# WEIGHTED WEAK GROUP INVERSE IN A RING WITH PROPER INVOLUTION

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Abstract. In this paper, we introduce weighted weak group inverse in a ring with proper involution. This is a natural generalization of weak group inverse for a complex matrix and weighted weak group inverse for a Hilbert operator. We characterize this weighted generalized by using a kind of decomposition involving weighted group inverses and nilpotents. The relations among weighted weak group inverse, weighted Drazin inverse and weighted core-EP inverse are thereby presented.

## 1. Introduction

Let $R$ be an associative ring with an identity. An involution of $R$ is an anti-automorphism whose square is the identity map 1. A ring $R$ with involution $*$ is called a *-ring. An element $a$ in a *-ring $R$ has group inverse provided that there exists $x \in R$ such that

$$ax^2 = x, ax = xa, a = xa^2.$$

Such $x$ is unique if exists, denoted by $a^{\#}$, and called the group inverse of $a$.

An element $a \in R$ has core-EP inverse (i.e., pseudo core inverse) if there exist $x \in R$ and $n \in \mathbb{N}$ such that

$$ax^2 = x, (ax)^* = ax, xa^{n+1} = a^n.$$

If such $x$ exists, it is unique, and denote it by $a^{\textcircled{D}}$. The core-EP inverse has been investigated from many different views, e.g., [3, 4, 10, 12, 17].

Wang and Chen (see [13]) introduced and studied a weak group inverse for square complex matrices. A square complex matrix $A$ has weak group inverse $X$ if it satisfies the equations

$$AX^2 = X, AX = A^{\textcircled{D}}A.$$

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The involution $*$ is proper, that is, $x^*x = 0 \Longrightarrow x = 0$ for any $x \in R$, e.g., in a Rickart *-ring, the involution is always proper. In [15], Zou et al. extend the notion of weak group inverse to elements in a ring with proper involution. An element $a \in R$ has weak group inverse if there exist $x \in R$ and $n \in \mathbb{N}$ such that

$$ax^2 = x, (a^*a^2x)^* = a^*a^2x, xa^{n+1} = a^n.$$

If such $x$ exists, it is unique, and denote it by $a^{\textcircled{W}}$. The weak group inverse has been extensively studied from many different views, e.g., [2, 5, 10, 11, 13, 15, 16].

Let $a, w \in R$. We recall that

**Definition 1.1.** *An element $a \in R$ has $w$-Drzin inverse if there exist $x \in R$ such that*

$$xwawx = x, awx = xwa, (aw)^n = xw(aw)^{n+1}$$

*for some $n \in \mathbb{N}$. The preceding $x$ is unique if it exists, and we denote it by $a^{D,w}$. The set of all $w$-Drazin invertible elements in $R$ is denoted by $R^{D,w}$.*

We say that $a$ has Drazin inverse if $w = 1$ and denote $a^{D,1}$ by $a^D$. In particular, for the preceding index $k = 1$, we have

**Definition 1.2.** *An element $a \in R$ has $w$-group inverse if there exist $x \in R$ such that*

$$xwawx = x, awx = xwa, awxwa = a.$$

*The preceding $x$ is unique if it exists, and we denote it by $a_w^{\#}$. The set of all weak $w$-group invertible elements in $R$ is denoted by $R_w^{\#}$.*

Recently, weighted weak group inverse for Hilbert space operators was studied by Mosić and Zhang in [9]. The motivation of this paper is to introduce and study a new kind of weak inverse with a wight as a natural generalization of weak group inverse for complex matrices and weighted weak group inverse for Hilbert operators mentioned above. In Section 2, we introduce weighted weak group inverse in terms of a new kind of weighted group decomposition. Many properties of the weak group inverse are thereby extended to the general cases.

Let $R_w^{nil} = \{x \in R \mid xw \in R \text{ is nilpotent }\}$.

**Definition 1.3.** *An element $a \in R$ has weak $w$-group decomposition if there exist $x, y \in R$ such that*

$$a = x + y, x^*y = ywx = 0, x \in R_w^{\#}, y \in R_w^{nil}.$$

We shall prove that $a \in R$ has weak $w$-group decomposition if and only if there exists unique $x \in R$ such that

$$x = ax^2, (a^n)^* a^2 x = (a^n) a^* a, a^k = xa^{k+1} \text{ for some } n \in \mathbb{N}.$$

We call the preceding $x$ the weak $w$-group inverse of $a$, and denote it by $a_w^{\textcircled{w}}$. The set of all weak $w$-group invertible elements in $R$ is denoted by $R_w^{\textcircled{w}}$.

In Section 3, we investigate elementary equivalent characterizations of weighted weak group inverse in a ring. We prove that $a \in R_w^{\textcircled{w}}$ if and only if $a \in R^{D,w}$ and there exists some $y \in R$ such that

$$(a^{D,w}w)^* a^{D,w} wy = (a^{D,w}w)^* a.$$

In this case, $a_w^{\textcircled{w}} = (a^{D,w}w)^3 y$.

In Section 4, the relations between weighted core-EP inverses and weighted weak group inverses are presented. The conditions under which an element and its weak weighted group inverse commute with the weight are given.

Throughout the paper, all rings are associative ring with a proper involution $*$. We use $R^{\#}, R^D, R^{\textcircled{D}}$ and $R^{\textcircled{w}}$ to denote the sets of all group invertible, Drazin invertible, core-EP invertible weak group invertible elements in $R$, respectively. $\mathbb{N}$ denotes the set of all natural numbers.

## 2. WEAK $w$-GROUP INVERSE

The purpose of this section is to introduce a new generalized inverse which is a natural generalization of weak group inverse in a *-Banach algebra. We begin with

**Lemma 2.1.** *Let $a, w \in R$. Then the following are equivalent:*

(1) $a \in R_w^{\#}$.
(2) $aw, wa \in R^{\#}$.
(3) There exist $x \in R$ such that

$$x(wa)^2 = a, a(wx)^2 = x, awx = xwa.$$

In this case, $a_w^{\#} = (aw)^{\#} a (wa)^{\#}$.

*Proof.* Straightforward. □

**Lemma 2.2.** *Let $a \in R_w^{\#}$. Then $(aw)^2 a_w^{\#} = a$ and $a[wa_w^{\#}]^2 = a_w^{\#}$.*

*Proof.* Let $x = a_w^{\#}$. Then

$$awxwa = a, xwawx = x, awx = xwa.$$

Hence, $(aw)^2 x = aw(awx) = (aw)x(wa) = a$. Moreover, we have $a[wa_w^{\#}]^2 = a(wx)^2 = x$ by Lemma 2.1. □

We are ready to prove:

**Theorem 2.3.** *Let $a, w \in R$. Then the following are equivalent:*

(1) $a \in R$ has weak $w$-group decomposition.
(2) There exist $x \in R$ and $n \in \mathbb{N}$ such that

$$x = a(wx)^2, \big((aw)^n\big)^*(aw)^2x = \big((aw)^n\big)^*a, (aw)^n = xw(aw)^{n+1}.$$

*Proof.* (1) $\Rightarrow$ (2) Let $a = a_1 + a_2$ be the weak $w$-group decomposition of $a$.
Let $x = (a_1)^{\#}_w$. By virtue of Lemma 2.2, we have

$$\begin{array}{rcl} awx & = & (a_1w + a_2w)(a_1)^{\#}_w = a_1w(a_1)^{\#}_w, \\ a(wx)^2 & = & a_1[w(a_1)^{\#}_w]^2 = (a_1)^{\#}_w = x, \end{array}$$

Since $a_2 \in R^{nil}_w$, $(a_2w)^n = 0$ for some $n \in \mathbb{N}$. As $a_2wa_1 = 0$, we see that

$$\begin{array}{rl} & aw - xw(aw)^2 \\ = & (a_1w + a_2w) - [(a_1)^{\#}_w wa_1w + (a_1)^{\#}_w wa_2w](a_1w + a_2w) \\ = & [1 - (a_1)^{\#}_w wa_1w - (a_1)^{\#}_w wa_2w]a_2w. \end{array}$$

Hence,

$$\begin{array}{rl} & (aw)^n - xw(aw)^{n+1} \\ = & [aw - xw(aw)^2](aw)^{n-1} \\ = & [1 - (a_1)^{\#}_w wa_1w - (a_1)^{\#}_w wa_2w]a_2w(aw)^{n-1} \\ = & [1 - (a_1)^{\#}_w wa_1w - (a_1)^{\#}_w wa_2w](a_2w)^n \\ = & 0. \end{array}$$

Thus, $(aw)^n = xw(aw)^{n+1}$.
Since $aw = a_1w + a_2w, (a_2w)(a_1w) = 0$ and $(a_2w)^n = 0$, we have

$$(aw)^n = \sum_{i=0}^{n}(a_1w)^i(a_2w)^{n-i} = (a_1w)^n + \sum_{i=1}^{n}(a_1w)^i(a_2w)^{n-i}.$$

As $(a_1)^*a_2 = 0$, we deduce that

$$\big((aw)^n\big)^*a_2 = \big((a_1w)^n\big)^*a_2 + \sum_{n=1}^{n-1}[(a_1w)^i(a_2w)^{n-i}]^*a_2 = 0;$$

and then $\big((aw)^n\big)^*a_1 = \big((aw)^n\big)^*a$. Accordingly,

$$\begin{array}{rcl} \big((aw)^n\big)^*(aw)^2x & = & \big((aw)^n\big)^*(a_1w + a_2w)(a_1w + a_2w)a_1^{\#} \\ & = & \big((aw)^n\big)^*(a_1w)^2a_1^{\#} = \big((aw)^n\big)^*a_1 \\ & = & \big((aw)^n\big)^*a. \end{array}$$

Therefore we derive

$$x = a(wx)^2, \big((aw)^n\big)^*(aw)^2x = \big((aw)^n\big)^*a, (aw)^n = xw(aw)^{n+1}.$$

(2) $\Rightarrow$ (1) By hypothesis, there exist $x \in R$ and $n \in \mathbb{N}$ such that

$$x = a(wx)^2, \big((aw)^n\big)^*(aw)^2x = \big((aw)^n\big)^*a, (aw)^n = xw(aw)^{n+1}.$$

Let $a_1 = (aw)^2x$ and $a_2 = a - (aw)^2x$.

Claim 1. $a_2wa_1 = 0$.

Clearly, $x = (aw)x(wx) = (aw)^2x(wx)^2 = (aw)^{n-1}x(wx)^{n-1}$. Then

$$\begin{array}{rcl} a_2wa_1 & = & [a-(aw)^2x]w(aw)^2x \\ & = & (aw)^3x - (aw)^2xw(aw)^2x \\ & = & (aw)^3x - (aw)^2xw(aw)^2x \\ & = & (aw)^3x - (aw)^2xw(aw)^2(aw)^{n-1}x(wx)^{n-1} \\ & = & (aw)^3x - (aw)^2[xw(aw)^{n+1}]x(wx)^{n-1} \\ & = & (aw)^3x - (aw)^{n+2}x(wx)^{n-1} \\ & = & (aw)^3x - (aw)^3[(aw)^{n-1}x(wx)^{n-1}] \\ & = & (aw)^3x - (aw)^3x = 0. \end{array}$$

Claim 2. $a_1^*a_2 = 0$.

Obviously, $(aw)^2x = (aw)^2[(aw)^{n-2}x(wx)^{n-2}] = (aw)^nx(wx)^{n-2}$. Then

$$\begin{array}{rcl} a_1^*a_2 & = & [(aw)^2x]^*[a-(aw)^2x] \\ & = & [(aw)^nx(wx)^{n-2}]^*[a-(aw)^2x] \\ & = & [x(wx)^{n-2}]^*[(aw)^n]^*[a-(aw)^2x] \\ & = & 0. \end{array}$$

Claim 3. $a_1 \in R_w^{\#}$. Evidently, we verify that

$$\begin{array}{rcl} a_1wx & = & (aw)^2xwx = awa(wx)^2 = awx, \\ xwa_1 & = & xw(aw)^2x = xw(aw)^2[(aw)^{n-1}x(wx)^{n-1}] \\ & = & [xw(aw)^{n+1}]x(wx)^{n-1} = (aw)^nx(wx)^{n-1} = awx, \end{array}$$

and then $a_1wx = xwa_1$. Moreover, we have

$$\begin{array}{rcl} a_1wxwa_1 & = & (a_1wx)wa_1 = awxw(aw)^2x \\ & = & awxw(aw)^2[(aw)^{n-1}x(wx)^{n-1}] \\ & = & aw[xw(aw)^{n+1}]x(wx)^{n-1} \\ & = & (aw)^{n+1}x(wx)^{n-1} = (aw)^2x = a_1, \\ xwa_1wx & = & (xwa_1)wx = a(wx)^2 = x. \end{array}$$

Hence, $a_1 \in R_w^{\#}$ and $(a_1)_w^{\#} = x$.

Claim 2. $a_2 \in R_w^{nil}$. It is easy to verify that

$$\begin{array}{rcl} [aw - xw(aw)^2]x & = & [aw - xw(aw)^2][(aw)^{n-1}x(wx)^{n-2}] \\ & = & [(aw)^n - xw(aw)^{n+1}]x(wx)^{n-2} = 0. \end{array}$$

Hence,

$$\begin{array}{rcl} [aw - xw(aw)^2]^2 & = & [aw - xw(aw)^2]aw - [aw - xw(aw)^2]xw(aw)^2 \\ & = & [aw - xw(aw)^2](aw). \end{array}$$

This implies that

$$\begin{array}{rcl} [aw - xw(aw)^2]^n & = & [aw - xw(aw)^2]^{n-2}[aw - xw(aw)^2]^2 \\ & = & [aw - xw(aw)^2]^{n-3}[aw - xw(aw)^2]^2(aw) \\ & = & [aw - xw(aw)^2]^{n-3}[aw - xw(aw)^2](aw)^2 \\ & \vdots & \\ & = & [aw - xw(aw)^2](aw)^{n-1} \\ & = & (aw)^n - xw(aw)^{n+1} = 0. \end{array}$$

Thus, $[1 - xw(aw)]aw \in R^{nil}$, and then $aw[1 - xw(aw)] = aw - aw(xw)aw \in R^{nil}$. Hence, $[1 - aw(xw)]aw \in R^{nil}$. This implies that $aw[1 - aw(xw)] \in R^{nil}$. That is, $aw - (aw)^2xw \in R^{nil}$; hence, $a_2 = a - (aw)^2x \in R_w^{nil}$.

Therefore $a = x + y$ is weak $w$-group decomposition of $a$, as required. □

**Corollary 2.4.** *Let $a, w \in R$. Then the following are equivalent:*

(1) $a \in R$ has weak $w$-group decomposition.
(2) There exists $x \in R$ such that

$$x = a(wx)^2, \big((aw)^m\big)^*(aw)^2x = \big((aw)^m\big)^*a, (aw)^n = xw(aw)^{n+1}$$

for some $m, n \in \mathbb{N}$.

*Proof.* (1) ⇒ (2) This is trivial by Theorem 2.3.

(2) ⇒ (1) By hypothesis, there exists $x \in R$ such that

$$x = a(wx)^2, \big((aw)^m\big)^*(aw)^2x = \big((aw)^m\big)^*a, (aw)^n = xw(aw)^{n+1}$$

for some $m, n \in \mathbb{N}$. Then

$$\begin{array}{rcl}
\big((aw)^n\big)^*(aw)^2x & = & \big(xw(aw)^{n+1}\big)^*(aw)^2x\\
& = & \big(a(wx)^2w(aw)^{n+1}\big)^*(aw)^2x\\
& = & \big((aw)(xw)^2(aw)^{n+1}\big)^*(aw)^2x\\
& = & \big((aw)^m(xw)^{m+1}(aw)^{n+1}\big)^*(aw)^2x\\
& = & \big((xw)^{m+1}(aw)^{n+1}\big)^*[((aw)^m)^*(aw)^2x]\\
& = & \big((xw)^{m+1}(aw)^{n+1}\big)^*[\big((aw)^m\big)^*a]\\
& = & \big((aw)^m(xw)^{m+1}(aw)^{n+1}\big)^*a\\
& = & \big((aw)(xw)^2(aw)^{n+1}\big)^*a\\
& & \\
& = & \big(a(wx)^2w(aw)^{n+1}\big)^*a\\
& = & \big(xw(aw)^{n+1}\big)^*a\\
& = & \big((aw)^n\big)^*a
\end{array}$$

In light of Theorem 2.3, we complete the proof. □

**Theorem 2.5.** *Let $a, w \in R$. Then the following are equivalent:*

(1) $a \in R$ has weak $w$-group decomposition.
(2) There exists unique $x \in R$ such that

$$x = a(wx)^2, \big((aw)^n\big)^*(aw)^2x = \big((aw)^n\big)^*a, (aw)^n = xw(aw)^{n+1}$$

for some $n \in \mathbb{N}$.

*Proof.* (2) ⇒ (1) Suppose that there exist $x, y \in R$ such that

$$\begin{array}{c}
x = a(wx)^2, \big((aw)^n\big)^*(aw)^2x = \big((aw)^n\big)^*a, (aw)^n = xw(aw)^{n+1};\\
y = a(wy)^2, \big((aw)^m\big)^*(aw)^2y = \big((aw)^m\big)^*a, (aw)^m = yw(aw)^{m+1}.
\end{array}$$

Choose $k = \max(m, n)$. Then

$$\begin{array}{c}
x = a(wx)^2, \big((aw)^k\big)^*(aw)^2x = \big((aw)^k\big)^*a, (aw)^k = xw(aw)^{k+1};\\
y = a(wy)^2, \big((aw)^k\big)^*(aw)^2y = \big((aw)^k\big)^*a, (aw)^k = yw(aw)^{k+1}.
\end{array}$$

Claim 1. $xw = yw$.
By hypothesis, we have

$$xw = aw(xw)^2, \big((aw)^n\big)^*(aw)^2x = \big((aw)^n\big)^*a, (aw)^n = xw(aw)^{n+1}.$$

Thus $xw$ is weak group inverse of $aw$. Likewise, $yw$ is weak group inverse of $aw$. In view of [15, Theorem 3.5], we have $xw = yw$.

Claim 2. $(aw)^2x = (aw)^2y$.

Since $x = a(wx)^2$, we have $x = (aw)^{n-2}x(wx)^{n-2}$, and so $(aw)^2x = (aw)^n x(wx)^{n-2}$. Then

$$\begin{array}{rl} & ((aw)^2x)^*(aw)^2x - ((aw)^2x)^*a \\ = & (x(wx)^{n-2})^*[((aw)^n)^*a^2x - ((aw)^n)^*a] \\ = & 0. \end{array}$$

Therefore $[((aw)^2x)^*(aw)^2x]^* = ((aw)^2x)^*a$. Similarly, we have

$$\begin{array}{rcl} (((aw)^2x)^*(aw)^2y)^* & = & ((aw)^2x)^*a, \\ (((aw)^2y)^*(aw)^2x)^* & = & ((aw)^2x)^*a, \\ (((aw)^2y)^*(aw)^2y)^* & = & ((aw)^2x)^*a. \end{array}$$

Let $z = (aw)^2x - (aw)^2y$. Then we check that

$$\begin{array}{rcl} z^*z & = & ((aw)^2x - (aw)^2y)^*((aw)^2x - (aw)^2y) \\ & = & ((aw)^2x)^*(aw)^2x - ((aw)^2x)^*(aw)^2y \\ & - & ((aw)^2y)^*(aw)^2x + ((aw)^2y)^*(aw)^2y \\ & = & ((aw)^2x)^*a - ((aw)^2x)^*a - ((aw)^2y)^*a^2 + ((aw)^2y)^*a^2 \\ & = & 0. \end{array}$$

Since $\mathcal{A}$ is a proper *-Banach algebra, we have $z = 0$; hence, $(aw)^2x = (aw)^2y$.

Claim 3. $x = y$.

We see that

$$\begin{array}{rcl} (xw)(awx) & = & xwaw[(aw)^k x(wx)^k]w = [xw(aw)^{k+1}]x(wx)^k \\ & = & (aw)^k x(wx)^k = x. \end{array}$$

Moreover, we have

$$\begin{array}{rcl} (xw)^2(aw)^2x & = & xw[xw(aw)^2](aw)^{k-1}x(wx)^{k-1} \\ & = & xw[xw(aw)^{k+1}]x(wx)^{k-1} \\ & = & xwaw[(aw)^{k-1}x(wx)^{k-1}] \\ & = & (xw)(awx). \end{array}$$

Therefore $x = (xw)^2(aw)^2x$. Likewise, $y = (yw)^2(aw)^2y$. By the preceding discussion, we have

$$xw = yw \text{ and } (aw)^2x = (aw)^2y.$$

Therefore $x = y$, as desired. $\square$

We denote $x$ in Theorem 2.5 by $a_w^{\textcircled{W}}$, and call it the weak $w$-group inverse of $a$. $R_w^{\textcircled{W}}$ denotes the sets of all weak $w$-group invertible elements in $R$.

**Corollary 2.6.** *Let $a \in R_w^{\textcircled{W}}$. Then the following hold.*

(1) $a_w^{\textcircled{W}} = a_w^{\textcircled{W}} wawa_w^{\textcircled{W}}$.
(2) $(aw)(a_w^{\textcircled{W}} w) = (aw)^m(a_w^{\textcircled{W}} w)^m$ for any $m \in \mathbb{N}$.

*Proof.* (1) Let $x = a_w^{\textcircled{W}}$. In view of Theorem 2.5, we have $x = a(wx)^2$ and $(aw)^n = xw(aw)^{n+1}$ for some $n \in \mathbb{N}$. Then $x = (aw)^n x(wx)^n$. Hence,

$$\begin{array}{rcl} a_w^{\textcircled{W}} wawa_w^{\textcircled{W}} & = & (xwaw)[(aw)^n x(wx)^n] \\ & = & [xw(aw)^{n+1}]x(wx)^n = (aw)^n x(wx)^n \\ & = & x, \end{array}$$

as required.

(2) We easily see that

$$(aw)^2(a_w^{\textcircled{W}} w)^2 = aw[a(w(a_w^{\textcircled{W}})^2]w = (aw)(a_w^{\textcircled{W}} w).$$

By induction, we complete the proof. □

**Theorem 2.7.** *Let $a, x \in R$ and $w \in R^{-1}$. Then the following are equivalent:*

(1) $x = a_w^{\textcircled{W}}$.
(2) There exists some $n \in \mathbb{N}$ such that

$$x = a(wx)^2, [(aw)^*(aw)^2xw]^* = (aw)^*(aw)^2xw, (aw)^n = xw(aw)^{n+1}.$$

In this case, $x = (aw)^{\textcircled{W}} w^{-1}$.

*Proof.* (1) ⇒ (2) By hypothesis, $a$ has the weak w-group decomposition $a = a_1 + a_2$. Let $x = (a_1)_w^{\#}$. As in the proof of Theorem 2.3, we see that

$$x = a(wx)^2, (aw)^n = xw(aw)^{n+1}.$$

Moreover, we have

$$\begin{array}{rcl} (aw)^*(aw)^2xw & = & (a_1w + a_2w)^*(a_1w + a_2w)^2(a_1)_w^{\#}w \\ & = & (a_1w + a_2w)^*(a_1w + a_2w)a_1w(a_1)_w^{\#}w \\ & = & (a_1w + a_2w)^*(a_1w)^2(a_1)_w^{\#}w \\ & = & (a_1w + a_2w)^*a_1w \\ & = & (a_1w)^*a_1w. \end{array}$$

Therefore

$$[(aw)^*(aw)^2xw]^* = [(a_1w)^*a_1w]^* = (a_1w)^*a_1w = (aw)^*(aw)^2xw,$$

as desired.

(2) ⇒ (1) By hypotheses, there exist $z \in R$ and $n \in \mathbb{N}$ such that

$$z = a(wz)^2, [(aw)^*(aw)^2zw]^* = (aw)^*(aw)^2zw, (aw)^n = zw(aw)^{n+1}.$$

Let $a_1 = (aw)^2x$ and $a_2 = a - (aw)^2x$. As in the proof of Theorem 2.3, we prove that

$$a_2wa_1 = 0, a_1 \in R_w^{\#} \text{ and } a_2 \in R_w^{nil}.$$

Moreover, we verify that

$$\begin{array}{rcl} a_1^*a_2w & = & [(aw)^2x]^*[a-(aw)^2x]w \\ & = & [(aw)^2x]^*aw-[awx]^*[(aw)^*(aw)^2xw] \\ & = & [(aw)^2x]^*aw-[awx]^*[(aw)^*(aw)^2xw]^* \\ & = & [(aw)^2x]^*aw-[(aw)^*(aw)^2xwawx]^* \\ & = & [(aw)^2x]^*aw-[(aw)^2xwawx]^*aw \\ & = & [(aw)^2x]^*aw-[(aw)^2xwaw((aw)^nx(xw)^n)]^*aw \\ & = & [(aw)^2x]^*aw-[(aw)^2(xw(aw)^{n+1})x(xw)^n)]^*aw \\ & = & [(aw)^2x]^*aw-[(aw)^2((aw)^nx(xw)^n)]^*aw \\ & = & [(aw)^2x]^*aw-[(aw)^2x]^*aw \\ & = & 0. \end{array}$$

As $w\in R^{-1}$, we deduce that $a_1^*a_2=0$. Therefore $a=x+y$ is weak $w$-group decomposition of $a$.

Obviously, we have

$$xw=aw(xw)^2,[(aw)^*(aw)^2xw]^*=(aw)^*(aw)^2xw,(aw)^n=xw(aw)^{n+1}.$$

Hence, $xw=(aw)^{\textcircled{w}}$. As $w\in R^{-1}$, we have $x=(aw)^{\textcircled{w}}w^{-1}$, as required. $\square$

## 3. EQUIVALENT CHARACTERIZATIONS

In this section we establish some equivalent characterizations of weak weighted group inverses. We now derive

**Theorem 3.1.** *Let $a\in R$. Then $a\in R_w^{\textcircled{w}}$ if and only if*

(1) $a\in R^{D,w}$;
(2) There exists $x\in R$ such that

$$xR=a^{D,w}R,((aw)^n)^*(aw)^2x=((aw)^n)^*a,(aw)^n=xw(aw)^{n+1}$$

for some $n\in\mathbb{N}$.

In this case, $a_w^{\textcircled{w}}=x$.

*Proof.* $\Longrightarrow$ In view of Theorem 2.3, there exist $x\in R$ and $n\in\mathbb{N}$ such that

$$x=a(wx)^2,((aw)^n)^*(aw)^2x=((aw)^n)^*a,(aw)^n=xw(aw)^{n+1}.$$

Here, $x=a_w^{\textcircled{w}}$. Hence, $xw=(aw)(wx)^2$ and $(aw)^n=(xw)(aw)^{n+1}$. By virtue of [17, Lemma 2.2], $aw\in R^D$. This implies that $a\in R^{D,w}$.

We claim that $xR=a^{D,w}R$.

By virtue of Theorem 2.3, there exist $z,y\in R$ such that

$$a=z+y,z^*y=ywz=0,z\in R_w^{\#},y\in R_w^{nil}.$$

Let $x = x_w^{\#}$. In view of Lemma 2.1, $zw \in R^{\#}$. Write $(yw)^{k+1} = 0$ for some $k \in \mathbb{N}$. Since $aw = zw + yw, yw \in R^{qnil}$ and $(yw)(zw) = 0$, it follows by [1, Corollary 3.5] that $aw \in R^D$ and

$$(aw)^D = (zw)^{\#} + \sum_{n=1}^{k}((zw)^{\#})^{n+1}(yw)^n.$$

We directly verify that

$$\begin{array}{rcl} (aw)(aw)^D x & = & (aw)^D(zw + yw)(zw)^{\#} \\ & = & (aw)^D zw(zw)^{\#} \\ & = & [(zw)^{\#} + \sum\limits_{n=1}^{k}((zw)^{\#})^{n+1}(yw)^n](zw)(zw)^{\#} \\ & = & (zw)^{\#}(zw)(zw)^{\#} \\ & = & x. \end{array}$$

Moreover, we have

$$\begin{array}{rcl} x(aw)(aw)^D & = & a_1^{\#}(a_1 + a_2)[a_1^{\#} + \sum\limits_{n=1}^{k}(a_1^{\#})^{n+1}a_2^n] \\ & = & a_1^{\#}a_1[a_1^{\#} + \sum\limits_{n=1}^{k}(a_1^{\#})^{n+1}a_2^n] \\ & = & a_1^{\#} + \sum\limits_{n=1}^{k}(a_1^{\#})^{n+1}a_2^n \\ & = & (aw)^D. \end{array}$$

Accordingly, $xR = a^{D,w}R$, as asserted.

$\Longleftarrow$ We directly check that

$$\begin{array}{rcl} awxw(aw)^D & = & awxw(aw)^{n+1}[(aw)^D]^{n+2} \\ & = & aw[xw(aw)^{n+1}][(aw)^D]^{n+2} \\ & = & aw(aw)^n[(aw)^D]^{n+2} = (aw)^D. \end{array}$$

Since $a^{D,w} = (aw)^D a(wa)^D$, we have $awxwa^{D,w} = a^{D,w}$, and so $(1-awxw)a^{D,w} = 0$. As $xR = a^{D,w}R$, we derive that $(1 - awxw)x = 0$. Therefore $x = awxwx = a(wx)^2$. By virtue of Theorem 2.3, $a \in R_w^{\textcircled{W}}$. In this case, $a_w^{\textcircled{g}} = x$, required. $\square$

**Corollary 3.2.** *Let $a \in R$. Then $a \in R_w^{\textcircled{W}}$ if and only if*

(1) $a \in R^{D,w}$;
(2) There exists $x \in R$ such that

$$xwawx = x, xR = (aw)^m R = (aw)^{m+1}R, a^*(aw)^m R \subseteq x^* R$$

for some $m \in \mathbb{N}$.

In this case, $a_w^{\textcircled{W}} = x$.

*Proof.* $\Longrightarrow$ In view of Theorem 3.1, $a \in R^{D,w}$ and there exist $x \in R$ and $m \in \mathbb{N}$ such that

$$x = a(wx)^2, \big((aw)^m\big)^*(aw)^2x = \big((aw)^m\big)^*a, (aw)^m = xw(aw)^{m+1}.$$

Then $x = awxwx = (aw)^m(xw)^mx$; hence, $xwawx = xwaw(aw)^m(xw)^mx = [xw(aw)^{m+1}](xw)^mx = (aw)^m(xw)^mx = (aw)(xw)x = a(wx)^2 = x$. Thus, $x = awxwx = (aw)^mx(wx)^m$; whence $xR \subseteq (aw)^mR$. On the other hand, $(aw)mR \subseteq xR$. Thus, $xR = (aw)^mR$. Obviously, $(aw)^{m+1}R \subseteq (aw)^mR$. On the other hand, $(aw)^m = xw(aw)^{m+1} = (aw)^{m+1}x(wx)^{m+1}w(aw)^{m+1}$; hence, $(aw)^mR \subseteq (aw)^{m+1}R$. This implies that $(aw)^mR = (aw)^{m+1}R$. Moreover, we get $\big((aw)^m\big)^*(aw)^2x = \big((aw)^m\big)^*a$, and so $a^*(aw)^m = x^*[\big((aw)^m\big)^*(aw)^2]^*$. Accordingly, $a^*(aw)^mR \subseteq x^*R$, as required.

$\Longleftarrow$ By hypothesis, $a \in R^{D,w}$ and there exists $x \in R$ such that

$$xwawx = x, xR = (aw)^mR = (aw)^{m+1}R, a^*(aw)^mR \subseteq x^*R$$

for some $m \in \mathbb{N}$.

Claim 1. $xR = a^{D,w}R$. Let $k = i(aw)$. Then $xR = (aw)^{m+k}R$. Since $(aw)^k = (aw)^D(aw)^{k+1}$, we have $xR = (aw)^D(aw)^{m+k+1}R = (aw)^DR = a^{D,w}wR = a^{D,w}R$, as desired.

Claim 2. $((aw)^m)^*(aw)^2x = ((aw)^m)^*a$. Since $x = xwawx$, we have $x(1 - wawx) = 0$, and so $(1 - wawx)^*x^* = 0$. Hence $(1 - wawx)^*a^*(aw)^m = 0$. Therefore $[(aw)^m]^*(aw)^2x = [(aw)^m]^*a$, as required.

Claim 3. $(aw)^m = xw(aw)^{m+1}$.

Since $(1 - xwaw)x = 0$, we see that $(1 - xwaw)(aw)^m = 0$. Thus $(aw)^m = xw(aw)^{m+1}$.

Therefore $a \in R_w^{\textcircled{W}}$ by Theorem 3.1. $\square$

**Theorem 3.3.** *Let $a, w \in R$. Then the following are equivalent:*

(1) $a \in R_w^{\textcircled{W}}$.

(2) $a \in R^{D,w}$ and there exist $n \in \mathbb{N}, x \in R$ such that

$$x = a(wx)^2, ((aw)^n)^*(aw)^2x = ((aw)^n)^*a.$$

(3) $a \in R^{D,w}$ and there exists some $y \in R$ such that

$$(a^{D,w}w)^*a^{D,w}wy = (a^{D,w}w)^*a.$$

In this case, $a_w^{\textcircled{W}} = (aw)(aw)^Dx = (a^{D,w}w)^3y$.

*Proof.* (1) $\Rightarrow$ (2) By virtue of Theorem 3.1, $a \in R^{D,w}$ and there exists $x \in R$ such that $\big((aw)^n\big)^*(aw)^2x = \big((aw)^n\big)^*a$, as desired.

(2) $\Rightarrow$ (3) By hypothesis, $a \in R^{D,w}$ and there exist $n \in \mathbb{N}, x \in R$ such that $x = a(wx)^2, ((aw)^n)^*(aw)^2x = ((aw)^n)^*a$. Then we see that $((aw)^D)^*(aw)^2x = ((aw)^D)^*a$. Obviously, we have

$$\begin{array}{rcl} a^{D,w}w & = & (aw)^Da(wa)^Dw = (aw)^D[a((wa)^D)^2w]aw \\ & = & [(aw)^D]^2aw = (aw)^D. \end{array}$$

Hence, $(a^{D,w}w)^*(aw)^2x = (a^{D,w}w)^*a$.

Since $x = a(wx)^2 = (aw)x(wx) = (aw)^nx(wx)^n$ for any $n \in \mathbb{N}$, we observe that

$$\begin{array}{rl} & (a^{D,w}w)^*(aw)^2x - (a^{D,w}w)^*(aw)^D(aw)^3x \\ = & (a^{D,w}w)^*(aw)^{n+2}x(wx)^n - (a^{D,w}w)^*(aw)^D(aw)^{n+3}x(wx)^n \\ = & (a^{D,w}w)^*[(aw)^n - (aw)^D(aw)^{n+1}](aw)^2x(wx)^n. \end{array}$$

Since $(aw)^n = (aw)^D(aw)^{n+1}$, we get

$$(a^{D,w}w)^*(aw)^2x - (a^{D,w}w)^*(aw)^D(aw)^3x = 0;$$

hence, $(a^{D,w}w)^*(aw)^D(aw)^3x = (a^{D,w}w)^*a$. Set $y = (aw)^3x$. Then we verify that

$$\begin{array}{rcl} (a^{D,w}w)^*a^{D,w}wy & = & (a^{D,w}w)^*[a^{D,w}w]awx \\ & = & (a^{D,w}w)^*(aw)^Dawx = (a^{D,w}w)^*a, \end{array}$$

as desired.

(3) $\Rightarrow$ (1) By hypothesis, $(a^{D,w}w)^*a^{D,w}wy = (a^{D,w}w)^*a$ for some $y \in R$. Then $((aw)^D)^*(aw)^Dy = ((aw)^D)^*a$. It is easy to verify that

$$\begin{array}{rcl} [aw(aw)^D]^*aw(aw)^D & = & (aw)^*[((aw)^D)^*a]w(aw)^D \\ & = & (aw)^*[((aw)^D)^*(aw)^Dy]w(aw)^D \\ & = & [aw(aw)^D]^*(aw)^Dyw(aw)^D \\ & = & [aw(aw)^D]^*aw[(aw)^D]^2yw(aw)^D. \end{array}$$

Since the involution $*$ is proper, we get $aw(aw)^D = (aw)^D yw(aw)^D$. Let $z = ((aw)^D)^3 y$. Then we verify that

$$\begin{array}{rcl} a(wz)^2 & = & aw((aw)^D)^3 yw((aw)^D)^3 y \\ & = & ((aw)^D)^2 yw((aw)^D)^3 y \\ & = & (aw)^D[(aw)^D yw(aw)^D]((aw)^D)^2 y \\ & = & (aw)^D aw(aw)^D((aw)^D)^2 y \\ & = & ((aw)^D)^3 y = z; \\ \big((aw)^D\big)^*(aw)^2 z & = & \big((aw)^D\big)^*(aw)^2((aw)^D)^3 y \\ & = & \big((aw)^D\big)^*(aw)^D y \\ & = & \big((aw)^D\big)^* a. \end{array}$$

Set $n = i(aw)$. Then $(aw)^n = (aw)^D(aw)^{n+1}$. Thus,

$$\begin{array}{cl} & (aw)^n - zw(aw)^{n+1} \\ = & [(aw)^n - (aw)^D(aw)^{n+1}] + [(aw)^D(aw)^{n+1} - zw(aw)^{n+1}] \\ = & (aw)^D(aw)^{n+1} - ((aw)^D)^3 yw(aw)^{n+1} \\ = & -((aw)^D)^3 yw[1 - (aw)^D(aw)](aw)^{n+1} \\ + & [(aw)^D(aw)^{n+1} - ((aw)^D)^3 yw(aw)^D(aw)^{n+2}] \\ = & (aw)^D(aw)^{n+1} - ((aw)^D)^3 yw(aw)^D(aw)^{n+2} \\ = & [(aw)^D]^2[(aw)(aw)^D](aw)^{n+2} - ((aw)^D)^3 yw(aw)^D(aw)^{n+2} \\ = & [(aw)^D]^2[(aw)^D yw(aw)^D](aw)^{n+2} - ((aw)^D)^3 yw(aw)^D(aw)^{n+2} \\ = & 0. \end{array}$$

That is, $(aw)^n = zw(aw)^{n+1}$. Accordingly, $a \in R_w^{\textcircled{W}}$. In this case,

$$a_w^{\textcircled{W}} = ((aw)^D)^3 y = (a^{D,w}w)^3 y = (a^{D,w}w)^3(aw)^3 x = (aw)(aw)^D x,$$

as asserted. $\square$

**Corollary 3.4.** *Let $a \in R$. Then $a \in R_w^{\textcircled{W}}$ if and only if*

1. $a \in R^{D,w}$;
2. There exists an idempotent $q \in R$ such that

$$a^{D,w}R = qR \text{ and } (a^{D,w})^* awq = (a^{D,w})^* a.$$

*Proof.* $\Longrightarrow$ By using Theorem 3.3, $a \in R^{D,w}$ and there exists some $y \in R$ such that

$$(a^{D,w}w)^* a^{D,w} wy = (a^{D,w}w)^* a.$$

Then

$$(a^{D,w}w)^*(aw)[(aw)^D]^2 y = (a^{D,w}w)^* a.$$

Set $q = [(aw)^D]^2 y$. Then $(a^{D,w})^* awq = (a^{D,w})^* a$. Obviously, $qR \subseteq a^{D,w}wR$. Moreover, $a_w^{\textcircled{W}} = (a^{D,w}w)^3 y = [(aw)^D]^3 y$. Then $a_w^{\textcircled{W}} = (aw)^D q$, and so $q =$

$(aw)[(aw)^Dq] = awa_w^{\textcircled{W}}$. Hence, $qw = (aw)(a_w^{\textcircled{W}}w)$, and so $qw(aw)^D = (aw)(a_w^{\textcircled{W}}w)(aw)^D$. We observe that

$$
\begin{array}{rl}
 & (aw)^D - qw(aw)^D \\
= & (aw)^D - (aw)(a_w^{\textcircled{W}}w)(aw)^D \\
= & (aw)^{n+1}[(aw)^D]^{n+2} - (aw)(a_w^{\textcircled{W}}w)(aw)^{n+1}[(aw)^D]^{n+2} \\
= & aw[(aw)^n - (a_w^{\textcircled{W}}w)(aw)^{n+1}][(aw)^D]^{n+2}.
\end{array}
$$

Hence, Since $(aw)^n = (a_w^{\textcircled{W}}w)(aw)^{n+1}$, we get $(aw)^D - qw(aw)^D = 0$; hence, $a^{D,w}w = (aw)^D = qw(aw)^D$; hence, $a^{D,w}wR \subseteq qR$. Thus, we prove that $a^{D,w}R = a^{D,w}wR = qR$, as desired.

$\Longleftarrow$ By hypothesis, $a \in R^{D,w}$ and there exists an idempotent $q \in R$ such that

$$a^{D,w}R = qR \text{ and } (a^{D,w})^*awq = (a^{D,w})^*a.$$

Since $a^{D,w}R = a^{D,w}wR = (aw)^DR$, we write $q = (aw)^Dz$ with $z \in R$. Choose $y = awz$. Then

$$
\begin{array}{rcl}
(a^{D,w}w)^*a^{D,w}wy & = & (a^{D,w}w)^*a^{D,w}w(aw)z \\
 & = & (a^{D,w}w)^*(aw)^D(aw)z \\
 & = & (a^{D,w}w)^*aw[(aw)^Dz] \\
 & = & (a^{D,w}w)^*awq \\
 & = & (a^{D,w}w)^*a,
\end{array}
$$

the result follows by Theorem 3.3. $\square$

## 4. RELATIONS WITH WEIGHTED CORE-EP INVERSES

In this section we investigate relations between weighted weak group and weighted core-EP inverses. Our starting point is the following.

**Theorem 4.1.** *Let $a \in R_w^{\textcircled{D}}$. Then $a \in R_w^{\textcircled{W}}$ and $a_w^{\textcircled{W}} = (a_w^{\textcircled{D}})^2a$.*

*Proof.* By hypothesis, we have

$$a_w^{\textcircled{W}} = a(wa_w^{\textcircled{W}})^2, (awa_w^{\textcircled{D}}w)^* = awa_w^{\textcircled{D}}w, (aw)^n = a_w^{\textcircled{D}}w(aw)^{n+1}.$$

Set $x=(a_w^{\textcircled{D}}w)^2a$. Then we check that

$$\begin{array}{rcl}
a(wx)^2 & = & [aw(a_w^{\textcircled{D}}w)^2][aw(a_w^{\textcircled{D}}w)^2]a\\
& = & aw(a_w^{\textcircled{D}}w)(a_w^{\textcircled{D}}w)^2a=(a_w^{\textcircled{D}}w)(a_w^{\textcircled{D}}w)a\\
& = & (a_w^{\textcircled{D}}w)^2a=x,\\
(aw)^*(aw)^2xw & = & (aw)^*(aw)^2(a_w^{\textcircled{D}}w)^2aw=(aw)^*aw[a(wa_w^{\textcircled{D}})^2]waw\\
& = & (aw)^*[awa_w^{\textcircled{D}}w]aw,\\
((aw)^*(aw)^2xw)^* & = & \big((aw)^*[awa_w^{\textcircled{D}}w]aw\big)^*=(aw)^*[awa_w^{\textcircled{D}}w]^*aw\\
& = & (aw)^*[awa_w^{\textcircled{D}}w]aw=(aw)^*awa_w^{\textcircled{D}}waw\\
& = & (aw)^*aw[a(wa_w^{\textcircled{D}})^2]waw\\
& = & (aw)^*(aw)^2(a_w^{\textcircled{D}}w)^2aw=(aw)^*(aw)^2xw.
\end{array}$$

Moreover, we see that

$$\begin{array}{rl}
& (aw)^n-xw(aw)^{n+1}\\
= & (aw)^n-(a_w^{\textcircled{D}}w)^2(aw)^{n+2}\\
= & (aw)^n-a_w^{\textcircled{D}}w(aw)^{n+1}+a_w^{\textcircled{D}}w(aw)^{n+1}-(a_w^{\textcircled{D}}w)^2(aw)^{n+2}\\
= & [(aw)^n-a_w^{\textcircled{D}}w(aw)^{n+1}]+a_w^{\textcircled{D}}w[(aw)^n-(a_w^{\textcircled{D}}w)(aw)^{n+1})]aw\\
= & 0,
\end{array}$$

and so $(aw)^n=xw(aw)^{n+1}$ for some $n\in\mathbb{N}$. Therefore $a_w^{\textcircled{W}}=(a_w^{\textcircled{D}}w)^2a$. $\square$

**Corollary 4.2.** *Let $a\in R_w^{\textcircled{D}}$. Then $a_w^{\textcircled{W}}=x$ if and only if $a(wx)^2=x, awx=a_w^{\textcircled{D}}wa$.*

*Proof.* $\Longrightarrow$ In view of Theorem 4.1, $a\in R_w^{\textcircled{W}}$ and $x:=a_w^{\textcircled{W}}=(a_w^{\textcircled{D}}w)^2a$. Therefore

$$\begin{array}{rcl}
awx & = & aw(a_w^{\textcircled{D}}w)^2a=[a(wa_w^{\textcircled{D}})^2]wa=a_w^{\textcircled{D}}wa,\\
a(wx)^2 & = & (awx)wx=[a_w^{\textcircled{D}}wawa_w^{\textcircled{D}}]wa_w^{\textcircled{D}}wa\\
& = & [a_w^{\textcircled{D}}w]^2a=a_w^{\textcircled{W}}=x,
\end{array}$$

as required.

$\Longleftarrow$ By hypotheses, $a(wx)^2=x, awx=a_w^{\textcircled{D}}wa$. Then we have

$$\begin{array}{rcl}
x & = & a(wx)^2=(awx)wx=a_w^{\textcircled{D}}wa(wx)\\
& = & a_w^{\textcircled{D}}w(awx)=a_w^{\textcircled{D}}w(a_w^{\textcircled{D}}wa)\\
& = & (a_w^{\textcircled{D}}w)^2a.
\end{array}$$

In light of Theorem 4.1, $x=a_w^{\textcircled{W}}$, as desired. $\square$

**Corollary 4.3.** *Let $A,W\in\mathbb{C}^{n\times n}$. Then $X$ is the weak $W$-group of $A$ if and only if $X$ satisfies*

$$A(WX)^2=X, AWX=A_W^{\textcircled{D}}WA.$$

*Proof.* Lethal $\mathbb{C}^{n\times n}$ be the ring of all $n\times n$ complex matrices, with conjugate transpose $*$ as the involution. Then the involution $*$ is proper. Then the result follows by Corollary 4.2. $\square$

We turn to investigate when weighted group and core-EP inverse coincide with each other.

**Lemma 4.4.** *Let $a\in \mathcal{A}_w^{\textcircled{W}}$. Then $awa\in \mathcal{A}_w^{\textcircled{W}}$. In this case,*

$$(awa)_w^{\textcircled{W}} = (a_w^{\textcircled{W}}w)^3a.$$

*Proof.* Let $x=(a_w^{\textcircled{W}}w)^3a$ and $c=awa$. Then we check that

$$\begin{array}{rcl}
c(wx)^2 & = & awaw(a_w^{\textcircled{W}}w)^3aw(a_w^{\textcircled{W}}w)^3a\\
& = & aw[a(wa_w^{\textcircled{W}})^2w](a_w^{\textcircled{W}}w)aw(a_w^{\textcircled{W}}w)^3a\\
& = & a(wa_w^{\textcircled{W}})^2waw(a_w^{\textcircled{W}}w)^3a\\
& = & [a_w^{\textcircled{W}}wawa_w^{\textcircled{W}}]w(a_w^{\textcircled{W}}w)^2a\\
& = & a_w^{\textcircled{W}}wa_w^{\textcircled{W}}wa_w^{\textcircled{W}}wa\\
& = & (a_w^{\textcircled{W}}w)^3a\\
& = & x,\\
(cw)^*(cw)^2(xw) & = & ((aw)^2)^*(aw)^4(a_w^{\textcircled{W}}w)^3aw\\
& = & ((aw)^2)^*(aw)^2[(aw)^2(a_w^{\textcircled{W}}w)^2](a_w^{\textcircled{W}}w)aw\\
& = & ((aw)^2)^*(aw)^2(aw)(a_w^{\textcircled{W}}w)(a_w^{\textcircled{W}}w)aw\\
& = & ((aw)^2)^*(aw)^2[a(wa_w^{\textcircled{W}})^2]waw\\
& = & [((aw)^2)^*(aw)^2a_w^{\textcircled{W}}]waw\\
& = & [((aw)^2)^*a]waw\\
& = & \big((aw)^2\big)^*(aw)^2,
\end{array}$$

and then

$$((cw)^*(cw)^2(xw))^* = [\big((aw)^2\big)^*(aw)^2]^* = \big((aw)^2\big)^*(aw)^2 = (cw)^*(cw)^2(xw).$$

Moreover, we see that

$$\begin{array}{rcl}
(cw)^n - xw(cw)^{n+1} & = & (aw)^{2n} - (a_w^{\textcircled{W}}w)^3aw(aw)^{2n+2}\\
& = & (aw)^{2n} - (a_w^{\textcircled{W}}w)^2a_w^{\textcircled{W}}w(aw)^{2n+3}\\
& = & (aw)^{2n} - a_w^{\textcircled{W}}w(aw)^{2n+1}
\end{array}$$

Hence, we have $(cw)^n = xw(cw)^{n+1} = 0$. Therefore we complete the proof. $\square$

**Theorem 4.5.** *Let $a\in R_w^{\textcircled{D}}$. Then the following are equivalent:*

(1) $a_w^{\textcircled{W}} = a_w^{\textcircled{D}}$.
(2) $awa_w^{\textcircled{D}} = a_w^{\textcircled{D}}wa$.
(3) $(awa)_w^{\textcircled{W}} = a_w^{\textcircled{W}}wa_w^{\textcircled{W}}$.
(4) $a$ has weak $w$-group decomposition $a=a_1+a_2$ with $a_1wa_2=0$.

(5) $a$ has weak $w$-group decomposition $a=a_1+a_2$, $a_1\in R_w^{\textcircled{\#}}$ and $a_1w(a_1)_w^{\textcircled{\#}}=(a_1)_w^{\textcircled{\#}}wa_1$.

*Proof.* (1) $\Rightarrow$ (2) In view of Theorem 4.1,

$$a_w^{\textcircled{w}}=(a_w^{\textcircled{d}}w)^2a.$$

Hence,

$$\begin{array}{rcl}awa_w^{\textcircled{d}}&=&awa_w^{\textcircled{w}}=aw(a_w^{\textcircled{d}}w)^2a\\ &=&[a(wa_w^{\textcircled{d}})^2]wa=a_w^{\textcircled{d}}wa,\end{array}$$

as desired.

(2) $\Rightarrow$ (1) Since $awa_w^{\textcircled{d}}=a_w^{\textcircled{d}}wa$, it follows by Theorem 4.1 that

$$\begin{array}{rcl}a_w^{\textcircled{w}}&=&(a_w^{\textcircled{d}}w)^2a\\ &=&a_w^{\textcircled{d}}w(a_w^{\textcircled{d}}wa)\\ &=&a_w^{\textcircled{d}}w(awa_w^{\textcircled{d}})\\ &=&a_w^{\textcircled{d}},\end{array}$$

as required.

Let $a=a_1+a_2$ be the $w$-core-EP decomposition. Then it is the weak $w$-group decomposition.

(1) $\Leftrightarrow$ (4) We check that

$$\begin{array}{rcl}awa_w^{\textcircled{d}}&=&awa_w^{\textcircled{w}}=aw(a_1)_w^{\#}=(a_1w+a_2w)(a_1)_w^{\#}=a_1w(a_1)_w^{\#},\\ a_w^{\textcircled{d}}wa&=&(a_1)_w^{\#}wa=(a_1)_w^{\#}(wa_1+wa_2)=(a_1)_w^{\#}wa_1+(a_1)_w^{\#}wa_2.\end{array}$$

Thus, $(a_1)_w^{\#}wa_2=0$ if and only if $awa_w^{\textcircled{d}}=a_w^{\textcircled{d}}wa$.

If $a_1wa_2=0$, then

$$(a_1)_w^{\#}wa_2=[(a_1)_w^{\#}wa_1w(a_1)_w^{\#}]wa_2=[(a_1)_w^{\#}w]^2[a_1wa_2]=0.$$

If $(a_1)_w^{\#}wa_2=0$, then

$$a_1wa_2=[a_1w(a_1)_w^{\#}wa_1]wa_1=[a_1w]^2[(a_1)_w^{\#}wa_1]=0.$$

Thus, $a_1wa_2=0$ if and only if $(a_1)_w^{\#}wa_2=0$. Accordingly, $a_w^{\textcircled{w}}=a_w^{\textcircled{d}}$ if and only if $a_1wa_2=0$, as desired.

(3) $\Leftrightarrow$ (4) In light of Lemma 4.4, we verify that

$$\begin{array}{rcl}
(awa)^{\textcircled{W}}_w & = & (a^{\textcircled{W}}_w w)^3 a \\
& = & ((a_1)^{\#}_w w)^3 (a_1 + a_2) \\
& = & ((a_1)^{\#}_w w)^3 a_1 + ((a_1)^{\#}_w w)^3 a_2 \\
& = & ((a_1)^{\#}_w w)^2 [(a_1)^{\#}_w w a_1] + ((a_1)^{\#}_w w)^2 [(a_1)^{\#}_w w a_2] \\
& = & ((a_1)^{\#}_w w)^2 [a_1 w (a_1)^{\#}_w] + ((a_1)^{\#}_w w)^2 [(a_1)^{\#}_w w a_2] \\
& = & ((a_1)^{\#}_w w)[(a_1)^{\#}_w w a_1 w (a_1)^{\#}_w] + ((a_1)^{\#}_w w)^2 [(a_1)^{\#}_w w a_2] \\
& = & (a_1)^{\#}_w w (a_1)^{\#}_w + ((a_1)^{\#}_w w)^2 [(a_1)^{\#}_w w a_2] \\
& = & (a_1)^{\#}_w w (a_1)^{\#}_w + ((a_1)^{\#}_w w)^2 [(a_1)^{\#}_w w a_2] \\
& = & a^{\textcircled{W}}_w w a^{\textcircled{W}}_w + ((a_1)^{\#}_w w)^2 [(a_1)^{\#}_w w a_2].
\end{array}$$

If $a_1 w a_2 = 0$, as in the argument above, we have $(a_1)^{\#}_w w a_2 = 0$, and so $((a_1)^{\#}_w w)^2 [(a_1)^{\#}_w w a_2] = 0$.

If $((a_1)^{\#}_w w)^2 [(a_1)^{\#}_w w a_2] = 0$, then $[a_1 w (a_1)^{\#}_w w (a_1)^{\#}_w] w [(a_1)^{\#}_w w a_2] = 0$; hence, $[(a_1)^{\#}_w w a_1 w (a_1)^{\#}_w] w [(a_1)^{\#}_w w a_2] = 0$. This implies that $(a_1)^{\#}_w w [(a_1)^{\#}_w w a_2] = 0$. Moreover, $[a_1 (w (a_1)^{\#}_w)^2] w a_2 = 0$. We infer that $(a_1)^{\#}_w w a_2 = 0$. Similarly to the preceding argument, we have $a_1 w a_2 = 0$. Then $a_1 w a_2 = 0$ if and only if $((a_1)^{\#}_w w)^2 [(a_1)^{\#}_w w a_2] = 0$. Therefore $a_1 w a_2 = 0$ if and only if $(awa)^{\textcircled{W}}_w = a^{\textcircled{W}}_w w a^{\textcircled{W}}_w$, as desired.

(1) $\Rightarrow$ (5) Since $(a_1)^{\#}_w = a^{\textcircled{\dagger}}_w$, we see that $a_1 \in R^{\textcircled{\#}}_w$. Moreover, we have

$$a_1 w (a_1)^{\textcircled{\#}}_w = a_1 w (a_1)^{\#}_w = (a_1)^{\#}_w w a_1 = (a_1)^{\textcircled{\#}}_w w a_1.$$

(5) $\Rightarrow$ (1) Since $a_1 w (a_1)^{\textcircled{\#}}_w = (a_1)^{\textcircled{\#}}_w w a_1$, we see that $(a_1)^{\textcircled{\#}}_w = (a_1)^{\#}_w$. Thus, we have

$$a^{\textcircled{W}}_w = (a_1)^{\#}_w = (a_1)^{\textcircled{\#}}_w = a^{\textcircled{\dagger}}_w,$$

as asserted. $\square$

**Lemma 4.6.** *Let $a \in R^{\textcircled{W}}_w$ and $i(a^{D,w}) = k$. Then the following hold:*

(1) $a^{\textcircled{W}}_w (wa)^k = a^{D,w} (wa)^k$.
(2) $(a^{D,w} w)^k = a^{\textcircled{W}}_w w (a^{D,w} w)^{k-1}$.

*Proof.* In view of Theorem 3.1, $a \in R^{D,w}$. Let $x = a^{\textcircled{W}}_w$. Then $a(wx)^2 = x, (aw)^k = (xw)(aw)^{k+1}$. Hence $(aw)(xw)^2 = xw, (aw)^k = (xw)(aw)^{k+1}$. In view of [17, Lemma 2.2], $aw \in R^D$ and $(aw)^D = (xw)^{k+1}(aw)^k$. Also we have $wa(wx)^2 = wx, (wa)^{k+1} = (wx)(wa)^{k+2}$. By using [17, Lemma 2.2] again,

$(wa)^D = (wx)^{k+2}(wa)^{k+1}$. Accordingly,

$$\begin{array}{rcl} a^{D,w} & = & (aw)^D a(wa)^D = (xw)^{k+1}(aw)^k a(wx)^{k+2}(wa)^{k+1} \\ & = & (xw)^{k+1}(aw)^{k+1}(xw)^{k+1}x(wa)^{k+1} \\ & = & (xw)^{k+1}(aw)(xw)x(wa)^{k+1} \\ & = & (xw)^{k+1}[a(wx)^2](wa)^{k+1} = (xw)^{k+1}x(wa)^{k+1} \\ & = & x(wx)^{k+1}(wa)^{k+1}. \end{array}$$

(1) We verify that

$$\begin{array}{rcl} a^{D,w}(wa)^k & = & x(wx)^{k+1}(wa)^{k+1}(wa)^k \\ & = & x(wx)^{k+1}(wa)^{2k+1} = x(wx)^k[(wx)(wa)^{k+1}](wa)^k \\ & = & x(wx)^k(wa)^{2k} = x(wx)^{k-1}[(wx)(wa)^{k+1}](wa)^{k-1} \\ & = & x(wx)^{k-1}(wa)^{2k-1} = \cdots = x(wx)(wa)^{k+1} \\ & = & x(wa)^k; \\ a_w^{\textcircled{W}}(wa)^k & = & x(wa)^k. \end{array}$$

Therefore $a_w^{\textcircled{W}}(wa)^k = a^{D,w}(wa)^k$.

(2) We easily check that

$$\begin{array}{rcl} (a^{D,w}w)^k & = & [x(wx)^{k+1}(wa)^{k+1}w][x(wx)^{k+1}(wa)^{k+1}w](a^{D,w}w)^{k-2} \\ & = & x(wx)^{k+1}(wa)^{k+1}(wx)^{k+2}(wa)^{k+1}w(a^{D,w}w)^{k-2} \\ & = & x(wx)^{k+1}[(wa)^{k+1}(wx)^{k+1}][(wx)(wa)^{k+1}]w(a^{D,w}w)^{k-2} \\ & = & x(wx)^{k+1}[(wa)(wx)](wx)(wa)^{k+1}w(a^{D,w}w)^{k-2} \\ & = & x(wx)^{k+1}w[a(wx)^2](wa)^{k+1}w(a^{D,w}w)^{k-2} \\ & = & x(wx)^{k+1}(wx)(wa)^{k+1}w(a^{D,w}w)^{k-2} \\ & = & xw[x(wx)^{k+1}(wa)^{k+1}]w(a^{D,w}w)^{k-2} \\ & = & xw(a^{D,w}w)^{k-1} = a_w^{\textcircled{W}}w(a^{D,w}w)^{k-1}. \end{array}$$

$\square$

Finally, we present various conditions under which an element and its weak weighted group inverse commute with the weight.

**Theorem 4.7.** *Let $a \in R_w^{\textcircled{W}}$ and $i(a^{D,w}) = k$. Then the following are equivalent:*

(1) $(aw)(a_w^{\textcircled{W}}w) = (a_w^{\textcircled{W}}w)(aw)$.
(2) $(aw)^k(a_w^{\textcircled{W}}w) = (aw)^k(a^{D,w}w)$.
(3) $(aw)^k = (aw)^{k+1}(a_w^{\textcircled{W}}w)$.
(4) $(a_w^{\textcircled{W}}w)(a^{D,w}w) = (a^{D,w}w)(a_w^{\textcircled{W}}w)$.

*Proof.* (1) $\Rightarrow$ (2) Since $(aw)(a_w^{\textcircled{W}}w) = (a_w^{\textcircled{W}}w)(aw)$, we verify that $(aw)^2(a_w^{\textcircled{W}}w) = (aw)[(aw)(a_w^{\textcircled{W}}w)] = (aw)[(a_w^{\textcircled{W}}w)(aw)] = [(aw)(a_w^{\textcircled{W}}w](aw) = [(a_w^{\textcircled{W}}w)(aw)](aw) =$

$(a_w^{\textcircled{W}}w)(aw)^2$. By iteration of this process, we prove that $(aw)^k(a_w^{\textcircled{W}}w) = (a_w^{\textcircled{W}}w)(aw)^k$. In light of Lemma 4.6, $(aw)^k(a_w^{\textcircled{W}}w) = [a_w^{\textcircled{W}}(wa)^k]w = (aw)^k(a^{D,w}w)$.

(2) $\Rightarrow$ (3) It is easy to check that

$$(aw)^{k+1}a_w^{\textcircled{W}}w = (aw)[(aw)^k(a_w^{\textcircled{W}}w] = (aw)^{k+1}a^{D,w}w = (aw)^k,$$

as required.

(3) $\Rightarrow$ (1) Since $(aw)^k = (aw)^{k+1}a_w^{\textcircled{W}}w$, we have

$$\begin{array}{rcl} a_w^{\textcircled{W}}w(aw)^k & = & [a_w^{\textcircled{W}}w(aw)^{k+1}]a_w^{\textcircled{W}}w \\ & = & (aw)^k a_w^{\textcircled{W}}w. \end{array}$$

In view of Theorem 2.3, there exist $z, y \in R$ such that

$$a = z + y, z^*y = ywz = 0, z \in R_w^{\#}, y \in R_w^{nil}.$$

Explicitly, $a_w^{\textcircled{W}} = z_w^{\#}$ and $(yw)^k = 0$. Since $aw = zw + yw$ and $(yw)(zw) = 0$, then we have

$$(aw)^k = \sum_{i=0}^{k}(zw)^i(yw)^{k-i} = \sum_{i=1}^{k}(zw)^i(yw)^{k-i}.$$

We check that

$$\begin{array}{rcl} (aw)^k a_w^{\textcircled{W}}w & = & [\sum\limits_{i=1}^{k}(zw)^i(yw)^{k-i}]z_w^{\#}w \\ & = & (zw)^k z_w^{\#}w. \end{array}$$

Moreover, we have

$$\begin{array}{rcl} a_w^{\textcircled{W}}w(aw)^k & = & z_w^{\#}w[\sum\limits_{i=1}^{k}(zw)^i(yw)^{k-i}] \\ & = & z_w^{\#}w(zw)^k + z_w^{\#}w\sum\limits_{i=1}^{k-1}(zw)^i(yw)^{k-i}]. \end{array}$$

Since $(zw)(z_w^{\#}w) = (z_w^{\#}w)(zw)$, by induction, we have $(zw)^k(z_w^{\#}w) = (z_w^{\#}w)(zw)^k$. This implies that $z_w^{\#}w[\sum\limits_{i=1}^{k}(zw)^i(yw)^{k-i}] = 0$. Accordingly, we have

$$\sum_{i=1}^{k}(zw)^i(yw)^{k-i} = (zw)(z_w^{\#}w)[\sum_{i=1}^{k-1}(zw)^i(yw)^{k-i}] = 0.$$

That is,

$$(zw)(yw)^{k-1} + (zw)^2(yw)^{k-2} + \cdots + (zw)^{k-1}(yw) = 0.$$

Since $(yw)^k=0$, we see that

$$\begin{array}{lll}(zw)^{k-1}(yw)^{k-1} & = & -[(zw)(yw)^{k-1}+(zw)^2(yw)^{k-2}\\ & + & \cdots+(zw)^{k-2}(yw)^2](yw)^{k-2}=0.\end{array}$$

As $zw\in R^{\#}$, we see that $zw(yw)^{k-1}=[(zw)^{\#}]^{k-2}[(zw)^{k-1}(yw)^{k-1}]=0$; and then

$$(zw)^2(yw)^{k-2}+\cdots+(zw)^{k-1}(yw)=0.$$

Furthermore, we have $zw(yw)^{k-2}=0$. By iteration of this process, we have $(zw)(yw)=0$.

Accordingly, we have

$$\begin{array}{lll}(aw)(a_w^{\textcircled{w}}w) & = & (zw+yw)z_w^{\#}w\\ & = & zwz_w^{\#}w+(ywz)[wz_w^{\#}]^2w=zwz_w^{\#}w,\\ (a_w^{\textcircled{w}}w)(aw) & = & z_w^{\#}w(z+y)w=z_w^{\#}wzw+z_w^{\#}wyw\\ & = & z_w^{\#}wzw+(z_w^{\#}w)^2(zwyw)=z_w^{\#}wzw.\end{array}$$

Hence, $(aw)(a_w^{\textcircled{w}}w)=(a_w^{\textcircled{w}}w)(aw)$, as required.

(1) $\Rightarrow$ (4) Let $x=a_w^{\textcircled{w}}$. Then $awxw=xwaw$ and $xw(aw)^{n+1}=(aw)^n(n\in\mathbb{N}), xwawx=(xwa)wx=(awx)wx=a(wx)^2=x$. Hence, $x=a^{D,w}$. This implies that $a_w^{\textcircled{w}}wa^{D,w}w=a^{D,w}wa_w^{\textcircled{w}}w$.

(4) $\Rightarrow$ (1) By using Cline's formula (see [5, Theorem 2.1]), we have $a^{D,w}w=(aw)^D$. Hence, $(a_w^{\textcircled{w}}w)(aw)^D=(aw)^D(a_w^{\textcircled{w}}w)$.

By virtue of Theorem 2.3, there exist $z,y\in R$ such that

$$a=z+y, z^*y=ywz=0, z\in R_w^{\#}, y\in R_w^{nil}.$$

Explicitly, $a_w^{\textcircled{w}}=z_w^{\#}$ and $(yw)^k=0$. Since $aw=zw+yw$ and $(yw)(zw)=0$, it follows by [1, Corollary 3.5] that

$$(aw)^D=(zw)^{\#}+\sum_{i=1}^{k-1}((zw)^{\#})^{n+1}(yw)^i.$$

It is easy to verify that

$$\begin{array}{lll}(aw)^Da_w^{\textcircled{w}}w & = & [(zw)^{\#}+\sum\limits_{i=1}^{k-1}((zw)^{\#})^{n+1}(yw)^i]z_w^{\#}w\\ & = & (zw)^{\#}z_w^{\#}w.\end{array}$$

Moreover, we have

$$\begin{aligned} a_w^{\textcircled{w}}w(aw)^D &= z_w^{\#}w[(zw)^{\#}+\sum_{i=1}^{k-1}((zw)^{\#})^{i+1}(yw)^i] \\ &= z_w^{\#}w(zw)^{\#}+z_w^{\#}w[\sum_{i=1}^{k-1}((zw)^{\#})^{i+1}(yw)^i]. \end{aligned}$$

Sine $(zw)^{\#}z_w^{\#}w=z_w^{\#}w(zw)^{\#}$, we have

$$z_w^{\#}w[\sum_{i=1}^{k-1}((zw)^{\#})^{i+1}(yw)^i]=0.$$

Thus,

$$\sum_{i=1}^{k-1}((zw)^{\#})^{i+1}(yw)^i=0.$$

That is,

$$[(zw)^{\#}]^2(yw)+[(zw)^{\#}]^3(yw)^2+\cdots+[(zw)^{\#}]^k(yw)^{k-1}=0.$$

Since $(yw)^k=0$, we have $(zw)(yw)^{k-2}=0$; hence, $(zw)(yw)^{k-3}=0$. By iteration of this process, we see that $(zw)(yw)=0$.

Therefor we have

$$\begin{aligned} (aw)(a_w^{\textcircled{w}}w) &= (zw+yw)z_w^{\#}w \\ &= zwz_w^{\#}w+(ywz)[wz_w^{\#}]^2w=zwz_w^{\#}w, \\ (a_w^{\textcircled{w}}w)(aw) &= z_w^{\#}w(z+y)w=z_w^{\#}wzw+z_w^{\#}wyw \\ &= z_w^{\#}wzw+(z_w^{\#}w)^2(zwyw)=z_w^{\#}wzw. \end{aligned}$$

Hence, $(aw)(a_w^{\textcircled{w}}w)=(a_w^{\textcircled{w}}w)(aw)$, as asserted. □

As an immediate consequence of Theorem 4.7, we derive

**Corollary 4.8.** *Let $a\in R_w^{\textcircled{w}}, w\in R^{-1}$ and $i(a^{D,w})=k$. Then the following are equivalent:*

(1) $awa_w^{\textcircled{w}}=a_w^{\textcircled{w}}wa$.
(2) $(aw)^ka_w^{\textcircled{w}}=(aw)^ka^{D,w}$.
(3) $(aw)^k=(aw)^{k+1}a_w^{\textcircled{w}}w$.
(4) $a_w^{\textcircled{w}}wa^{D,w}=a^{D,w}wa_w^{\textcircled{w}}$.

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