then by (1.6) and (1.13) we have

$$2^{2m_1+1}(t_1 - 1) + \frac{2^{2m_1} - 1}{3} = 2^{2m_2} t_2 - \frac{2^{2m_2-1} + 1}{3}, \tag{1.18}$$

or

$$12 \cdot 4^{m_1}(t_1 - 1) + 2 \cdot 4^{m_1} = 6 \cdot 4^{m_2} t_2 - 4^{m_2},$$

that is

$$4^{m_1}(12t_1 - 10) = 4^{m_2}(t_2 - 1). \tag{1.19}$$

If $m_1 = m_2$, by (1.19) we have

$$12t_1 - 10 = 6t_2 - 1, \tag{1.20}$$

that is

$$2(2t_1 - t_2) = 3, \tag{1.21}$$

based on parity properties, we can see that (1.21) does not hold for positive integers. If $m_1 > m_2$, by (1.19) we have

$$4^{m_1 - m_2}(12t_1 - 10) = 6t_2 - 1, \tag{1.22}$$

we can also see that (1.22) does not hold for positive integers.
If $m_1 < m_2$, by (1.19) we have

$$12t_1 - 10 = 4^{m_2 - m_1}(6t_2 - 1), \tag{1.23}$$

as $4^{m_2-m_1}(6t_2 - 1) \equiv 0\ (mod 4)$, but $12t_1 - 10 \equiv 2\ (mod 4)$, thus (1.23) does not hold for positive integers.
Thus, by (1.21), (1.22) and (1.23) we have $n_1 \neq n_2$. In the same way, it can be proven that any two inverse path numbers from two different sequences which are both obtained from two different left non-triples or right non-triples is also unequal. □

Theorem 1.5. *There are all of the odds in the continuous inverse operation paths and the Collatz conjecture holds.*

Proof. From Lemma 1.4 we can see that there has not any cycle in any continuous inverse operation path starting with the odd 1 or another (if it is a triple, to take its next inverse path number using (1.17)), thus any path will tends to infinity, and to do continuous forward operations for any odd obtained in the paths, it will return to the odd 1 or the starting odd. If there is an odd missed in one of the paths starting with the odd 1, it also will return to the odd 1 when doing continuous forward operations for it as its continuous inverse operation paths tend to infinity, thus it must also be at one of the inverse operation paths. From this we can draw a last conclusion that we can obtain any odd in the inverse operation paths and thus the conclusion in the remark holds, that is, n_1 and n_2 exactly contain all odds. From all odds obtained we can get all evens if to multiply with 2^k in (1.1) for each odd, thus we can get any positive integer. As to do continuous forward operations for any positive integer obtained in the paths, it will return to the odd 1, thus the Collatz conjecture holds. □

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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