A Solution to the “Snellius-Pothenot” Problem via Rotations and Reflections in Geometric Algebra (GA)

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Abstract

Using GA’s capacities for rotating and reflecting vectors, we solve the classic 2-D version of the Snellius-Pothenot surveying problem. The method used here provides two solutions, which can be averaged to better estimate the location of the unknown point \( P \). A link to a GeoGebra worksheet of the solutions is provided so that the reader may test the validity of the method.

The vector \( \mathbf{p} \) is the reflection of \( \mathbf{a} \) with respect to \( \mathbf{w}_a \), where \( \mathbf{w}_a = \mathbf{a} + \mathbf{b} + [(\mathbf{a} - \mathbf{b}) \mathbf{i}] / \tan \alpha \). The vector \( \mathbf{p} \) is also the reflection of \( \mathbf{c} \) with respect to \( \mathbf{w}_c \), where \( \mathbf{w}_c = \mathbf{c} + \mathbf{b} - [(\mathbf{c} - \mathbf{b}) \mathbf{i}] / \tan \beta \). The vector \( s = \mathbf{v} \mathbf{i} / \tan (\alpha + \beta) \).
1 Statement of the Problem

Fig. 1 shows the problem statement.

Figure 1: Problem statement: Express the location of point \( P \) in terms of the locations of points \( A, B, C \) and the angles \( \alpha \) and \( \beta \).

2 Ideas that We Will Use

See also Macdonald [1].

1. For any two vectors \( q \) and \( t \), \( q \land t = [(q\mathbf{i}) \cdot t] \mathbf{i} = -[q \cdot (t\mathbf{i})] \mathbf{i} \).

2. For any two vectors \( q \) and \( t \), \( q \land t = (qt)_2 \).

3. The vectors \( q e^{i\theta} \) and \( q e^{-i\theta} \) are rotations of \( q \) by the same angle \( \theta \), but in opposite directions.

4. The reflection of a vector \( q \) with respect to vector \( t \) can be written as the product \( tqt^{-1} \), which is equal to \( [tqt]/\|t\|^2 \) (Fig. 2).
Figure 2: The reflection of a vector $q$ with respect to vector $t$ can be written as the product $tqt^{-1}$, which is equal to $\frac{|tqt|}{\|t\|^2}$.

3 Formulation in GA Terms

Fig. 3 Shows the formulation. Note the sign convention for the angles $\alpha$ and $\beta$.

Figure 3: Formulation in terms of GA. As our origin, we use the midpoint of $AC$ (point $O$). The vector $s$ is from $O$ to the center of $K$, and the vector $v$ is from $O$ to point $A$. Vector $p$ is from $S$ to point $P$. Regarding the algebraic signs of $\alpha$ and $\beta$: Angles are measured in the same direction as the rotation of $i$. In this diagram, $\alpha$ — the angle of rotation from $PB$ to $PA$ — is in the same sense as $i$. Therefore, the angle $\alpha$ in this diagram is positive. The angle $\beta$ is the rotation from $PC$ to $PB$. This rotation, too, is in the same sense as $i$. Therefore, the angle $\beta$ in this diagram is positive.
4 Solution Strategy

We will begin by determining the location of the center of circle $K$ (Fig. 3).

Then, we will right-multiply the vector $a - p$ by $e^{-i\alpha}$ to make it parallel to $b - p$, after which we use the fact that $(b - p) \wedge [(a - p) e^{-i\alpha}] = 0$ to obtain an equation for $p$. We will obtain a separate equation for $p$ by right-multiplying the vector $c - p$ by $e^{i\beta}$ to make it parallel to $b - p$, then using the fact that $(b - p) \wedge [(c - p) e^{i\beta}] = 0$.

5 Solution

Readers who wish to test the solutions that are derived here can access the associated interactive GeoGebra worksheet (2).

5.1 Determining the Location of the Center of Circle $K$

The vector $s$, from the midpoint of $AC$ to the center of $K$, is

$$s = \frac{vi}{\tan(\alpha + \beta)}.$$  \hfill (5.1)

Figs. 4 and 5 present two cases that show why this relationship holds.

5.2 Finding $p$ from the Rotations of the Vectors $a - p$ and $c - p$.

We will see that each rotation provides a separate solution. These could be averaged to better estimate the location of $P$.

5.2.1 Finding $p$ from the Rotation of $a - p$.

The vector $(a - p) e^{-i\alpha}$ is parallel to $b - p$. Therefore,

$$(b - p) \wedge [(a - p) e^{-i\alpha}] = 0,$$  
$$\langle (b - p) [(a - p) e^{-i\alpha}] \rangle_2 = 0.$$

Expanding the exponential and the geometric product $(b - p) (a - p)$,

$$\langle [b \cdot a + b \wedge a - b \cdot p - b \wedge p \cdot a - p \wedge a + p^2] (\cos \alpha - i \sin \alpha) \rangle_2 = 0.$$

Because the product of any outer product with $i$ is a scalar, the preceding equation simplifies to

$$[b \wedge a - b \wedge p \wedge a] \cos \alpha - (b \cdot a - b \cdot p - p \cdot a + p^2) \sin \alpha = 0.$$
Figure 4: One of the arrangements of points used in deriving an equation for the location of the center of the circle $K$. Point $P$ is on the arc $APC$, and $B$ is on the opposite side of chord $AC$. To find the length of $s$, we must first find the central angle that subtends $v$. To do so, we note that the magnitude of the central angle that subtends the same arc as $\angle CPA$ is $2(\alpha + \beta)$. (See Fig. 3 regarding the sign convention for $\alpha$ and $\beta$.) Thus, the magnitude of the central angle that subtends $AC$ is $2\pi - 2(\alpha + \beta)$, and the magnitude of the angle that subtends $v$ is $\frac{1}{2}[2\pi - 2(\alpha + \beta)] = \pi - (\alpha + \beta)$. Using the trigonometric identity $\parallel \tan [\pi - (\theta + \phi)] \parallel = \parallel \tan (\theta + \phi) \parallel$, we find that the length of $s$ is $\parallel v \parallel / \parallel \tan (\alpha + \beta) \parallel$. Vector $s$ is perpendicular to $v$ because the bisector of any chord (in this case, $AC$) is perpendicular to that chord. The sense of rotation from $v$ to $s$ is contrary to the sense of $i$. For that reason, $s = -\|v\|i / \|\tan (\alpha + \beta)\|$. Because $\alpha + \beta$ is a positive angle between $\pi/2$ and $\pi$, $\tan (\alpha + \beta)$ is a negative number. From the foregoing, we can see that the equation $s = \frac{vi}{\tan (\alpha + \beta)}$ captures both the magnitude of $s$ and the sense of vector $s$’s rotation with respect to $v$. 
Figure 5: Another arrangement of points used in deriving an equation for the location of the center of circle $\mathcal{K}$. Point $P$ is on the arc $APC$, and $B$ is on the same side of chord $AC$ as point $P$. (Compare to Fig. 4.) As in the case of Fig. 4 we must find the magnitude of the angle that subtends $\mathbf{v}$. From elementary geometry, that magnitude is the same as the magnitude of $\angle APC$. Because the angles $\alpha$ and $\beta$ are of opposite signs, the magnitude of that angle is $|\alpha + \beta|$. The negative angle ($\beta$) is larger than the positive one ($\alpha$); therefore, $\alpha + \beta$ is negative. In addition, $|\alpha + \beta| < \pi/2$ because the arc that is subtended by $\angle APC$ is smaller than $\pi$. Thus, just as in Fig. 4 $\tan(\alpha + \beta)$ is negative, leading once again to $s = \mathbf{v}\lambda / \tan(\alpha + \beta)$. 
Now, we divide by $\cos \alpha$, then use the identities $\mathbf{q} \land \mathbf{t} = [(\mathbf{q} \cdot \mathbf{t}) \mathbf{i} \text{ and } = \mathbf{(q \cdot t)}] - [\mathbf{q} \cdot (\mathbf{t} \mathbf{i})]$:

$$[(\mathbf{b} \cdot \mathbf{a}) \mathbf{i} - [(\mathbf{b} \cdot \mathbf{p}) \mathbf{i} - [(\mathbf{p} \cdot \mathbf{a}) \mathbf{i} - (\mathbf{b} \cdot \mathbf{a} - \mathbf{b} \cdot \mathbf{p} - \mathbf{p} \cdot \mathbf{a} + p^2) \mathbf{i} \tan \alpha = 0$$

$$[\mathbf{a} \cdot (\mathbf{b}) \mathbf{i} - [\mathbf{p} \cdot (\mathbf{b}) \mathbf{i} + [\mathbf{p} \cdot (\mathbf{a}) \mathbf{i} - (\mathbf{a} \cdot \mathbf{b} - \mathbf{p} \cdot \mathbf{a} - \mathbf{p} \cdot \mathbf{b} + p^2) \mathbf{i} \tan \alpha = 0.$$

Multiplying both sides by $-i$, then rearranging,

$$\mathbf{p} \cdot \left[ \mathbf{a} + \mathbf{b} + \frac{(\mathbf{a} - \mathbf{b}) \mathbf{i}}{\tan \alpha} \right] = p^2 + \mathbf{a} \cdot \left[ \mathbf{b} + \frac{-\mathbf{b} \mathbf{i}}{\tan \alpha} \right].$$

Now, we define $\mathbf{w}_a = \mathbf{a} + \mathbf{b} + \frac{(\mathbf{a} - \mathbf{b}) \mathbf{i}}{\tan \alpha}$, in order to transform the right-hand side:

$$\mathbf{p} \cdot \mathbf{w}_a = p^2 + \mathbf{a} \cdot \left[ \mathbf{b} - \frac{2 \mathbf{a} \cdot \mathbf{w}_a - \mathbf{w}_a \mathbf{w}_a}{\|\mathbf{w}_a\|^2} \right] - \frac{\mathbf{a} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{w}_a}{\|\mathbf{w}_a\|^2} = 0.$$

Because $\mathbf{a}$ and $\mathbf{p}$ are radii of the same circle ($K$), $a^2 = p^2$. Therefore,

$$\mathbf{p} \cdot \mathbf{w}_a = \mathbf{a} \cdot \mathbf{w}_a.$$

Next, we recognize that because $\|\mathbf{a}\| = \|\mathbf{p}\|$, and because $\mathbf{p} \neq \mathbf{a}$, $\mathbf{p}$ must be the reflection of $\mathbf{a}$ with respect to $\mathbf{w}_a$. That is,

$$\mathbf{p} = [\mathbf{w}_a] \mathbf{a} [\mathbf{w}_a^{-1}]$$

$$= \frac{[\mathbf{w}_a] \mathbf{a} [\mathbf{w}_a]}{\|\mathbf{w}_a\|^2}$$

$$= \frac{\mathbf{w}_a \{2 \mathbf{a} - \mathbf{w}_a \mathbf{a} \} \|\mathbf{w}_a\|^2}{\|\mathbf{w}_a\|^2}$$

$$= 2 \frac{\mathbf{w}_a \cdot \mathbf{a}}{\|\mathbf{w}_a\|^2} \mathbf{w}_a - \mathbf{a}. \tag{5.2}$$

Therefore, the location of $P$ with respect to the midpoint of $AC$ is given by the vector $\mathbf{p}^*$ (Fig. 6):

$$\mathbf{p}^* = \mathbf{s} + \mathbf{p}$$

$$= \frac{\mathbf{v} \mathbf{i}}{\tan (\alpha + \beta)} + 2 \frac{\mathbf{w}_a \cdot \mathbf{a}}{\|\mathbf{w}_a\|^2} \mathbf{w}_a - \mathbf{a}. \tag{5.3}$$

### 5.2.2 Finding $\mathbf{p}$ from the Rotation of $\mathbf{c} - \mathbf{p}$.

The vector $(\mathbf{c} - \mathbf{p}) e^{i \beta}$ is parallel to $\mathbf{b} - \mathbf{p}$. Therefore,

$$(\mathbf{b} - \mathbf{p}) \land [(\mathbf{c} - \mathbf{p}) e^{i \beta}] = 0,$$

and

$$((\mathbf{b} - \mathbf{p}) [(\mathbf{c} - \mathbf{p}) e^{i \beta}])_2 = 0.$$
Figure 6: Vector $\mathbf{p}$ is the reflection of $\mathbf{a}$ with respect to $\mathbf{w}_a$, where $\mathbf{w}_a = \mathbf{a} + \mathbf{b} + [\mathbf{a} - \mathbf{b}]i / \tan \alpha$. Vector $\mathbf{p}$ is also the reflection of $\mathbf{c}$ with respect to $\mathbf{w}_c$, where $\mathbf{w}_c = \mathbf{c} + \mathbf{b} - [(\mathbf{c} - \mathbf{b})i \tan \beta]$. Therefore, the location of $P$ with respect to the midpoint of $AC$ is given by the vector $\mathbf{p}^* = \frac{\mathbf{vi}}{\tan (\alpha + \beta)} + 2 \left( \frac{\mathbf{w}_a \cdot \mathbf{a}}{||\mathbf{w}_a||^2} \right) \mathbf{w}_a - \mathbf{a}$, and also by $\mathbf{p}^* = \frac{\mathbf{vi}}{\tan (\alpha + \beta)} + 2 \left( \frac{\mathbf{w}_c \cdot \mathbf{c}}{||\mathbf{w}_a||^2} \right) \mathbf{w}_c - \mathbf{c}$.

Expanding the exponential and the geometric product $(\mathbf{b} - \mathbf{p})(\mathbf{c} - \mathbf{p})$,

$\langle [\mathbf{b} \cdot \mathbf{c} + \mathbf{b} \wedge \mathbf{c} - \mathbf{b} \cdot \mathbf{p} - \mathbf{p} \cdot \mathbf{c} - \mathbf{p} \wedge \mathbf{c} + \mathbf{p}^2] (\cos \beta + i \sin \beta) \rangle = 0$.

Because the product of any outer product with $i$ is a scalar, the preceding equation simplifies to

$[\mathbf{b} \wedge \mathbf{c} - \mathbf{b} \wedge \mathbf{p} - \mathbf{p} \wedge \mathbf{c}] \cos \alpha + (\mathbf{b} \cdot \mathbf{c} - \mathbf{b} \cdot \mathbf{p} - \mathbf{p} \cdot \mathbf{c} + \mathbf{p}^2) i \sin \beta = 0$.

Dividing by $\cos \beta$, then using the identities $\mathbf{q} \wedge \mathbf{t} = [(\mathbf{q}i) \cdot \mathbf{t}] i$ and $[(\mathbf{q}i) \cdot \mathbf{t}] - [\mathbf{q} \cdot (\mathbf{t}i)]$,

$[\mathbf{c} \cdot (\mathbf{b}i)] i - [\mathbf{p} \cdot (\mathbf{b}i)] i + [\mathbf{p} \cdot (\mathbf{c}i)] i + (\mathbf{c} \cdot \mathbf{b} - \mathbf{p} \cdot \mathbf{b} - \mathbf{p} \cdot \mathbf{c} + \mathbf{p}^2) i \tan \beta = 0$.

Multiplying both sides by $-i$, then rearranging,

$\mathbf{p} \cdot \left[ \frac{\mathbf{c} + \mathbf{b} - (\mathbf{c} - \mathbf{b})i}{\tan \beta} \right] = \mathbf{p}^2 + \mathbf{c} \cdot \left[ \mathbf{b} + \frac{\mathbf{bi}}{\tan \beta} \right]$. 

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Continuing as we did when finding $p$ from the rotation of $a - p$,

$$p \cdot w_c = p^2 + c \cdot \left[ c + b - \frac{(c - b) \mathbf{i}}{\tan \beta} \right] \left[ w_c - \frac{c \cdot c}{c^2} \right] \left[ c \cdot c \right] \tan \beta,$$

and

$$p \cdot w_c = c \cdot w_c,$$

leading to

$$p = [w_c] c \left[ w_c^{-1} \right]$$

$$= 2 \left[ \frac{w_c \cdot c}{||w_c||^2} \right] w_c - c. \quad (5.4)$$

Therefore, the derivation that starts from the rotation of $c - p$ finds that the location of $P$ with respect to the midpoint of $AC$ is given by the same vector $p^*$ that we found when starting from the rotation of $a - p$. However, the vector $p$ is now expressed in terms of $c$ (Fig. 6):

$$p^* = s + p$$

$$= \frac{vi}{\tan (\alpha + \beta)} + 2 \left[ \frac{w_c \cdot c}{||w_c||^2} \right] w_c - c. \quad (5.5)$$

References
