RIEMANN HYPOTHESIS VIA NICOLAS CRITERION

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Abstract. The Robin's Theorem with Nicolas criterion were used to prove the Riemann Hypothesis in a straightforward way.
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1. Introduction

There is a vivid interest in the Riemann Hypothesis proposed by Bernhard Riemann in 1859. While there are no reasons to doubt the validity of the Riemann Hypothesis [1], many colleagues consider it the most important unsolved problem in pure mathematics [2]. The Riemann Hypothesis is of great interest in number theory because it implies results about the distribution of prime numbers. In this short note, I offer a proof of the Riemann hypothesis via the Robin theorem.

Let us define $d(n) = \mathrm{e}^\gamma \log\log n - \sigma(n)/n$, where $\sigma(n)$ is the sum of divisors function. Robin's theorem [3] tells us that if $d(n) \geq 0$ for all $n > 5040$, the Riemann Hypothesis is true.

Is known [4] that the hypothetical counter-example (one with $d(n) < 0$) is of form

$$n = \prod_{i=1}^{k} p_i^{x_i}\,, \tag{1}$$

where $x_1 \geq x_2 \geq ... \geq x_k$, $x_k = 1$, are integers and $p_i = 2, 3, 5, 7, \ldots, p_k$ are the first k successive primes.

Nicolas has shown [5] that if

$$\frac{N_k}{\varphi(N_k)} > \mathrm{e}^\gamma \log\log N_k\,, \tag{2}$$

where the primorial of order k is given by

$$N_k = \prod_{i=1}^{k} p_i\,, \tag{3}$$

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the Riemann Hypothesis is true. Here, $\varphi(N)$ is Euler's totient function, i.e., the number of integers less than N that are not coprime to N.

Note that Nicolas' criterion ignores all powers $x_i > 1$. Therefore, it can be concluded, that if $d(N_k) > 0$ for all k, the Riemann hypothesis is true. But because of Ref. [4], we know that $x_1 \neq 1$ has to be in order to violate $d(n) > 0$. Hence, $d(N_k) > 0$ with the case $x_1 = 1$ being true.

2. Proof in detail

Is known that

$$\varphi(m) = m \prod_{p|m} \left(1 - \frac{1}{p}\right) , \tag{4}$$

which, in my case,

$$\varphi(N_k) = N_k \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right) . \tag{5}$$

Then, using the Taylor series,

$$\frac{N_k}{\varphi(N_k)} = \prod_{i=1}^{k} \left(1 + \frac{1}{p_i} + \frac{1}{p_i^2} + O(1/p_i^3)\right) > \prod_{i=1}^{k} \left(1 + \frac{1}{p_i}\right) . \tag{6}$$

Since this Taylor series is convergent (function is $f(x) = 1/(1-1/x)$), this Taylor series development is valid for any p_i.

On the other hand, the element in the Robin's theorem is

$$\frac{\sigma(N_k)}{N_k} = \prod_{i=1}^{k} \left(1 + \frac{1}{p_i}\right) . \tag{7}$$

Hence,

$$\frac{N_k}{\varphi(N_k)} > \frac{\sigma(N_k)}{N_k} . \tag{8}$$

Please, consider inequality

$$\frac{N_k}{\varphi(N_k)} > \mathrm{e}^{\gamma} \log\log N_k . \tag{9}$$

Comparing the latter two expressions (8), (9), I conclude that Eq. (8) is necessary for $\mathrm{e}^{\gamma} \log\log N_k > \frac{\sigma(N_k)}{N_k}$ to take place. If the strength of Eq. (8) is not sufficient, then

$$\frac{N_k}{\varphi(N_k)} > \frac{\sigma(N_k)}{N_k} > \mathrm{e}^{\gamma} \log\log N_k \tag{10}$$

happens. But the latter is impossible if $d(N_k) > 0$.

Variant

$$\frac{\sigma(N_k)}{N_k} > \frac{N_k}{\varphi(N_k)} > e^{\gamma} \log\log N_k \,, \tag{11}$$

is not possible, because Eq. (8) is a fact.

Now, please, consider inequality

$$e^{\gamma} \log\log N_k > \frac{N_k}{\varphi(N_k)} \,. \tag{12}$$

Then, from Eq. (8),

$$e^{\gamma} \log\log N_k > \frac{N_k}{\varphi(N_k)} > \frac{\sigma(N_k)}{N_k} \,. \tag{13}$$

This means, $d(N_k) > 0$, and, essentially, $\frac{N_k}{\varphi(N_k)}$ takes up the role of $\frac{\sigma(N_k)}{N_k}$ in the Robin's theorem. However, there are no such N_k, because high exponents in $\frac{N_k}{\varphi(N_k)}$, which are seen in Eq. (6), are pushing the exponents of N_k in $e^{\gamma} \log\log N_k$ of Eq. (13) higher than 1, which is not possible because of N_k definition.

References

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