Pi's Irrationality Using Maclaurin Polynomials

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Abstract

Using the Maclaurin series and polynomials for $\sin(x)$, we give a simple proof that π is irrational.

Background

First, here are a few observations. The Maclaurin series for $\sin(x)$ is

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^{2k-1}}{(2k-1)!}.$$

Consider the zeros for the first few Maclaurin polynomials for $\sin(x)$ [2, 3]:

$$T_3(x) = x - \frac{x^3}{3!},$$

$$T_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!},$$

and

$$T_7(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}.$$

Note that Maclaurin polynomials add one term to previous polynomials. This implies that the roots can't be the same. The roots of each $T_j(x)$ is different than all others; each is adjusted by a new term.

The roots of these $T_j(x)$ will have to be the same as

$$3!T_3(x) = -x(x^2 - 3!),$$

$$5!T_5(x) = x(x^4 - 5 \cdot 4x^2 + 5!),$$

and

$$7!T_7(x) = -x(x^6 - 7 \cdot 6x^4 + 7 \cdot 6 \cdot 5 \cdot 4x^2 - 7!).$$

Zero times even a large factorial number is still 0.

The non-zero zeros of these will have to be the same as those of

$$\hat{T}_3(x) = x^2 - 3!,$$

$$\hat{T}_5(x) = x^4 - 5 \cdot 4x^2 + 5!,$$

and

$$\hat{T}_7(x) = x^6 - 7 \cdot 6x^4 + 7 \cdot 6 \cdot 5 \cdot 4x^2 - 7!.$$

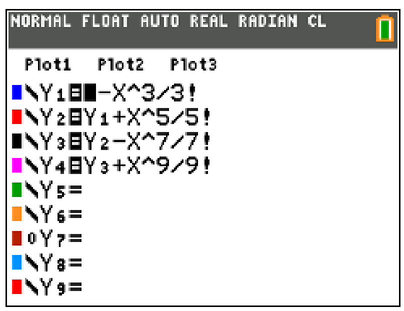

Figure 1: The first few Maclaurin polynomials for $\sin(x)$.

A TI84-CE calculator can be used to graph these functions. The first few, T_3 through T_9 are given in Figure 1 and graphed in Figure 2. Notice that the $\sin(x)$ curve is slowly being formed. As the degree of the polynomial grows the number of turning points in the curve increases [1] and, we can assume, the accuracy of the zero estimates of $\sin(x)$ get better. In Figure 3 we can see that $T_9(\pi) = 0.006$, almost zero. Per the periodicity of $\sin(x)$, the roots of $\sin(x)$ are of the form $n\pi$ for integer n. As any calculus book will attest the series converges to $\sin(x)$ for all of the reals; an infinite circle of convergence.

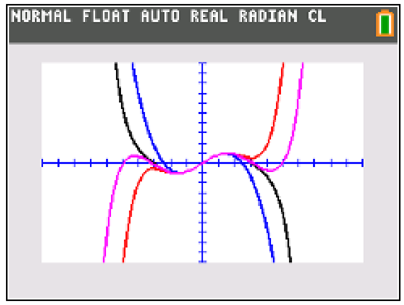

Figure 2: A few Maclaurin polynomials.

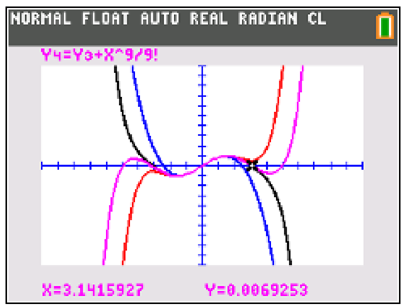

Figure 3: The calc feature of this TI84 calculator gives the value of $T_4(\pi)$.

Proof

Define

$$T_j(x) = \sum_{k=1}^{j} \frac{(-1)^{k-1}x^{2k-1}}{(2k-1)!}$$

and

$$\hat{T}_j(x) = \frac{j!T_j(x)}{x} \quad \text{where } x \neq 0 \ .$$

Then $\hat{T}_j(x)$ is a integer polynomial. Also the non-zero roots of $\sin(x)$ are the same as

$$\lim_{j\to\infty} T_j(x)$$

and

$$\lim_{j\to\infty} \hat{T}_j(x).$$

These roots are integer multiples of π.

Assume for a contradiction that $\pi = p/q$ then $q\pi = p$ and $\sin(p) = 0$. This means

$$\lim_{j \to \infty} \hat{T}_j(p) = 0.$$

But this implies that given an ϵ such that $0 < \epsilon < 1$, there exists N such that for all $j > N$,

$$0 < |\hat{T}_j(p)| < \epsilon,$$

but all $\hat{T}_j(x)$ are integer polynomials and when evaluated at the integer p have to be an integer or 0. The latter is precluded: non-zero roots of $\hat{T}_j(x)$ can't be the same as $\sin(x)$; they're always different approximations.

References

[1] Blitzer, R. (2010). *Algebra and Trignometry*, 3rd ed., Pearson.

[2] Larson, R. and Edwards, B.H. (2010). *Calculus*, 9th ed., Belmont, CA: Brooks/Cole.

[3] Thomas, G.B. (1968). *Calculus and Analytic Geometry*, 4th ed., Reading, MA: Addison-Wesley.