Interest Rate Curve Simulation

David Lee

ABSTRACT

Interest rate curves play an important role in financial market. Curve forecasting is used for risk management, hedge, and arbitrage. The article proposes a model for simulating forward interest rate. The number of drivers is to be three to adequately capture the short-, medium-, and long-term rates since each is driven by different mechanism, e.g., short-end is driven primarily by policy maker; long-end driven by market, although policy maker could also assert influence through Operation "Twist".

Key Words: interest rate, counterparty credit risk, simulation, calibraton

There are different types of interest rates, mainly government rates and interbank rates, such as LIBOR (London Interbank Offered Rate) or OIS (Overnight Indexed Swap) rate. LIBOR, as the name implies, is the rate of interest that one London bank will offer to pay on a deposit by another. There will, in general, be a different LIBOR for each of the standard deposit maturities. Whereas government rates are determined by government issued bonds.

An interest rate curve is the graph of function between maturities and associated interest rates. Curves can be bootstrapped from government bonds or index rates. An interest rate curve is also called the term structure of interest rates.

LIBOR curves have more advantages than government curves. They are more liquid and have stronger correlation among financial products. Therefore, the base LIBOR curve become funding curve and more efficient for hedging and pricing.

The manipulation scandal with LIBOR and other benchmarks was uncovered in 2012 that had negative impact on the reliability and robustness of financial markets. The Financial Stability Board (FSB) recommended to replace LIBOR or more general IBOR with risk free reference rates. The most popular risk reference rates are SOFR, ESTR, SONIA, TONA, CORRA, etc.

There is a rich literature on interest rate. Medova al et. (2006) use a three-factor interest rate curve model and the Kalman filter to study EU swap yield data from 1997 to 2002 and capture the salient features of the whole term structure in forward simulation.

Bolder and Streliski (1999) create a framework to construct a historical data base of zero coupon and forward yield curves estimated from Government of Canada securities and have a better understanding of the behaviors of a class of parametric yield curve models.

Madsen (2012) defines a general model for the shift function and specify a risk model that uses the shift function. Fisher (2004) examines the forces impacting interest rates in the context of an extremely simple model in which all uncertainty was resolved by the simple flip of a coin.

Akram (2020) proposes a long-term interest rate model to represent John Maynard Keynes's conjecture that the central bank's actions influence the long-term interest rate primarily through the short-term interest rate.

Li and Su (2021) apply a rolling-window strategy to determine the dynamic linear and nonlinear Granger causality relationship between short- and long-term interest rates over time and visualize the results.

Crummp and Gospodinov (2024) introduce a nonparametric bootstrap for the yield curve that is agnostic to its true factor structure. They deconstruct the yield curve into primitive objects, which weak cross-sectional and time-series dependence,

Levrero and Matteo [2019] study the causal relationship between short- and long-term interest rates and outline an asymmetry in the relationship. Bauer and Hamilton (2019) propose a new bootstrap procedure to test the spanning hypothesis and conclude that conventional tests of whether variables other than the level, slope and curvature can help predict bond returns have significant size distortions.

This article presents a new interest rate model. Three risk factors are designed to capture the dynamic of short-, medium-, and long-term interest rate. Given a scenario of these three risk factors, we can construct a curve at 3M, 5y3M, and 15y3M points. Linear interpolation may lead to some discrepancy with market curve which has more granular points on the term structure.

Mean reversion of interest rates is considered a desirable property of a model because it is perceived that interest rates tend to trade within a fairly tightly defined rage. This indeed true, but when pricing exotic derivative (see https://finpricing.com/lib/EqWarrant.html) it is the effect of mean reversion on the correlation of interest rates at different time that is more important.

The total amount of interest that the depositing bank will receive is calculated by multiplying the LIBOR by the amount of time, as a proportion of a year, for which this money has been on deposit This amount if tune us caked the accrual factor or day count fraction,

Intuitively, multi-factor short rate or instantaneous forward rate driven term structure, is thought to be unlikely to guarantee non-negativity of forward rate on the curve, since the two points on the curve are driven by factors not necessarily meeting that condition. Forward rate model can satisfy that condition, but LMM is not mean-reverting. So we need to make it mean-reverting.

we assume that the logarithm of each of these risk factors follows Vasicek model. The initial value can be derived from today's yield curve. That guarantees the non-negativity and mean-reversion of the forward rate. Also, the lognormal distribution of the forward rate at future simulation time buckets is consistent with the Black model condition for cap/floor.

The only desirable feature this model fails to capture is the fat-tail of daily return (i.e. relative change) distribution of the risk factor. However, the daily return measured in absolute change is not normal and have fatter tail, as opposed to HW model where the absolute change is normal. Anyways, that's not the primary concern of the model devised for CCR given long simulation horizon.

if we want to reduce the number of risk factors, we could do principal component analysis (PCA), and select, say 3 risk drivers. Now, it requires a nx3 matrix to recover the n tenor points in simulation. This matrix is the transpose of the eigenvector of the corresponding PC. Again, there will be discrepancy at the initial curve since PCA is only a projection, and the system is over-determined (a smaller number of risk drivers than the number of tenor nodes)

The rest of this paper is organized as follows: The model is presented in Section 1; Section 2 elaborates calibration. Numerical results are discussed in Section 3; the conclusions are given in Section 4.

1. Model

We propose to model the forward rate in a PCA setting. The number of drivers is to be three to adequately capture the short-, medium-, and long-term rate since each is driven by different mechanism (short-end driven primarily by policy maker, long-end driven by market, although policy maker could also assert influence through Operation “Twist”). Each driver is assumed to follow exponential OU process.

Intuitively, multi-factor short rate or instantaneous forward rate driven term structure, is thought to be unlikely to guarantee non-negativity of forward rate on the curve, since the two points on the curve are driven by factors not necessarily meeting that condition. Forward rate model can satisfy that condition, but LMM is not mean-reverting. So we need to make it mean-reverting.

Now, we assume that the logarithm of each of these risk factors follows Vasicek model. The initial value can be derived from today’s yield curve. That guarantees the non-negativity and mean-reversion of the forward rate. Also, the lognormal distribution of the forward rate at future simulation time buckets is consistent with the Black model condition for cap/floor.

The only desirable feature this model fails to capture is the fat-tail of daily return (i.e. relative change) distribution of the risk factor. However, the daily return measured in absolute change is not normal and have fatter tail, as opposed to HW model where the absolute change is normal. Anyways, that’s not the primary concern of the model devised for CCR given long simulation horizon.

If we want to reduce the number of risk factors, we could do PCA, and select, say 3 risk drivers. Now, it requires a nx3 matrix to recover the n tenor points in simulation. This matrix is the transpose of the eigenvector of the corresponding PC. Again, there will be discrepancy at the initial curve since PCA is only a projection, and the system is over-determined (less number of risk drivers than the number of tenor nodes)

Scenarios for an interest rate curve are generated by simulating the correlated returns of risk factors on the curve, which are currently computed as either relative returns or absolute returns. Since it is the returns and not the rates that are simulated, the returns scenarios need to be converted back to rates for re-pricing under each scenario. The rate scenarios, x_s, can be expressed in terms of the simulated returns as:

$$x_s = (1 + REL)x_0 \qquad \text{for relative returns} \qquad (1)$$

$$x_s = ABS + x_0 \qquad \text{for absolute returns} \qquad (2)$$

where x_0 is the closing rate, ABS is the absolute return scenario (i.e., $ABS = \Delta x$), and REL is the relative return scenario (i.e., $REL = \frac{\Delta x}{x_0}$).

For large shocks, such as those for stress scenarios, the rate scenarios x_s can potentially be negative-valued. We can see from the above expressions that negative values of x_s would occur for relative return scenarios REL less than -1 and for absolute return scenarios ABS less than the closing rate x_0. Under the current stressed VaR framework, the approach taken to ensure non-negativity of rates is to floor the rate scenarios at 0, that is to say, the corrected rates $\tilde{x}_s$ are given by:

$$\tilde{x}_s = \max(0, x_s). \qquad (3)$$

An alternative approach to dealing with the problem of negative-valued rates for the case of relative returns is to instead use log returns. The rate scenarios x_s would then be given by:

$$x_s = x_0 e^{LR} \qquad (4)$$

where LR is the log return scenario (i.e., $LR = log\left(\frac{x}{x_0}\right)$). We can clearly see that using log returns would ensure that the rate scenarios x_s are always non-negative.

In addition to negative rates potentially resulting from simulation, there is also the possibility of negative forward rates occurring. To correct for negative forward rates, an algorithm described in [1] is used to adjust the negative-valued rate scenarios. The method involves first checking the condition that all forward rates of the given interest rate curve are above a minimal constant level $\varepsilon > 0$, and then applying a perturbation to the key interest rates so that the condition is satisfied.

For an interest rate curve that is described by a set of key zero coupon rates, $\{r_1, \dots, r_i, \dots, r_n\}$, and for the method of linear interpolation of interest rates between key rates, the necessary and sufficient condition for all forward rates of an interest rate curve to be above a minimal constant level $\varepsilon > 0$ is the following set of inequalities:

$$r_{i+1} - \frac{t_{i+1}}{2t_{i+1}-t_i} r_i \geq \varepsilon \frac{t_{i+1}-t_i}{2t_{i+1}-t_i}\ , \quad i = 1, \dots, n-1 \qquad (5)$$

If the above condition does not hold for a given Monte Carlo scenario of an interest rate curve, the key interest rates for the scenario need to be adjusted in such a way that the 'optimally' adjusted set of key rates $\{\tilde{r}_1, .., \tilde{r}_n\}$ satisfies the above condition. As described in [1], the adjustment of the key interest rates to satisfy the above condition can be formulated as the following optimization problem:

Find a solution $\{\tilde{r}_1, \dots, \tilde{r}_n\}$ *of the weighted least squares problem*

$$F(\tilde{r}_1, \dots, \tilde{r}_n) = \sum_{i=1}^{n} \frac{1}{\sigma_i^2} (\tilde{r}_i - r_i)^2 \longrightarrow \min \qquad (6)$$

with the linear constraints

$$\tilde{r}_{i+1} - \frac{t_{i+1}}{2t_{i+1}-t_i} \tilde{r}_i \geq \varepsilon \frac{t_{i+1}-t_i}{2t_{i+1}-t_i}\ , \quad i = 1, \dots, n-1 \qquad (7)$$

where the σ_i are the volatilities (see https://finpricing.com/lib/FxVolIntroduction.html) corresponding to the key interest rates. We note that under the current stressed VaR framework, ε is set to 1 bp. A detailed description of the algorithm used to solve the above optimization problem is provided.

Each curve is as a linear interpolation and/or extrapolation of the following nine interest rates:

$$\{ r_{1m} ,\ r_{3m} , \ldots\ldots,\ r_{10y} \} \tag{8}$$

For convenience, denote

$$t_1 = 1\text{m},\ \ t_2 = 3\text{m},\ \ldots..,\ t_9 = 10\text{y} \tag{9}$$

A correlated random movement of nine term rates for a given currency determines a random movement of their represented yield curve. A random movement for the nine term rates { r_{t_1} , r_{t_2} , …., r_{t_9} } at time $t+\Delta t$ is simulated by the equation [III.7] given a set of nine random numbers { $\varepsilon_{t+\Delta t,t_1}$, $\varepsilon_{t+\Delta t,t_2}$, …, $\varepsilon_{t+\Delta t,t_9}$ }. Only three of these nine random numbers are drawn randomly. Each $\varepsilon_{t+\Delta t,t_i}$ of the nine random numbers is determined by three key numbers $\varepsilon_{t+\Delta t,t_1}$, $\varepsilon_{t+\Delta t,t_5}$ and $\varepsilon_{t+\Delta t,t_9}$. It is given by the following formula:

$$\varepsilon_{t+\Delta t,t_i} = (B_c)_i * \begin{bmatrix} \varepsilon_{t+\Delta t,t_1} \\ \varepsilon_{t+\Delta t,t_5} \\ \varepsilon_{t+\Delta t,t_9} \end{bmatrix} \tag{10}$$

where $(B_c)_i$ is the i^{th} row of the following matrix:

$$B_c = \begin{bmatrix} 1 & 0 & 0 \\ \frac{21}{\sqrt{445+84\rho_{1m,2y}}} & \frac{2}{\sqrt{445+84\rho_{1m,2y}}} & 0 \\ \frac{18}{\sqrt{349+180\rho_{1m,2y}}} & \frac{5}{\sqrt{445+84\rho_{1m,2y}}} & 0 \\ \frac{12}{\sqrt{265+264\rho_{1m,2y}}} & \frac{11}{\sqrt{265+264\rho_{1m,2y}}} & 0 \\ 0 & 1 & 0 \\ 0 & \frac{7}{\sqrt{50+14\rho_{2y,10y}}} & \frac{1}{\sqrt{50+14\rho_{2y,10y}}} \\ 0 & \frac{5}{\sqrt{34+30\rho_{2y,10y}}} & \frac{3}{\sqrt{34+30\rho_{2y,10y}}} \\ 0 & \frac{3}{\sqrt{34+30\rho_{2y,10y}}} & \frac{5}{\sqrt{34+30\rho_{2y,10y}}} \\ 0 & 0 & 1 \end{bmatrix} \qquad (11)$$

and where $\rho_{1m,2y}$ and $\rho_{2y,10y}$ are the correlation coefficients between r_{t_1} and r_{t_5} and between r_{t_5} and r_{t_9} respectively.

The spot rate is given by:

$$r_\tau = R_\tau \exp\left(-\frac{\sigma_\tau{}^2}{2} \left(\frac{1-e^{-2\,\kappa_\tau\,t}}{2\,\kappa_\tau}\right) + \sigma_\tau\, Y_\tau(t) \right), \quad Y_\tau(t) = \int_0^t e^{-\kappa_\tau\,(t-s)} dWs \quad (12)$$

There are four calibration approaches available for the model: 1) Historical quantile match: we match the historical ratio of 95th and 5th and one of the 2 quantiles at ∞, the long-term mean is then implied by those 2 quantiles. 2) Historical 95th – 5th width match at ∞ and historical mean for long term mean. 3) Match the historical ratio of 95th and 5th and choose the long term mean independently. For example, using historical average or forward rates. κ will be the same as in 1. 4)The last calibration is matching the historical 5th and mean of the distribution at ∞ and letting the lognormal distribution decide on the 95th location. This approach is conservative as the implied distance 95th - mean is always bigger than mean – 5th for a lognormal distribution, resulting in a wider envelope of simulated rates.

2. Calibration

The mean reversion speed κ is calibrated assuming the interest rate distribution is constant and not time dependent which theoretically happens at infinity.

2.1 Quantile match

In this calibration,

- κ is obtained from matching the ratio of $\frac{r_{95}}{r_5}$ at ∞ which yields one solution

It is given by:

$$\mathrm{WLN_{hist}} = \ln\left(r_{95}(t)\right) - \ln\left(r_5(t)\right) = \sigma\sqrt{\frac{1-e^{-2\kappa_\tau t}}{2\kappa_\tau}}\left(N^{-1}(0.95) - N^{-1}(0.05)\right)$$

At $t = \infty$, we get the simple equation:

$$\mathrm{WLN_{hist}} = \ln\left(r_{95}(\infty)\right) - \ln\left(r_5(\infty)\right) = \frac{\sigma}{\sqrt{2\kappa_\tau}}\left(N^{-1}(0.95) - N^{-1}(0.05)\right)$$

which has only one positive solution. θ is obtained from κ and the condition to match the 5th or 95th quantile (which produces logically the same result)

$$r_5(\infty) = \theta_\tau \exp\left(-\frac{\sigma_\tau^2}{4\kappa_\tau} + \sigma_\tau \frac{N^{-1}(0.05)}{\sqrt{2\kappa_\tau}}\right)$$

Therefore, they are no stability or multiple solution issues encountered during calibration.

2.2 Width match

The width match relies on matching $r_{95} - r_5$ which yields the equation at $t = \infty$:

$$r_{95}(\infty) - r_5(\infty) = \theta \left(e^{-\frac{\sigma_\tau^2}{2\kappa_\tau} + \sigma_\tau \frac{N^{-1}(0.95)}{\sqrt{2\kappa_\tau}}} - e^{-\frac{\sigma_\tau^2}{2\kappa_\tau} + \sigma_\tau \frac{N^{-1}(0.05)}{\sqrt{2\kappa_\tau}}} \right)$$

This equation yields one none degenerated solution κ_τ^* when $\theta_\tau$is chosen as the average of the historical rates. When θ is chosen to be too small (e.g. set to the current forwards), no solution to the minimization will be returned.

2.3 Ratio match and flexible mean

- κ is obtained the same way as in calibration 1
- θ is obtained independently, e.g. set to the forwards or historical mean

As per calibration 1, this calibration does not have instability of solutions except if κ is calibrated at a specific horizon instead of the proposed $t = \infty$.

2.4 Mean and one quantile match

This last calibration matches

- The mean is obtained easily by setting $\theta = avg_{hist}$. It could also be chosen to correspond to the forward rates.

- κ needs to solve the following

  $r_5(\infty) = \theta \left(e^{-\frac{\sigma_\tau^2}{2\kappa_\tau} + \sigma_\tau \frac{N^{-1}(0.05)}{\sqrt{2\kappa_\tau}}} \right)$ Which is a polynomial of degree 2 in κ.

Hence it has 2 solutions. Both solutions are positive in this case. One solution is extremely close to zero for all tenors; we will discard it here since small κ (e.g. < 5%) indicate a complete diffusion behavior and create numerical problems during simulation. The other solution gives very reasonable κ values as it can be seen in the numerical examples hereafter.

3. Numerical Results

An example of an interest rate curve scenario for which an adjustment needs to be applied to remove negative forward rates is shown in Figures 1 and 2. The closing rates of the USD_STUB curve along with two stress scenarios using simulated 10-day returns are shown in Figure 1. 10-day returns were simulated as $\sqrt{G\left(\sqrt{10}, \sqrt{10}\right)} \cdot Z$, where $G\left(\sqrt{10}, \sqrt{10}\right)$ is a gamma-distributed random variable with both shape and scale parameters of $\sqrt{10}$, and Z is a standard normal random variable.

The scenario labeled ‘scenario 1’ does not exhibit any negative forward rates, while the scenario labeled ‘scenario 2’ requires an adjustment to remove negative forward rates that occur at short maturities.

The short maturity region of ‘scenario 2’ is shown for the original and adjusted curves in Figure 2, where we see that only a small adjustment is required for two tenor points to correct for the negative forward rates. Even for the larger shocks associated with stress scenarios, the occurrence of negative forward rates is fairly low, as it was found that for a simulation run of 20,000 scenarios only 30 required adjustments. Interest rate curve is important for pricing derivatives, such as callable exotics.

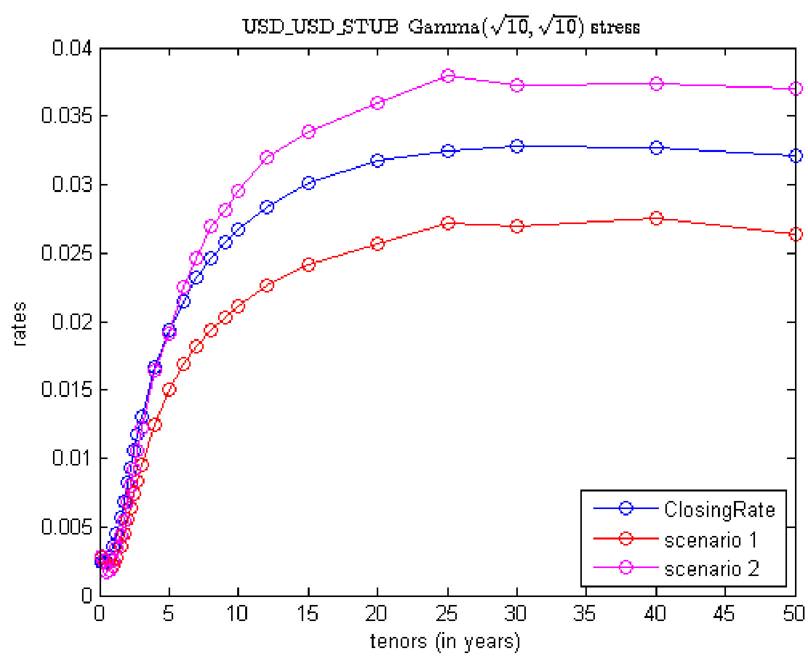


Figure 1: 10-day scenarios for USD_STUB curve.

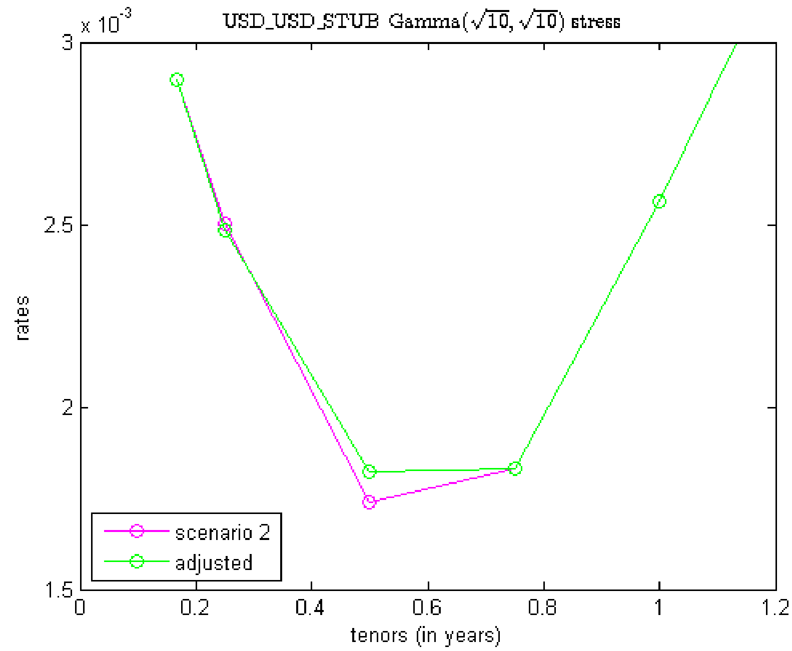


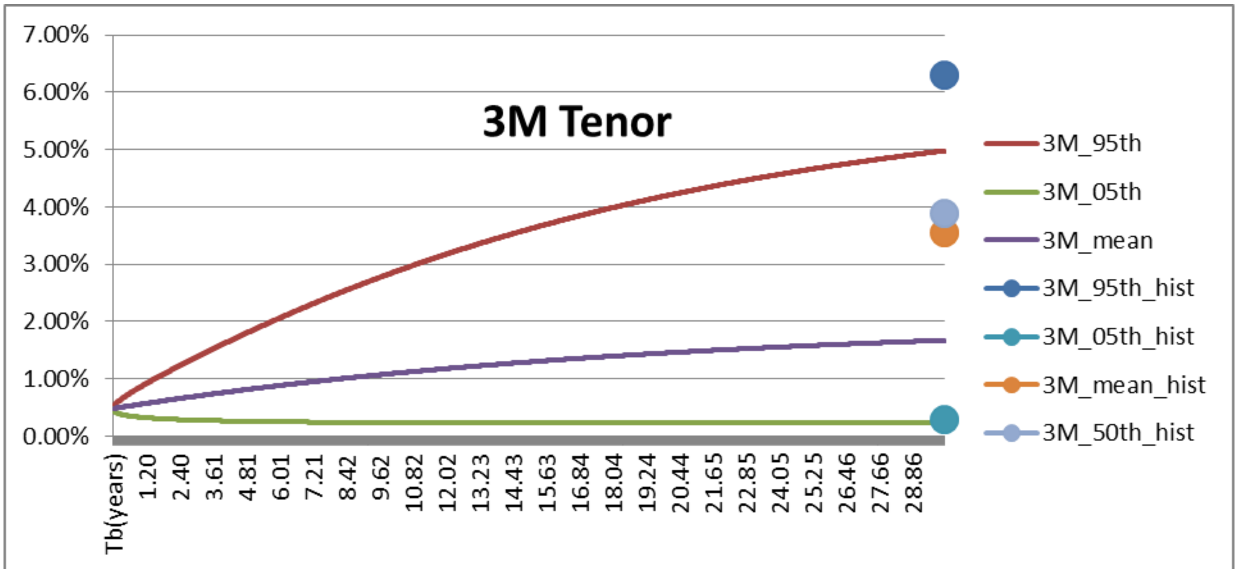
Figure 2: Adjustment to USD_STUB curve scenario to remove negative forward rates.

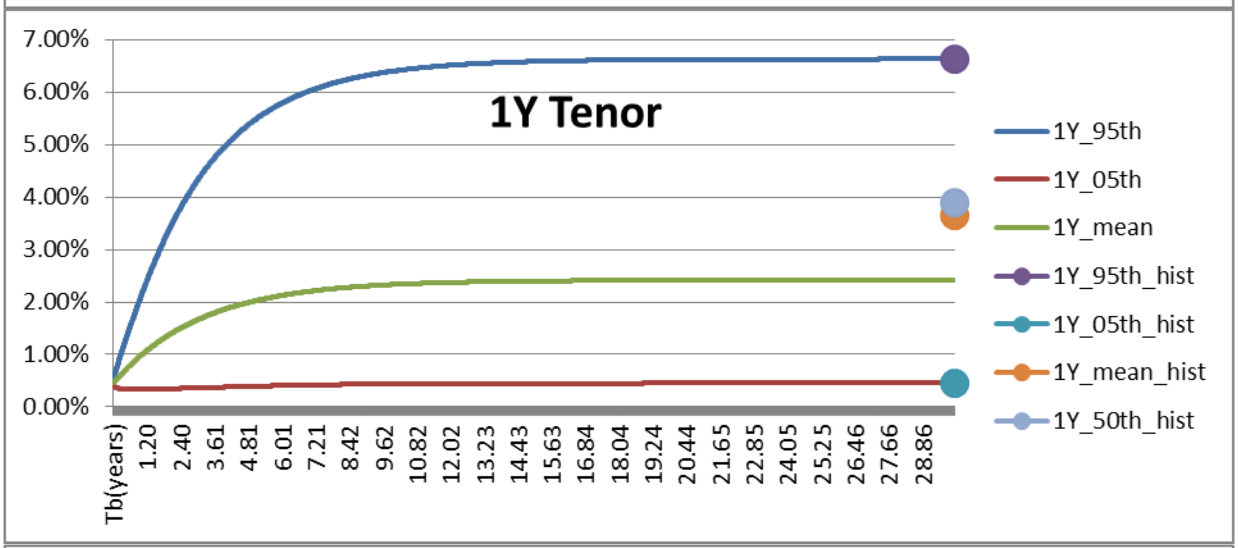
The simulated rates are produced by the following parameters:

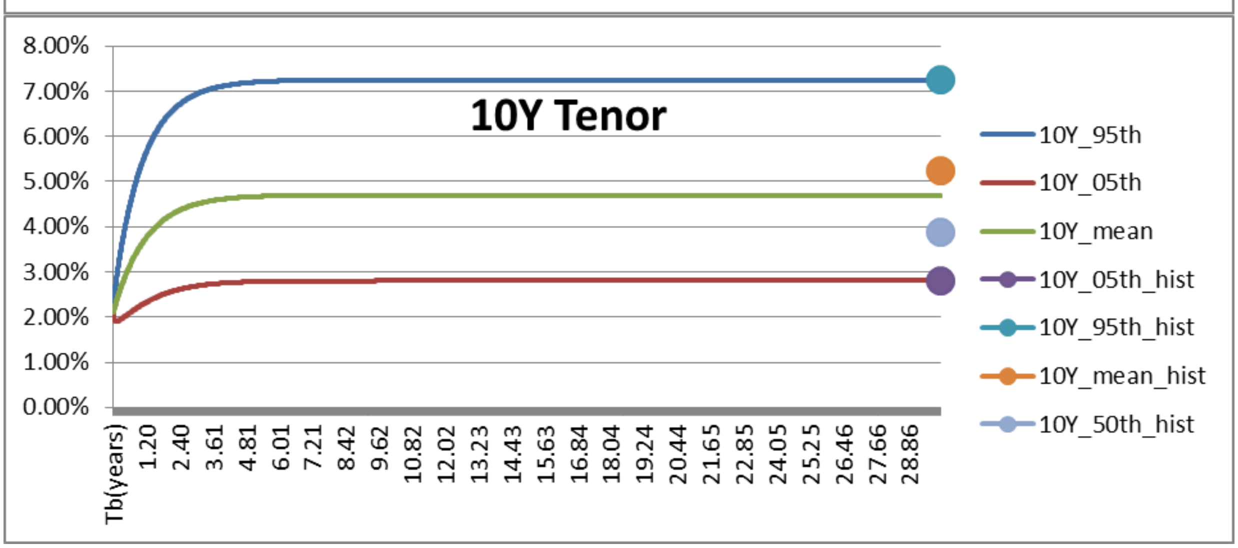
	Tenors	1M	3M	6M	1Y	2Y	3Y	5Y	7Y	10Y	30Y
USD	Sigma	24.23%	30.20%	53.03%	65.11%	64.43%	58.22%	47.40%	41.36%	37.72%	32.91%
	Theta	2.00%	2.08%	2.20%	2.41%	2.73%	3.08%	3.81%	4.28%	4.70%	5.16%
	Kappa	2.91%	5.12%	17.49%	31.78%	42.77%	49.00%	62.25%	70.80%	85.16%	100.20%
	Whist	6.06%	6.03%	6.10%	6.17%	6.05%	5.84%	5.28%	4.86%	4.45%	3.94%
	Hist 95th	6.29%	6.31%	6.43%	6.63%	6.73%	6.82%	7.01%	7.13%	7.25%	7.36%
	Hist 05th	0.23%	0.28%	0.34%	0.45%	0.68%	0.99%	1.73%	2.27%	2.80%	3.43%
	Hist avg	3.45%	3.54%	3.56%	3.66%	3.94%	4.21%	4.64%	4.93%	5.23%	5.68%
CAD	Sigma	33.3%	24.4%	35.2%	50.8%	52.0%	45.9%	41.7%	33.8%	27.5%	20.2%
	Theta	2.1%	2.2%	2.2%	2.5%	3.1%	3.4%	3.9%	4.3%	4.9%	5.6%
	Kappa	8.3%	4.8%	10.8%	30.8%	52.0%	50.5%	56.5%	45.6%	41.8%	37.6%
	Whist	5.48%	5.44%	5.38%	5.22%	5.05%	5.04%	4.93%	4.99%	4.83%	4.27%
	Hist 95th	5.88%	5.89%	5.87%	5.92%	6.21%	6.49%	6.81%	7.26%	7.70%	7.97%
	Hist 05th	0.40%	0.44%	0.49%	0.70%	1.16%	1.44%	1.87%	2.27%	2.87%	3.71%
	Hist avg	3.34%	3.39%	3.41%	3.54%	3.88%	4.15%	4.55%	4.84%	5.18%	5.59%

EUR	Sigma	65.36%	32.01%	31.05%	39.80%	44.24%	34.26%	29.66%	26.29%	24.22%	25.35%
	Theta	1.89%	2.24%	2.30%	2.47%	2.72%	3.04%	3.46%	3.80%	4.12%	4.36%
	Kappa	39.57%	14.65%	14.78%	31.02%	51.94%	46.34%	52.30%	53.58%	59.62%	67.81%
	Whist	4.41%	4.27%	4.23%	4.05%	3.85%	3.54%	3.28%	3.17%	3.00%	3.11%
	Hist 95th	4.84%	4.98%	4.99%	4.99%	5.06%	5.13%	5.34%	5.59%	5.79%	6.09%
	Hist 05th	0.43%	0.71%	0.76%	0.95%	1.21%	1.59%	2.06%	2.42%	2.79%	2.97%
	Hist avg	2.72%	2.86%	2.85%	2.93%	3.13%	3.38%	3.72%	3.99%	4.29%	4.67%
JPY	Sigma	40.09%	24.32%	39.20%	26.54%	33.08%	47.82%	50.18%	49.10%	39.25%	24.01%
	Theta	0.24%	0.31%	0.30%	0.33%	0.44%	0.57%	0.81%	1.06%	1.44%	2.32%
	Kappa	10.39%	3.92%	11.43%	6.16%	13.27%	34.25%	46.33%	62.83%	63.35%	113.50%
	Whist	0.67%	0.85%	0.78%	0.79%	0.90%	1.06%	1.37%	1.51%	1.64%	1.22%
	Hist 95th	0.70%	0.90%	0.84%	0.86%	1.02%	1.25%	1.67%	1.98%	2.41%	2.98%
	Hist 05th	0.04%	0.05%	0.06%	0.07%	0.12%	0.19%	0.30%	0.47%	0.76%	1.76%
	Hist avg	0.24%	0.30%	0.32%	0.35%	0.45%	0.60%	0.88%	1.17%	1.54%	2.37%
CHF	Sigma	106.3%	88.8%	162.2%	320.9%	112.8%	75.4%	40.9%	36.5%	32.2%	40.7%
	Theta	1.0%	1.2%	1.2%	1.3%	1.6%	1.9%	2.3%	2.7%	3.0%	3.0%
	Kappa	48.4%	47.5%	168.8%	830.0%	195.8%	139.9%	78.3%	95.5%	108.6%	173.5%
	Whist	3.3%	3.2%	3.2%	3.1%	3.0%	2.8%	2.5%	2.3%	2.2%	2.2%
	Hist 95th	3.4%	3.4%	3.4%	3.4%	3.5%	3.6%	3.8%	4.0%	4.2%	4.2%
	Hist 05th	0.1%	0.2%	0.2%	0.3%	0.5%	0.8%	1.3%	1.7%	2.1%	2.1%
	Hist avg	1.3%	1.4%	1.5%	1.6%	1.9%	2.1%	2.5%	2.8%	3.1%	3.1%

These parameters are obtained with calibration 1 described above. The plots to follow, beyond displaying the resulting simulated shape produced by the model, stress the flexibility that the model possesses due to the various calibrations proposed.


3M Tenor
7.00%
6.00%
5.00%
4.00%
3.00%
2.00%
1.00%
0.00%
Tb(years)
1.20
2.40
3.61
4.81
6.01
7.21
8.42
9.62
10.82
12.02
13.23
14.43
15.63
16.84
18.04
19.24
20.44
21.65
22.85
24.05
25.25
26.46
27.66
28.86
3M_95th
3M_05th
3M_mean
3M_95th_hist
3M_05th_hist
3M_mean_hist
3M_50th_hist


1Y Tenor
7.00%
6.00%
5.00%
4.00%
3.00%
2.00%
1.00%
0.00%
Tb(years)
1.20
2.40
3.61
4.81
6.01
7.21
8.42
9.62
10.82
12.02
13.23
14.43
15.63
16.84
18.04
19.24
20.44
21.65
22.85
24.05
25.25
26.46
27.66
28.86
1Y_95th
1Y_05th
1Y_mean
1Y_95th_hist
1Y_05th_hist
1Y_mean_hist
1Y_50th_hist


10Y Tenor
8.00%
7.00%
6.00%
5.00%
4.00%
3.00%
2.00%
1.00%
0.00%
Tb(years)
1.20
2.40
3.61
4.81
6.01
7.21
8.42
9.62
10.82
12.02
13.23
14.43
15.63
16.84
18.04
19.24
20.44
21.65
22.85
24.05
25.25
26.46
27.66
28.86
10Y_95th
10Y_05th
10Y_mean
10Y_05th_hist
10Y_95th_hist
10Y_mean_hist
10Y_50th_hist

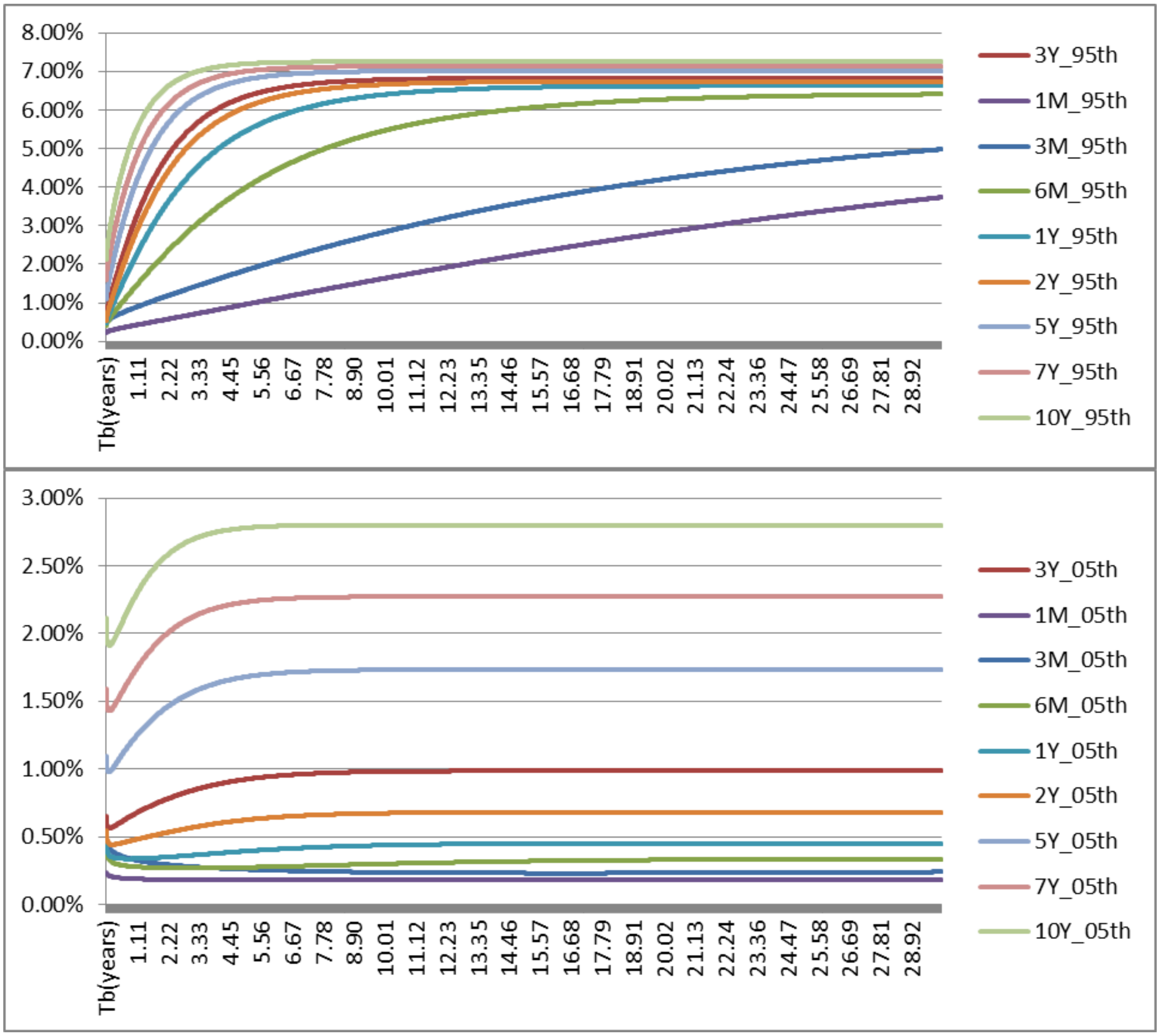


Figure 3: USD results

It can be seen that each tenor has a very distinct behavior due to each tenor its own volatility, mean reversion speed and long-term mean. The historical 95^{th}-5^{th}sits right at the edges of the envelop as per the calibration. In this simulation there is no intermediary regime changes, but rather a convergence to the long-term mean driven by κ, σ and the level of θ. USD exhibits very few crossings of tenors at the 5^{th} level and none at the 95^{th}. Due to the data not being log-normally distributed and the calibration matching the 2 extreme quantiles, the historical mean sits typically above the simulated mean. This behavior is most apparent for the 3M rate and fades as tenors increase.

4. Conclusion

In this article we present simulation model under the physical measure using various calibration techniques. The model does not have particular restrictions in terms of the shape it may produce. Any shapes may be observed during simulations. This may lead to “wavy” shapes which are not possible or extremely unlikely in economic terms. Without explicit rejection rules at simulation run-time, negative forwards may be observed, although they should be rare as long term means and spot are usually not negative and consequently expected future spot rates.

The model forecast log-normally distributed interest rates; therefore, interest rates cannot become negative. As observed in the simulation plots, rates are floored around zeros when volatility is sufficiently high or long-term 5th quantile is sufficiently low.

Having more tenors available does not influence previously calibrated parameters since each tenor is calibrated separately. The model will simply gain accuracy without disrupting the previously calibrated parameters.

References:

Akram, T., 2020, A simple model of the long-term interest rate, working paper, 2020.

Akram, T., and Li, H, 2020, An inquiry concerning long-term U.S. interest rates using monthly data, Applied Economics, 52 (24), 2594-2621.

Bauer, M.D. and Hamilton, J.D., 2018, Robust bond risk premia, Review of Financial Studies, 31, 399-448.

Bolder D. and Streliski D. 1999, Curve modelling at the Bank of Canada, working paper, 1999.

Crmmp, R.K. and Gospodinov, N., 2024, Deconstructing the yield curve, Federal Reserve Bank of New York Staff Reports, 884, 2024.

Fisher, M., 2004, Modeling the term structure of interest rates: an introduction, Economic Review, 2004, 41-62.

Madsen, C., 2012, Empirical yield curve dynamics, scenario simulation and risk measures, DTU Management Engineering.

Medova, E.A., Rietbergen, M.I., Villaverde, M. and Yong, Y.S., 2006, Modelling the long-term dynamics of yield curves, working paper, 2006.

Levrero, E.S., and Deleidi, M., 2019 The causal relationship between short- and long-term interest rates: an empirical assessment of the United States,” MPRA paper No. 93608.

Li, H. and Su, Y., 2021, The nonlinear causal relationship between short- and long-term interest rates: an empirical assessment of the United States, the United Kingdom, and Japan, International Finance, 24 (3), 332-355, 2021.